HODGE DECOMPOSITION

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Abstract. This is an expository paper on the Hodge Decomposition Theorem. The aim is to give a proof of this theorem. Along the way we will discuss some machinery involving Sobolev spaces and differential operators and an application to de Rham cohomology.

Contents

1. Introduction

This paper is an exposition on the Hodge decomposition theorem. We aim to study p-forms by considering the action of the Laplace-Beltrami operator. This is an extension of the Laplace operator in calculus. The kernel of this action are special forms called harmonic forms. The space of harmonic forms turns out to be finite dimensional which allows us to take orthogonal complements. This forms the basis of the Hodge decomposition theorem.

The Hodge decomposition theorem has many useful applications. We will discuss one application to de Rham cohomology which says that each cohomology class has a unique harmonic representative, i.e. we have a correspondence between de Rham cohomology groups H^p_{dR} and p-harmonic forms.

We will mostly follow [5]. For this proof, we will need some background in Sobolev spaces and differential operators. We will at least state all results needed to prove the Hodge decomposition theorem in this paper but may refer the reader to [5] for some of the proofs. For a more general exposition of this theory, one may check out [1].

2. Laplace-Beltrami Operator

We want to study p-forms by studying the action of a Laplacian operator. From calculus, recall the Laplacian operator

$$\sum_i \frac{\partial^2}{\partial x_i^2}.$$

We want to extend this operator to an operator on p-forms. In this section we will construct this operator, the Laplace-Beltrami operator.

2.1. **Hodge Star.**

Let V be a finite dimensional vector space with an inner product $\langle \cdot, \cdot \rangle$. We can extend this inner product to exterior powers $\bigwedge^k V$ as follows. On decomposable tensors, $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_k$ and $\beta = \beta_1 \wedge \cdots \wedge \beta_k$, define

$$\langle \alpha, \beta \rangle = \det(\langle \alpha_i, \beta_j \rangle)_{ij}.$$

One can check that this extends to a unique inner product on $\bigwedge^k V$.
We want to define an operator on the exterior algebra $\bigwedge V$. We will use the following lemma.

Lemma 2.1. *Let ϕ be a linear functional on V. Then there exists a unique $v \in V$ such that for all $w \in V$*

$$\phi(w) = \langle w, v \rangle.$$

Proof. See [3]. $\square$

Given $\alpha \in \bigwedge^p V$, we can define a linear functional as follows. Let $\beta \in \bigwedge^{n-p} V$, so $\alpha \wedge \beta \in \bigwedge^n V$, where n is the dimension of V. Let $\omega \in \bigwedge^n V$ be given by determinant. Then since $\bigwedge^n V$ is one dimensional, ω is a basis. We can then write

$$\alpha \wedge \beta = \phi_\alpha(\beta)\omega$$

where the expression on the right is unique. Note that ϕ_α is a linear functional on $\bigwedge^{n-p} V$.

Definition 2.2. Let $\alpha \in \bigwedge^p V$. We denote by $*\alpha$ the element of $\bigwedge^{n-p} V$ such that

$$\phi_\alpha(\beta) = \langle \beta, *\alpha \rangle,$$

as given by Lemma 2.1. We call this operator $*$ the Hodge star operator.

The case we are most interested in is when we have a compact orientable Riemannian manifold M. The Riemannian metric allows us to define the star operator $*$ taking p-forms on M to $(n-p)$-forms on M.

If we choose an orthonormal basis $e_1, ..., e_n$ of V, we can write the Hodge star operator more explicitly. Note that a choice of basis fixes an orientation on V. That is if we have another basis $f_1, ..., f_n$, we have a unique linear transformation T mapping $e_i \mapsto f_i$. If T has positive determinant, these bases have same orientation, otherwise, they have the opposite orientation.

Proposition 2.3. *The Hodge star operator*

$$* : \bigwedge V \to \bigwedge V$$

satisfies the property that for any orthonormal basis $f_1, ..., f_n$*,*

$$*(f_1 \wedge \cdots \wedge f_k) = \pm f_{k+1} \wedge \cdots \wedge f_n$$

where the sign depends on the orientation. Here we set

$$*(1) = \pm f_1 \wedge \cdots \wedge f_n$$

and

$$*(f_1 \wedge \cdots \wedge f_n) = \pm 1.$$

We can check that this follows from the original definition by expanding out with the chosen bases.

The definition of the Hodge star operator may seem rather opaque. However, there is geometric intuition behind this operation, particularly it encodes orthogonality. Suppose that $j_1, ..., j_k$ is an orthogonal basis of a subspace. By Gram-Schmidt we may extend this to an orthonormal basis $j_1, ..., j_n$ modulo some scaling factors. The properties in Definition 2.3 show that $*(j_1 \wedge \cdots \wedge j_k)$ is the element $j_{k+1} \wedge \cdots \wedge j_n$, where $j_{k+1}, ..., j_n$ spans a subspace orthogonal to the one spanned by $j_1, ..., j_k$.

This is most easily visualized on low dimensions of $\mathbb{R}^n$ which we illustrate in the following examples.

Example 2.4. We first consider the plane $\mathbb{R}^2$. We choose the ordered basis dx, dy on 1-forms. Following the properties in Proposition 2.3, we can compute the star operator explicitly.

- $*(1) = dx \wedge dy$
- $*(dx) = dy$
- $*(dy) = -dx$
- $*(dx \wedge dy) = 1$

If we abuse notation and think of dx as the standard basis vector $\hat{\imath}$ and dy by $\hat{\jmath}$, We see that $*(x\hat{\imath} + y\hat{\jmath}) = -y\hat{\imath} + x\hat{\jmath}$, which rotates counter clockwise by 90 degrees.

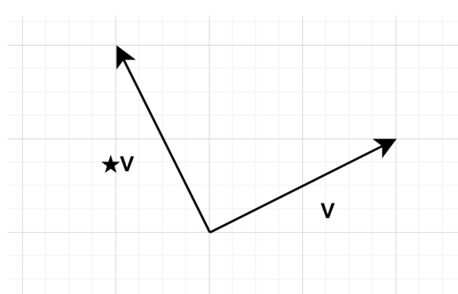

FIGURE 1. The star operator on $\mathbb{R}^2$ corresponds to rotating counterclockwise by 90 degrees

Example 2.5. Another easy case to visualize is $\mathbb{R}^3$. By choosing an ordered basis dx, dy, dz, we can again compute the star operator explicitly.

- $*(1) = dx \wedge dy \wedge dz$
- $*(dx) = dy \wedge dz$
- $*(dy) = dz \wedge dx$
- $*(dz) = dx \wedge dy$
- $*(dx \wedge dy) = dz$
- $*(dx \wedge dz) = -dy$
- $*(dy \wedge dz) = dx$
- $*(dx \wedge dy \wedge dz) = 1.$

In particular if we abuse notation again and view dx, dy, dz as $\hat{\imath}, \hat{\jmath}, \hat{k}$, we see that the star operator agrees with the cross product on $\mathbb{R}^3$. The cross product of two linearly independent vectors u, v gives the normal vector to the plane spanned u, v, which connects the star operator to orthogonality.

2.2. **Laplace-Beltrami.** Now we are ready to construct the Laplace-Beltrami operator. Recall the Laplacian operator

$$\sum_i \frac{\partial^2}{\partial x_i^2}. \tag{2.6}$$

We would like to extend this operator to the space of p-forms Ω^p. We will do this as follows.

Definition 2.7. Define $\delta : \Omega^p \to \Omega^{p-1}$ by

$$\delta = (-1)^{n(p+1)+1} * d * .$$

On 0-forms this is the 0 map. The -1 term is added to be compatible with the -1 term in $*^2$.

Definition 2.8. We define the Laplace-Beltrami operator, Δ, by

$$\Delta = \delta d + d\delta.$$

We will call this operator the Laplacian.

We should check that on 0-forms, the Laplacian Δ matches the Laplacian operator (2.6) on $\mathbb{R}^n$. The 0-forms are just C^∞ functions. Let f be such a function.

Then

$$\begin{aligned}
\Delta f &= \delta df \\
&= *d * df \\
&= *d * \left(\frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n \right) \\
&= *d \left(\frac{\partial f}{\partial x_1} dx_2 \wedge \cdots \wedge dx_n + \cdots + \frac{\partial f}{\partial x_n} dx_1 \wedge \cdots \wedge dx_{n-1} \right) \\
&= * \left(\frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} \right) dx_1 \wedge \cdots \wedge dx_n \\
&= \left(\frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} \right).
\end{aligned}$$

To get from the first line to the second line recall that δ is the zero operator in 0-forms. To get from the fourth to fifth line, after expanding the exterior derivative, the signs of terms with mixed partials cancel out due to the signs form $*$. Thus, it makes sense to consider the Laplace-Beltrami operator as an extension of the classical Laplacian.

The star operator can also be used to define an inner product on Ω^p by

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge *(\beta).$$

This can be extended linearly to $\sum \Omega^p$ by requiring p and q forms to be orthogonal for $p \neq q$. This inner product is symmetric and positive definite. The importance of this inner product is that it makes the Laplacian a self-adjoint operator.

Theorem 2.9. *If α, β are p-forms, then*

$$\langle \Delta\alpha, \beta \rangle = \langle \alpha, \Delta\beta \rangle.$$

We see that to prove this, it is enough to show that δ is the adjoint to d.

Proposition 2.10. *For α a $(p-1)$-form, β a p-form, we have that*

$$\langle d\alpha, \beta \rangle = \langle \alpha, \delta\beta \rangle.$$

Proof. We will compute $\int_M d(\alpha \wedge *\beta)$ in two ways. First, by Stokes' theorem and that M is compact, we know that

$$\int_M d(\alpha \wedge *\beta) = 0.$$

For the second computation, note that

$$*\delta = (-1)^{n(p+1)+1} * *d* = -1^{(p-1)(n-p-1)} d*,$$

where we use that $** = (-1)^{p(n-p)}$. Then we see that

$$\begin{aligned}\int_M d(\alpha \wedge *\beta) &= \int_M d\alpha \wedge *\beta + (-1)^{p-1} \alpha \wedge d * \beta \\ &= \int_M d\alpha \wedge *\beta - \alpha \wedge *\delta\beta.\end{aligned}$$

Then

$$0 = \int_M d\alpha \wedge *\beta - \int_M \alpha \wedge *\delta\beta = \langle d\alpha, \beta \rangle - \langle \alpha, \delta\beta \rangle,$$

which completes the proof. □

Our goal is to prove the Hodge decomposition theorem which says that the space of p-forms decomposes as the direct sum of the image and kernel of the Laplacian.

The elements of the kernel have a special name.

Definition 2.11. The p-forms in the kernel of the Laplacian, denoted

$$H^p = \{\omega \in \Omega^p(M) : \Delta\omega = 0\}$$

are called the harmonic p-forms.

Example 2.12. The harmonic 0-forms are constant functions. This is since if α is a p-form then,

$$0 = \langle \Delta\alpha, \alpha \rangle = \langle \delta\alpha, \delta\alpha \rangle + \langle d\alpha, d\alpha \rangle.$$

Hence , $d\alpha = 0$. In particular, if α is a 0-form, then α is a function, so it must be constant.

We would like to understand harmonic p-forms in general. Finding harmonic forms is related to solving equations of the form $\Delta\omega = 0$, or more generally equations of the form $\Delta\omega = \alpha$. Thus, we will be interested in solving such equations. This is not an easy task.

If we happen to have a solution ω to the equation $\Delta\omega = \alpha$, then we can form a bounded linear function ℓ on Ω^p by

$$\ell(\beta) = \langle \omega, \beta \rangle.$$

Now given any $\gamma \in \Omega^p$, we have that

$$\ell(\Delta\gamma) = \langle \omega, \Delta\gamma \rangle = \langle \Delta\omega, \gamma \rangle, = \langle \alpha, \gamma \rangle.$$

This observation gives the following definition.

Definition 2.13. A bounded linear operator ℓ on Ω^p satisfying

$$\ell(\Delta\gamma) = \langle \alpha, \gamma \rangle,$$

is called a weak solution of the equation $\Delta\omega = \alpha$.

Weak solutions come from the theory of distributions. Distributions will not be discussed in this paper but in a sense, they can be thought of as generalized functions. The idea here is that we think of functions as objects acting on a class of test functions. Then we can reformulate our differential equation using test functions and try to find a solution in that sense. This is what is being done by introducing the functional ℓ. Here our test function are p-forms β. We are defining the weak solution as a functional that acts on test functions in the same way a real solution ω acts on test functions by inner product.

In general, it is much easier to find a weak solution. One trick that we will use involves the Hahn-Banach theorem.

Definition 2.14. Let E be a vector space over $\mathbb{R}$ or $\mathbb{C}$. A function $p : E \to \mathbb{R}$ satisfying

$$p(\lambda x) = \lambda p(x)$$

for all $x \in E$, $\lambda > 0$ and

$$p(x + y) \leq p(x) + p(y)$$

for $x, y \in E$ is called a sublinear functional.

Theorem 2.15. *(Hahn-Banach) Let $G \subset E$ be a subspace, p a sublinear functional and $g : G \to \mathbb{R}$ a linear functional satisfying*

$$g(x) \leq p(x)$$

for all $x \in G$. Then we can extend g to a functional $f : E \to \mathbb{R}$ on all of E such that

$$f(x) \leq p(x)$$

for all $x \in E$.

Proof. The proof is found in [1, Thm. 1.1]. □

The Hahn-Banach theorem can be used as follows. We have a subspace $\Delta(\Omega^p)$. On this subspace, we can define a linear functional ℓ by

$$\ell(\Delta\gamma) = \langle \alpha, \gamma \rangle.$$

Then by proving some bound on the norm, which we use as the sublinear functional, we can apply Hahn-Banach to extend ℓ to a linear functional on all of Ω^p. By definition, this is a weak solution to $\Delta\omega = \alpha$.

Fortunately, it turns out that finding a weak solution is enough to find an actual solution to the equation. In the next few sections, we develop the tools necessary to prove this fact.

3. Sobolev Spaces

In this section, we introduce Sobolev spaces. Sobolev spaces will give us the framework and tools to understand the behavior of weak solutions. In particular to investigate regularity i.e differentiability of these solutions.

To define Sobolev spaces, we need to understand differentiability in the weak sense.

Definition 3.1. Let f be a function on $\mathbb{R}^n$. Let α be an index, i.e. a n-tuple of natural numbers. We say g is the α^{th} weak derivative of f if g satisfies integration by parts with all test functions $\phi \in C(\mathbb{R}^n)$ with compact support. That is

$$\int fD^\alpha\phi = (-1)^{|\alpha|}\int g\phi.$$

For notation, we will denote g by $D^\alpha f$ as well.

As a remark, weak differentiability is related to distributions as well as we are characterizing a derivative by how it acts on test functions. Here "acting" is satisfying integration by parts.

Example 3.2. A canonical example of a nondifferentiable function with a weak derivative is the absolute value function $f(x) = |x|$ on $\mathbb{R}$. From calculus, we know that f is not differentiable at 0. However, f has a weak derivative given by

$$f'(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0. \end{cases}$$

The idea here is that the nondifferentiable part of f is just a single point, but integration doesn't care about measure zero sets. By the same reasoning, we also see that weak derivatives are not unique. This is since we modify the weak derivative on a measure zero set but this modification can't be detected by integration.

Definition 3.3. We define the Sobolev space $W^{k,p}(\mathbb{R}^n)$ to be the space of all ($\mathbb{C}^m$ valued) functions $f \in L^p(\mathbb{R}^n)$ whose weak derivatives up to order k also belong to $L^p(\mathbb{R}^n)$.

The Sobolev space $W^{k,p}$ comes with a norm

$$|f|_{W^{k,p}} = |f|_p + \sum_{|\alpha|\leq k} |D^\alpha f|_p$$

given by taking the sum of the L^p norms of the function and its weak derivatives up to order k making it a Banach space.

We are particularly interested in the case when $p = 2$ For notation, we will denote

$$H_k := W^{k,2}.$$

Note that it is more standard to denote the indices by superscripts but we will use subscripts to avoid confusion with the notation for harmonic forms. We use the notation H to emphasize that in the $p = 2$ case, the Sobolev space is actually a Hilbert space. It has an inner product

$$\langle f, g\rangle_k = \langle f, g\rangle_2 + \sum_{|\alpha| \leq k} \langle D^\alpha f, D^\alpha g\rangle_2$$

induced by the inner product on the Hilbert space $L^2(\mathbb{R}^n)$.

Equivalently, we can define H_k in terms of Fourier series.

Denote by $\mathcal{S}$ the space of vectors in $\mathbb{C}^m$ indexed by n-tuples of integers $\xi = (\xi_1, ..., \xi_n)$.

Given $u = \{u_\xi\} \in \mathcal{S}$, we view u as a formal Fourier series

$$\sum_\xi u_\xi e^{ix\xi}.$$

In this situation, we view weak derivatives as taking formal derivatives of Fourier series

$$D^\alpha u = \sum_\xi \xi^\alpha u_\xi e^{ix\xi}.$$

We have inner products

$$\langle u, v\rangle_k = \sum_\xi (1 + |\xi|^2)^k |u_\xi| \cdot |v_\xi|,$$

which gives the norm

$$|u|_k^2 = \sum_\xi (1 + |\xi|^2)^k |u_\xi|^2.$$

Definition 3.4. We define the Sobolev space

$$H_k := \{u \in \mathcal{S} : |u|_k < \infty\}.$$

For notation, we will denote by $H_{-\infty}$ the union of all H_k.

The equivalence to the Sobolev spaces defined earlier can be seen as follows. Let u be a function and $\{u_\xi\}$ denote its Fourier coefficients. Then for a positive constant c, we have the inequality

$$c(1 + |\xi|^2)^k |u_\xi|^2 \leq \sum_{|\alpha|=0}^{k} |D^\alpha u|^2 \leq (1 + |\xi|^2)^k |u_\xi|^2$$

showing equivalence of norms.

There are a number of properties and inequalities that we will need. These will be stated without proof. Proofs of the following can be found in chapter 6 of [5].

Let $\mathcal{P}$ denote the space of $\mathbb{C}^m$ valued C^∞ functions on $\mathbb{R}^n$ which are 2π-periodic. Then by viewing each function by its Fourier coefficients, we see that $\mathcal{P}$ is a subspace of H_k for all k. Moreover, $\mathcal{P}$ is dense as it contains the sequences with only finitely many terms nonzero.

Proposition 3.5. *If $u \in H_{k+[\alpha]}$, then*

$$|D^\alpha u|_k \leq |u|_{k+[\alpha]}.$$

Proof. See [5, Thm. 6.18(h)]. □

The upshot here is that D^α is a bounded operator from $H_{k+[\alpha]}$ to H_k.

Proposition 3.6. *If $u \in H_{k+\ell}$, then*

$$|u|_{k+\ell} = \sup_{v \in H_{k-\ell}} \frac{|\langle u, v\rangle_k|}{|v|_{k-\ell}}.$$

Proof. See [5, Thm. 6.18(f)]. □

This result will be applied in the case when $\ell = 0$. This tells us that we can check for equality by testing against elements of H_k. That is, $u \in H_k$ is 0 if and only if $\langle u, v\rangle_k = 0$ for all $v \in H_k$.

Proposition 3.7. *(Peter-Paul Inequality) If $k' < k < k''$ and $u \in H_{k''}$, then for $\epsilon > 0$, there is a constant c depending on ϵ such that*

$$|u|_k^2 \leq \epsilon |u|_{k''}^2 + c|u|_{k'}^2.$$

Proof. See [5, Thm. 6.18(g)]. □

We have two more inequalities regarding periodic functions.

Proposition 3.8. *Let ω be a C^∞ complex valued periodic function on $\mathbb{R}^n$. Then for any $\varphi \in \mathcal{P}$, we have constants c, c' such that*

$$|\omega\varphi|_k \leq c|\omega|_\infty |\varphi|_k + c'|\varphi|_{k-1}.$$

Proof. See [5, Thm. 6.18(i)]. □

Proposition 3.9. *Let ω be as before and $u, v \in H_k$. Then there is a constant c such that*

$$|\langle \omega u, v\rangle_k - \langle u, \overline{\omega} v\rangle_k| \leq c(|u|_k |v|_{k-1} + |u|_{k-1}|v|_k).$$

Proof. See [5, Thm. 6.18(j)] □

3.1. **Sobolev Embedding.** Suppose we have some $u \in H_k$. We view u as a formal Fourier series $\sum_\xi u_\xi e^{ix\xi}$. We want to know when u is an actual function, i.e. when does the Fourier series converge, and moreover if it does converge, how differentiable the function it converges to is in the classical sense. These questions are answered by the Sobolev embedding theorem.

Theorem 3.10. *(Sobolev Embedding) If $u \in H_k$ where $k \geq [n/2] + 1 + m$, then u is a C^m function.*

This is a case of a more general Sobolev embedding theorem, see [1].
The proof follows from the following results.

Lemma 3.11. *(Sobolev Lemma) Let $u \in H_k(\mathbb{R}^n)$. If $k \geq [n/2] + 1$, where $[\cdot]$ denotes the least integer function, then $\sum_\xi u_\xi e^{ix\xi}$ converges uniformly.*

Proof. We will show absolute convergence. Note that $|u_\xi e^{ix\xi}| = |u_\xi|$. Then,

$$\begin{aligned}\sum_{|\xi|<N} |u_\xi| &= \sum_{|\xi|<N} (1+|\xi|^2)^{-k/2}(1+|\xi|^2)^{k/2}|u_\xi| \\ &\leq (\sum_{|\xi|<N} (1+|\xi|^2)^{-k/2})(\sum_{|\xi|<N} (1+|\xi|^2)^{k/2}|u_\xi|) \\ &= (\sum_{|\xi|<N} (1+|\xi|^2)^{-k})^{1/2}(\sum_{|\xi|<N} (1+|\xi|^2)^{k}|u_\xi|^2)^{1/2} \\ &= (\sum_{|\xi|<N} (1+|\xi|^2)^{-k})^{1/2}|u|_k.\end{aligned}$$

The lemma then follows from the fact that $\sum_{|\xi|}(1+|\xi|^2)^{-k}$ converges for $k \geq [n/2]+1$, and by sending N to ∞. □

If we apply the Sobolev lemma to derivatives, we get the following result.

Corollary 3.12. *Let $u \in H_k$ and $k \geq [n/2]+1+m$. By the Sobolev lemma, $u(x) = \sum_\xi u_\xi e^{ix\xi}$ is a continuous function. Then for $|\alpha| \leq m$, the derivatives $D^\alpha u := \sum_\xi \xi^\alpha u_\xi e^{ix\xi}$ converge uniformly.*

Proof. $D^\alpha u$ belongs to $H_{k-|\alpha|}$. Applying the Sobolev lemma to $D^\alpha u$ and using Proposition 3.5 shows that $\sum_\xi \xi^\alpha u_\xi e^{ix\xi}$ converges uniformly. □

The Sobolev embedding gives a partial converse to the following statement.

Theorem 3.13. *If f is a C^m function, then its Fourier coefficients, u_n, are $o(|n|^{-m})$.*

Proof. By integration by parts we know that the norm of the Fourier coefficients of the kth derivative of f is $|n|^k|u_n|$. Then $|n|^m|u_n|$ goes to zero since f is C^m. □

Then we know that $C^m \subset H_{m-1}$. This doesn't quite match up with the Sobolev embedding. To know a function is C^m from only its Fourier coefficients, we need the stronger condition that it is H_k for $k \geq [n/2]+1+m$. For example, the absolute value function has Fourier coefficients which are $o(|n|^{-2})$ but it is not differentiable.

3.2. **Difference Quotients.** A useful technique to see the differentiability of functions in terms of Sobolev spaces is to consider their difference quotient.

Recall from calculus that given a function f, we have the difference quotient

$$f^h(x) = \frac{f(x+h)-f(x)}{|h|} = \frac{T_h(f)-f}{|h|}$$

where

$$T_h(f)(x) := f(x+h)$$

is the translate of f by h.

If ϕ is a periodic function with Fourier coefficients ϕ_ξ, then $T_h(\phi)$ has Fourier coefficients $e^{ih\xi}\phi_\xi$.

Then it makes sense to define translates and difference quotients on H_k in the following way.

Definition 3.14. Let $u \in H_k$. We define the translate by

$$T_h(u) := \{e^{ih\xi}u_\xi\}$$

and the difference quotient by

$$u^h := \frac{T_h(u) - u}{|h|} = \{\left(\frac{e^{ih\xi} - 1}{|h|}\right)u_\xi\}.$$

Some computations show that T_h is an isometry, that is

$$|T_h(u)|_k = |u|_k$$

and that if $u \in H_{k+1}$ then

$$|u^h|_k \leq |u|_{k+1}.$$

We also have a converse to this statement which will be important.

Lemma 3.15. *If $u \in H_k$ and for all h the difference quotients*

$$|u^h|_k \leq C$$

are bounded by some constant C, then $u \in H_{k+1}$.

Proof. We use two tricks. The first trick is to truncate the sequence u. That is for every positive integer N, define u_N to be the sequence such that $(u_N)_\xi = u_\xi$ if $|\xi| \leq N$ and 0 otherwise. Then as long as $|u_N|_{k+1}$ is bounded uniformly in N, $|u|_{k+1}$ is bounded which will show that $u \in H_{k+1}$.

The second trick is to restrict to a line te_i where e_i is a standard basis vector of $\mathbb{R}^n$. Let $h = te_i$ for some constant t. We know that

$$\sum_{|\xi|\leq N}(1 + |\xi|^2)^k|u_\xi|^2|\frac{e^{ih\xi} - 1}{|h|}|^2 \leq |u^h|_k \leq C^2.$$

Now take $t \to 0$. Then $\lim_{t\to 0}\frac{e^{ih\xi}-1}{|h|}$ is just the derivative of $e^{it\xi_i}$ with respect to t so this is $\xi_i e^{it\xi_i}$ which has norm ξ_i. From this, we know that

$$\sum_{|\xi|\leq N}(1 + |\xi|^2)^k|u_\xi|^2|\xi_i|^2 \leq C^2.$$

Then

$$\begin{aligned}
|u_N|_{k+1}^2 &= \sum_{|\xi|\leq N}(1 + |\xi|^2)^{k+1}|u_\xi|^2 \\
&= \sum_{|\xi|\leq N}(1 + |\xi|^2)^k|u_\xi|^2 + \sum_{|\xi|\leq N}(1 + |\xi|^2)^k|u_\xi|^2|\xi|^2 \\
&\leq \sum_{|\xi|\leq N}(1 + |\xi|^2)^k|u_\xi|^2 + \sum_{i=1}^{n}\sum_{|\xi|\leq N}(1 + |\xi|^2)^k|u_\xi|^2|\xi_i|^2 \\
&\leq |u|_k^2 + nC^2.
\end{aligned}$$

This shows that $|u_N|_{k+1}$ is uniformly bounded which completes the proof. $\square$

4. Differential Operators

In this section, we will introduce differential operators. We are interested in a specific type of differentiable operator called an elliptic operator. The highest order part of these operators is strictly positive.

Definition 4.1. A differential operator L of order ℓ on complex valued $C^\infty(\mathbb{R}^n)$ functions is an operator of the form

$$L = \sum_{|\alpha|=0}^{\ell} a_\alpha D^\alpha,$$

where the coefficients a_α are complex valued C^∞ functions on $\mathbb{R}^n$ and for some $|\alpha| = \ell$, $a_\alpha \neq 0$.

A differential operator $L = \sum_{|\alpha|=0}^{\ell} a_\alpha D^\alpha$ is periodic if the coefficients a_α are periodic.

Definition 4.2. A differential operator L of order ℓ on $\mathbb{C}^m$-valued $C^\infty(\mathbb{R}^n)$ functions is a m by m matrix $\{L_{ij}\}$ where each L_{ij} is differential operator on complex valued $C^\infty(\mathbb{R}^n)$ functions, and is periodic if each matrix entry is periodic.

Remark 4.3. We can similarly define differential operators on other function spaces such as the periodic functions $\mathcal{P}$.

There are a couple of inequalities that will be useful.

Lemma 4.4. *Let L be a differential operator on $\mathcal{P}$, the periodic functions, of order ℓ. Let M be a bound on the norms of the coefficients of L. Then we can find constants c, c' such that for all $\varphi \in \mathcal{P}$,*

$$|L\varphi|_k \leq cM|\varphi|_{k+\ell} + c'|\varphi|_{s+\ell-1}.$$

Lemma 4.5. *Let L be a differential operator of order ℓ, and ω a C^∞ periodic function. Then for $u \in H_{k+\ell}$, there exists a constant c such that*

$$|\langle L(\omega^2 u), Lu\rangle_k - |L(\omega u)|_k^2| \leq c(|u|_{k+\ell}|u|_{k+\ell-1}).$$

For proofs see [5] 6.25, 6.27.

4.1. **Elliptic Operators.**

We now turn to elliptic operators.

Let L be a differential operator of order ℓ.

For each matrix entry L_{ij}, we look at the highest order part

$$\sum_{|\alpha|=\ell} a_\alpha^{ij} D^\alpha.$$

Replacing D^α with ξ^α, we get a function on $\mathbb{R}^n$

$$P_{ij}(\xi)(x) = \sum_{|\alpha|=\ell} a_\alpha^{ij}(x)\xi^\alpha.$$

Definition 4.6. The symbol of L is the function p on $\mathbb{R}^n \times \mathbb{R}^n$ given by $(\xi, x) \mapsto \{P_{ij}(\xi)(x)\}$.

Definition 4.7. A differential operator L is elliptic if, for all x and nonzero ξ, $p(\xi, x)$ is nonsingular.

Definition 4.8. If L is an operator on $\mathcal{P}$, with entries

$$L_{ij} = \sum_{|\alpha|=0}^{\ell} a_{\alpha}^{ij} D^{\alpha},$$

we define its adjoint L^* to be the operator with entries

$$L_{ij}^* = \sum_{|\alpha|=0}^{\ell} D^{\alpha} \overline{a_{\alpha}^{ji}}.$$

The adjoint L^* satisfies the adjoint property for the L_2 norm on $\mathcal{P}$. That is for all $\varphi, \psi \in \mathcal{P}$, we have that

$$\langle L\varphi, \psi \rangle = \langle \varphi, L^* \psi \rangle.$$

This follows from integration by parts. We restrict to operators on periodic functions $\mathcal{P}$ since the boundary term in integration by parts is 0.

We look at a couple of examples of elliptic operators.

Example 4.9. The simplest differential operator is $L = \frac{d}{dx}$. The symbol of this differential operator is the 1 by 1 matrix (ξ) which is nonsingular at each nonzero ξ. Thus, this is an example of an elliptic operator. To find its adjoint, we use integration by parts. We see that

$$\langle L\varphi, \psi \rangle = \int \varphi' \overline{\psi} = \int \varphi \overline{\psi}' = \langle \varphi, L\psi \rangle.$$

Thus, integrating by parts tells us that the adjoint $L^* = L$ which matches the definition given.

Example 4.10. Consider the operators

$$L_1 = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$$

and

$$L_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

The operator L_1 has symbol $(\xi^2 - \xi^2) = (0)$, so it is not elliptic. The operator L_2 has symbol $(2\xi^2)$ so it is an elliptic operator.

The second operator is the classical Laplacian operator. Since it is elliptic, we wonder if the generalization, the Laplace-Beltrami operator, is also elliptic. However, this question doesn't make any sense since the Laplace-Beltrami operator is defined on p-forms and isn't a differential operator according to our definition. Using the manifold structure, the Laplace-Beltrami operator locally defines a differential operator. We can then investigate the ellipticity of these induced operators.

Since M is a manifold we have a collection of coordinate charts. Fix a coordinate chart U. Using transition maps, each p-form can be thought of as a C^∞ function from $\mathbb{R}^n$ to $\mathbb{R}^m$ where $\mathbb{R}^m$ where $\mathbb{R}^m$ is identified as the p^{th} wedge of the space of 1-forms which has dimension n, so $m = \binom{n}{p}$.

Under this process, the Laplacian Δ induces an operator L on C^∞ functions from $\mathbb{R}^n$ to $\mathbb{R}^m$. Now we can talk about the ellipticity of the operator L.

Theorem 4.11. *The induced operator L is an order 2 elliptic differential operator.*

Proof. The proof is in 6.35 in [5]. $\square$

We need to be careful that this is invariant under coordinate changes.

Proposition 4.12. *Let $\varphi : U \to V$ be a diffeomorphism and L an order ℓ elliptic operator on U. Then $(\varphi^{-1})^* L \varphi^*$ is an order ℓ elliptic operator on V.*

Proof. The computation can be found in [4]. The idea is that changing coordinates doesn't affect the highest order part which preserves order and ellipticity. □

Why do we care about this specific class of differential operators? The main reason stems from the fundamental inequality which we prove next. In essence, it says that we can understand something about higher order derivatives using lower order ones.

This idea will be applied in the following framework. We are looking at some equation $Lu = f$, and we have a weak solution. We will see that this corresponds to finding some $u \in H_k$ satisfying the equation. But now we can use the special property of ellipticity to gain more derivatives of u in the weak sense. We will continuously repeat this process to get higher and higher derivatives. Then the Sobolev embedding theorem will tell us something about the regularity class, i.e. which C^m the function u belongs to.

Theorem 4.13. *(Fundamental Inequality) Let L be an elliptic operator on $\mathcal{P}$ of order ℓ. For all $u \in H_{k+\ell}$, there exists a constant $c > 0$ such that*

$$|u|_{k+\ell} \le c(|Lu|_k + |u|_k).$$

We will prove this inequality in three steps. We will first show that this inequality holds for elliptic operators which have constant coefficients. Step 2 will show that this inequality holds in a neighborhood of every point. Step 3 will show that we can patch these local solutions to a global one.

One piece of machinery we will use is partitions of unity.

Definition 4.14. Let (U_α) be cover of the manifold M. A partition of unity subordinate to this cover is a collection of C^∞ functions φ_i such that $\varphi_i \ge 0$, each φ_i has support in some U_α and for all $p \in M$, $\sum_i \varphi_i(p) = 1$. Another variation of partitions of unity has $\sum_i \varphi_i^2(p) = 1$.

Theorem 4.15. *If M is a differentiable manifold and (U_α) is an open cover, then a countable partition of unity (and the variation) subordinate to this cover exists.*

Proof. See [5, Thm. 1.11]. □

Now we will prove the fundamental inequality. We follow the proof in [5, Thm. 6.29].

Step 1: First consider the case when L is an elliptic operator of order ℓ, with only degree ℓ terms with constant coefficients and let p denote its symbol. Since we have constant coefficients, p only depends on ξ and not on x. We have that

$$|Lu|_k^2 = \sum_\xi |p(\xi)u_\xi|^2(1 + |\xi|^2)^k.$$

Let ξ', φ be arbitrary of norm 1. By ellipticity, we know that $|p(\xi)\varphi|^2 > 0$. By compactness of unit sphere, we have that $|p(\xi')\varphi|^2 \ge c$ for some constant c. Now applying this with $\xi' = \frac{\xi}{|\xi|}$ and $\varphi = \frac{u_\xi}{|u_\xi|}$ gives that

$$|p(\xi)u_\xi|^2 \ge c|\xi|^{2\ell}|u_\xi|^2.$$

Since the coefficients are constant, c doesn't depend on u. Thus, we get that

$$|Lu|_k^2 = \sum_\xi |p(\xi)u_\xi|^2(1+|\xi|^2)^k \geq c\sum_\xi |\xi|^{2\ell}|u_\xi|^2(1+|\xi|^2)^k.$$

Now we have that

$$\begin{aligned}(|Lu|_k + |u|_k)^2 &\geq |Lu|_k^2 + |u|_k^2 \\ &\geq c\sum_\xi |\xi|^{2\ell}|u_\xi|^2(1+|\xi|^2)^k + \sum_\xi |u_\xi|^2(1+|\xi|^2)^k \\ &= \sum_\xi |u_\xi|^2(1+|\xi|^2)^k(1+c|\xi|^{2\ell}) \\ &= c'\sum_\xi |u_\xi|^2(1+|\xi|^{k+\ell})^2 + \text{ additional terms} \\ &\geq c'\sum_\xi |u_\xi|^2(1+|\xi|^{k+\ell})^2 = c'|u|_{k+\ell}\end{aligned}$$

where c' a constant depending on c. This shows the fundamental inequality in this case.

Step 2: Now let L be any elliptic operator of order ℓ. Let us consider this operator locally at some point $p \in \mathbb{R}^n$. We can define an operator L_0 with constant coefficients and only order ℓ terms as follows. Suppose the order ℓ terms of L_{ij} are

$$\sum_{|\alpha|=\ell} a_\alpha^{ij} D^\alpha.$$

We can then define the operator L_0 with matrix entries

$$L_{0_{ij}} = a_\alpha^{ij}(p)D^\alpha.$$

Applying step 1 to the operator L_0, we see that

$$|u|_{k+\ell} \leq C(|L_0 u|_k + |u|_k)$$

for some constant C. Then,

$$|u|_{k+\ell} \leq C(|Lu|_k + |(L_0 - L)u|_k + |u|_k).$$

Using continuity of the coefficients of L, for a small enough neighborhood U. the coefficients of the operator $L_0 - L$ have norm less than $1/(2cC)$, where c is the constant in Lemma 4.4. By possibly shrinking we can choose a periodic operator $\widetilde{L}$ which agrees with $L_0 - L$ on U and the coefficients of $\widetilde{L}$ have norm less than $1/(2cC)$ everywhere. Now we have that by lemma 4.4

$$|\widetilde{L}u|_k \leq \frac{1}{2C}|u|_{k+\ell} + c'|u|_{k+\ell-1}.$$

Now if u has support in U, $\widetilde{L}u = (L_0 - L)u$ so

$$\begin{aligned} |u|_{k+\ell} &\leq C(|Lu|_k + |(L_0 - L)u|_k + |u|_k) \\ &= C(|Lu|_k + |\widetilde{L}u|_k + |u|_k) \\ &\leq C(|Lu|_k + \frac{1}{2C}|u|_{k+\ell} + c'|u|_{k+\ell-1} + |u|_k) \\ &= C|Lu|_k + \frac{1}{2}|u|_{k+\ell} + Cc'|u|_{k+\ell-1} + C|u|_k. \end{aligned}$$

By the Peter-Paul inequality,

$$|u|_{k+\ell-1} \leq \frac{1}{4Cc'}|u|_{k+\ell} + C'|u|_k$$

where C' is a constant. Plugging this in, we get that

$$\frac{1}{4}|u|_{k+\ell} \leq \text{const}(|Lu|_k + |u|_k).$$

The upshot is that for any point p, we can find a neighborhood U of p such that the fundamental inequality holds for all functions with support in U.

Step 3: Since we are looking at operators on 2π periodic functions $\mathcal{P}$, the domain is the torus T^n. At each point p, we have a neighborhood U_p such that the inequality holds for all functions with support in U_p. By compactness of T^n, finitely many of these open sets $U_1, ..., U_s$ cover T^n. We have a partition of unity $\omega_1, ..., \omega_s$ subordinate to this cover such that

$$\sum_{i=1}^{s} \omega_i^2 = 1.$$

Then

$$|\varphi|_{k+\ell}^2 = \langle \varphi, \varphi \rangle_{k+\ell} = \langle \sum_i \omega_i^2 \varphi, \varphi \rangle_{k+\ell} = \sum_i \langle \omega_i^2 \varphi, \varphi \rangle_{k+\ell}.$$

The second equality comes from the fact that $\sum_i \omega_i^2 = 1$. Now by Proposition 3.9, in $\langle \omega_i^2 \varphi, \varphi \rangle_{k+\ell}$ we can move one of the ω_i to the other side which introduces an error term of $c|\varphi|_{k+\ell}|\varphi|_{k+\ell-1}$. We now see that

$$|\varphi|_{k+\ell}^2 \leq \sum_i \langle \omega_i \varphi, \omega_i \varphi \rangle_{k+\ell} + c|\varphi|_{k+\ell}|\varphi|_{k+\ell-1}.$$

But now $\omega_i \varphi$ is supported in one of the neighborhoods constructed in step 2. We can then apply the fundamental inequality to each $\omega_i \varphi$. Some further computations show that the fundamental inequality is satisfied.

5. Regularity

5.1. **Periodic Elliptic Operators.** We start by proving a regularity result on periodic elliptic operators. This shows how ellipticity is used to gain more derivatives.

Theorem 5.1. *Let $u \in H_{-\infty}$, $v \in H_k$ and L a periodic elliptic operator of order ℓ. If*

$$Lu = v,$$

then $u \in H_{k+\ell}$.

Proof. By iterating, it is enough to show for the case when $u \in H_s$ and $v \in H_{s-\ell+1}$. Here we want to show that $u \in H_{s+1}$, i.e. we get one more derivative. The proof then follows from using difference quotients and the fundamental inequality. We define the operator L^h to be the operator L where we replace its coefficients by their difference quotients.

One can check that L^h satisfies

$$L(u^h) = (Lu)^h - L^h(T_h u). \tag{5.2}$$

□

The fundamental inequality states that

$$|u^h|_s \leq c|L(u^h)|_{s-\ell} + c|u^h|_{s-\ell}.$$

Using (5.2), we see that

$$|u^h|_s \leq c|(Lu)^h|_{s-\ell} + c|L^h(T_h u)|_{s-\ell} + c|u^h|_{s-\ell}. \tag{5.3}$$

Now since the coefficients of L are periodic and C^∞, their difference quotients are uniformly bounded. For $|\alpha| = \ell$, $|D^\alpha T_h u|_{s-\ell} \leq |T_h(u)|_s$ by 3.5. Then by using the triangle inequality we see that

$$|L^h(T_h u)|_{s-\ell} \leq M|T_h(u)|_s$$

where M is a constant depending on the bound of the coefficients of L^h. Also recall that T_h is an isometry and that $|Lu^h|_{s-\ell} \leq |Lu|_{s-\ell+1}$. Then the right side of (5.3) is bounded by $|Lu|_{s-\ell+1}, |u|_s$ and $|u|_{s-\ell+1}$ which are all finite by our assumptions. Then Lemma 3.15 tells us that $u \in H_{s+1}$.

5.2. **Laplacian.** The goal of this section is to prove the following regularity theorem. For notation, we will denote $\langle \cdot, \cdot \rangle'$ to be the inner product on Ω^p given by the $*$ operator. We will use $\langle \cdot, \cdot \rangle$ to denote the standard L_2 inner product.

Theorem 5.4. *(Regularity) Let f be a differentiable p-form and $\ell' : \Omega^p \to \mathbb{R}$ a bounded linear functional satisfying*

$$\ell'(\Delta\varphi) = \langle f, \varphi \rangle'$$

for all $\varphi \in \Omega^p$. Then there exists a differentiable p-form u such that $\ell'(t) = \langle u, t \rangle'$ for all $t \in \Omega^p$.

We will use the regularity theorem in the following way. This says that if we have a weak solution to the equation $\Delta\omega = \alpha$, we can find an actual solution.

Corollary 5.5. *Let α be a p-form and ℓ a weak solution to the equation $\Delta\omega = \alpha$. Then there exists a p-form u such that $\Delta u = \alpha$.*

Proof. Using the regularity theorem, there exists a p-form u such that

$$\ell(\beta) = \langle u, \beta \rangle$$

for all p-forms β. Now we compute $\ell(\Delta\beta)$ in two ways. By definition of weak solution

$$\ell(\Delta\beta) = \langle \alpha, \beta \rangle.$$

We also have that

$$\ell(\Delta\beta) = \langle u, \Delta\beta \rangle = \langle \Delta u, \beta \rangle.$$

Now for all β

$$\langle \Delta u, \beta \rangle = \langle \alpha, \beta \rangle$$

so $\Delta u = \alpha$.

$\square$

We now prove the regularity theorem. We follow the proof in [5, 6.32]. Recall we have the following local situation. We make a choice of coordinate charts for our manifold M. We work locally on a fixed chart U. Here we can view p-forms as vector valued C^∞ functions on $\mathbb{R}^n$. In this setting, the Laplacian Δ is an order 2 elliptic operator we denote by L.

We let C_0^∞ denote the space of C^∞ functions with compact support. For any subspace V, we denote by $C_0^\infty(V)$ to be the C^∞ functions with support in V. If $\overline{V}$ is contained in a 2π cube, we can identify $C_0^\infty(V)$ as a subspace of the periodic functions $\mathcal{P}$ by extension. Moreover, by extending by 0, any element of C_0^∞ is a complex-valued p-form.

An issue we run into is that we now have two inner products. One inner product $\langle \cdot, \cdot \rangle'$ is induced by the star operator (compose the inner product (2.2) by charts). The other is the standard L_2 inner product which we denote by $\langle \cdot, \cdot \rangle$. Note that these inner products are integrals of the metric on M and the standard dot product. Thus there exists a hermitian positive definite matrix A such that

$$\langle \varphi, \psi \rangle' = \langle \varphi, A\psi \rangle.$$

We denote by L^* the adjoint of L under the L_2 inner product.

Now we define the functional ℓ on C_0^∞ by

$$\ell(\varphi) = \ell'(A^{-1}\varphi).$$

We wish to show that there exists a C^∞ function u such that $\ell(t) = \langle u, t \rangle$ for all $t \in C^\infty$. Hence we have now reduced the problem on our fixed chart to one involving elliptic differential operators acting on functions. Here ℓ is a weak solution to $Lu = f$ where we view f as a function. We want to show that this weak solution can be represented by some $u \in C^\infty$.

The idea of the proof is as follows. At each point p, we wish to find a neighborhood W_p such that for all C^∞ functions t with support in W_p we have that $\ell(t) = \langle u_p, t \rangle$ for some $u_p \in C_0^\infty$. These u_p glue together to a C^∞ function u which solves the problem on the chart U.

Lemma 5.6. *Let $p \in \mathbb{R}^n$. There exists a neighborhood W_p of p and $u_p \in C_0^\infty$ such that for all $t \in C_0^\infty(W_p)$*

$$\ell(t) = \langle u_p, t \rangle.$$

Proof. We will use the general framework described earlier. First we show that on a neighborhood, ℓ corresponds too some element in $H_{-\infty}$. In the next step, we show using periodic elliptic regularity that after restricting to a possibly smaller neighborhood, this element actually belongs to H_k for all k. Finally, we use the Sobolev embedding theorem to show that this corresponds to a C^∞ function.

Step 1: Choose V to be an open set containing p such that $\overline{V}$ is contained in a 2π cube. Let $\widetilde{\ell}$ denote the restriction of ℓ to $C_0^\infty(V)$. We will use two facts about $\widetilde{\ell}$.

First, $\widetilde{\ell}$ is bounded. That is, for all $\varphi \in C_0^\infty(V)$, we have that

$$|\widetilde{\ell}(\varphi)| \leq c|\varphi|$$

for some constant c.

Next, we have that $\widetilde{\ell}$ is a weak solution to $Lu = f$. That is, for all $\varphi \in C_0^\infty(V)$, we have that

$$\widetilde{\ell}(L^*\varphi) = \langle f, \varphi \rangle.$$

Using that $\widetilde{\ell}$ is bounded, we can extend $\widetilde{\ell}$ to a bounded linear functional on H_0 using Hahn-Banach where the sublinear functional is the L_2 norm. Then we can find a $\widetilde{u} \in H_0$ such that

$$\widetilde{\ell}(t) = \langle \widetilde{u}, t \rangle$$

for $t \in H_0$. This is almost what we want, except we need differentiability. This is where ellipticity is needed. We will use the differentiability of $Lu = f$ to get more and more derivatives of $\widetilde{u}$.

Step 2: Let O_0 be a neighborhood of p contained in V such that on O_0 there is a periodic elliptic operator $\widetilde{L}$ which agrees with L. We then choose a sequence of neighborhoods as follows. Let O be a neighborhood of p such that $\overline{O} \subset O_0$. Now for each n choose a neighborhood O_n of p such that $\overline{O} \subset O_n$ and $\overline{O_n} \subset \overline{O_{n-1}}$. For each n, let ω_n be a C^∞ function, $0 \leq \omega_n \leq 1$, and ω_n equal to 1 on O_n with support in O_{n-1}.

Now set $v_1 = \omega_1 \widetilde{u} \in H_0$. Let M_1 denote the operator $\widetilde{L}\omega_1 - \omega_1 \widetilde{L}$. This is an order 1 operator since $\widetilde{L}$ is order 2. We see that

$$\widetilde{L} v_1 = \omega_1 \widetilde{L} \widetilde{u} + M_1 \widetilde{u}.$$

Since M_1 is order 1 and $\widetilde{u} \in H_0$, $M_1 \widetilde{u} \in H_{-1}$ using Proposition 3.5.

Next we want to show that $\omega_1 \widetilde{L} \widetilde{u} = \omega_1 f$. Using Proposition 3.6, we just need to show that for all $\varphi \in \mathcal{P}$, we have that

$$\langle \omega_1 \widetilde{L} \widetilde{u} - \omega_1 f, \varphi \rangle = 0.$$

Notice that

$$\langle \omega_1 \widetilde{L} \widetilde{u}, \varphi \rangle = \langle \widetilde{L} \widetilde{u}, \omega_1 \varphi \rangle = \langle \widetilde{u}, L^* \omega_1 \varphi \rangle = \widetilde{\ell}(L^* \omega_1 \varphi).$$

The first equality comes from Proposition 3.9. The second equality from adjoint and the final equality comes from the construction of $\widetilde{u}$.

We also have that

$$\langle \omega_1 f, \varphi \rangle = \langle f, \omega_1 \varphi \rangle = \widetilde{\ell}(L^* \omega_1 \varphi).$$

Here the first equality comes from Proposition 3.9 again, and the second equality comes from the fact that $\widetilde{\ell}$ is a weak solution to $Lu = f$.

Then by additivity

$$\langle \omega_1 \widetilde{L} \widetilde{u} - \omega_1 f, \varphi \rangle = 0,$$

so $\omega_1 \widetilde{L} \widetilde{u} = \omega_1 f$. In particular $\omega_1 \widetilde{L} \widetilde{u}$ is C_0^∞, so it belongs to every Sobolev space. But now we see that $\widetilde{L} v_1$ belongs to H_{-1}. Then periodic elliptic regularity tells us that v_1 belongs to H_1.

Now suppose that that $v_{n-1} = \omega_{n-1} \widetilde{u} \in H_{n-1}$. We want to show that $v_n = \omega_n \widetilde{u} \in H_n$. Set $M_n = \widetilde{L}\omega_n - \omega_n \widetilde{L}$. Now $\widetilde{u} = v_{n-1}$ on O_{n-1} since ω_{n-1} is 1 on O_{n-1}. Then since M_n has support on O_{n-1} we see that $M_n v_{n-1} = \widetilde{L} v_n - \omega_n \widetilde{L} \widetilde{u}$. Thus,

$$\widetilde{L} v_n = \omega_n \widetilde{L} \widetilde{u} + M_n v_{n-1}.$$

By similar computation as before, $\omega_n \widetilde{L}\widetilde{u} \in C_0^\infty$. The operator M_n has order 1 and by assumption $v_{n-1} \in H_{n-1}$, so $M_n v_{n-1} \in H_{n-2}$. Then $\widetilde{L} v_n \in H_{n-2}$ so by periodic elliptic regularity, $v_n \in H_n$.

Step 3: Now choose W_p to be a neighborhood of p such that $\overline{W_p} \subset O$, and ω a function between 0 and 1 with support in O such that it is exactly 1 on W_p. Then $\omega\widetilde{u} = \omega\omega_n\widetilde{u}$ so $\omega\widetilde{u} \in H_n$ for all n. Then the Sobolev embedding theorem says that $\omega\widetilde{u}$ is a C^∞ function which we denote by u. The function u has support in a 2π cube, so $u \in C_0^\infty$. Now if $t \in C_0^\infty(W_p)$, then

$$\ell(t) = \widetilde{\ell}(t) = \langle \widetilde{u}, t\rangle = \langle \omega\widetilde{u}, t\rangle = \langle u, t\rangle.$$

Take u_p to be this u.

□

With this lemma we are ready to complete the proof to Theorem 5.4. Note that for any two points p, q, and any $t \in \mathbb{C}_0^\infty(W_p \cap W_q)$ we have that

$$\langle u_p, t\rangle = \ell(t) = \langle u_q, t\rangle$$

, so by proposition 3.6, we have that $u_p = u_q$ on $W_p \cap W_q$. Then these u_p glue to form a C^∞ function u which restricts to u_p on each W_p.

Now let ϕ_i be a partition of unity for the cover given by the open sets W_p. For any $t \in C_0^\infty$,

$$\ell(t) = \sum \ell(\phi_i t) = \sum \langle u_p, \phi_i t\rangle = \sum \langle u, \phi_i t\rangle = \langle u, t\rangle.$$

Remember that each $u_p \in C_0^\infty$ so it is viewed as a p-form. These glue together to a p-form u with support in the coordinate chart U. For any φ a p-form with support on U, we see that

$$\ell'(\varphi) = \ell(A\varphi) = \langle u, A\varphi\rangle = \langle u, \varphi\rangle'.$$

Thus, we have solved the problem locally on the chart U. But by similar argument as before, the p-forms u we get on each chart agree on the overlaps so they glue to a global p-form.

6. Hodge Decomposition

With the regularity theorem, we are now ready to prove the Hodge decomposition theorem. We need one additional lemma.

Lemma 6.1. *Let (α_n) be a sequence of p-forms on M such that for a constant c, $|\alpha_n| \leq c$ and $|\Delta\alpha_n| \leq c$. Then there exists a Cauchy subsequence of (α_n).*

Proof. See [5, Prop. 6.33]. □

Corollary 6.2. *The space of harmonic p-forms H^p is finite dimensional.*

Proof. Suppose that H^p is not finite dimensional. Then we can find an orthonormal basis of infinite length (α_n). The norms of elements of this basis are 1, and since they are harmonic for all n, $\Delta\alpha_n = 0$. Then this sequence satisfies the conditions in Lemma 6.1, so there is a Cauchy subsequence. However, this is absurd since the distance between any two basis elements is 1. □

Corollary 6.3. *Let $\beta \in (H^p)^\perp$. Then there exists a constant c such that*

$$|\beta| \leq c|\Delta\beta|.$$

Proof. Suppose no such constant exists. Then we can find a sequence (β_i) in $(H^p)^\perp$ such that $|\beta_i| = 1$ and $|\Delta\beta_i| \to 0$. Using Lemma 6.1, we have a Cauchy subsequence. Without loss of generality, we suppose (β_i) is Cauchy. We can define a functional ℓ by

$$\ell(\phi) = \lim_i \langle \beta_i, \phi\rangle$$

and the limit exists since (β_i) is Cauchy. Using self-adjointness of the Laplacian

$$\ell(\Delta\alpha) = \lim_i \langle \beta_i, \Delta\alpha\rangle = \lim_i \langle \Delta\beta_i, \alpha\rangle = 0,$$

so ℓ is a weak solution of the equation $\Delta\omega = 0$. Then regularity tells us that there exists an actual solution β such that $\Delta\beta = 0$, so $\beta \in H^p$. On the other hand, we know $\lim\langle \beta_i, \Delta\alpha\rangle = \langle \beta, \Delta\alpha\rangle$ for all α, so $\beta = \lim \beta_i$. But then $\beta \neq 0$ since $|\beta_i| = 1$, so $|\beta| = 1$ and $\beta \in (H^p)^\perp$ since $\beta_i \in (H^p)^\perp$. This is a contradiction. □

Theorem 6.4. *(Hodge Decomposition) Let M be a compact, Riemannian n-manifold and $0 \leq p \leq n$. Then we have an orthogonal direct sum decomposition*

$$\Omega^p(M) = \Delta(\Omega^p) \oplus H^p.$$

Proof. Since H^p is finite dimensional, we have an orthogonal decomposition

$$\Omega^p = H^p \oplus (H^p)^\perp.$$

Then it is enough to show that $(H^p)^\perp = \Delta(\Omega^p)$. One inclusion is easy to see. Suppose that $\omega \in \Omega^p$ and $\alpha \in H^p$. Then by self-adjointess of Δ we see that

$$\langle \Delta\omega, \alpha\rangle = \langle \omega, \Delta\alpha\rangle = \langle \omega, 0\rangle = 0.$$

Thus, $\omega \in (H^p)^\perp$ so $(H^p)^\perp \supset \Delta(\Omega^p)$.

It remains to show that $(H^p)^\perp \subset \Delta(\Omega^p)$. Let $\alpha \in (H^p)^\perp$. We can define a operator ℓ on $\Delta(\Omega^p)$ by

$$\ell(\Delta\phi) = \langle \alpha, \phi\rangle.$$

Let H denote the projection operator to the space of harmonic forms. Consider $\psi = \phi - H(\phi)$. In particular, $\psi \in (H^p)^\perp$ and $\Delta\phi = \Delta\psi$. Then

$$|\ell(\Delta\psi)| = |\langle \alpha, \psi\rangle| \leq |\alpha||\psi|.$$

Since $\psi \in (H^p)^\perp$, we can use corollary 6.3, so there exists a constant c such that

$$|\psi| \leq c|\Delta\psi|.$$

Now

$$|\ell(\Delta\phi)| = |\ell(\delta\psi)| \leq c|\alpha||\Delta\psi| = c|\alpha||\Delta\phi|.$$

This shows that we can apply the Hahn-Banach theorem with the sublinear functional $p(\phi) = c|\alpha||\phi|$ to extend ℓ to all of Ω^p. Then ℓ is a weak solution of $\Delta u = \alpha$, so by regularity, there exists a p-form ω such that $\Delta\omega = \alpha$. Thus, $\alpha \in \Delta(\Omega^p)$ which shows the other inclusion. □

We end with an application to de Rham cohomology.

Let $\alpha \in \Omega^p$ be a p-form. From Hodge decomposition, we can write $\alpha = \Delta\beta + H\alpha$ where H is the projection to harmonic p-forms. We denote β by $G(\alpha)$. The operator G is called a Green's operator.

Lemma 6.5. *The operator G commutes with d.*

Proof. See [5, Prop. 6.10]. □

Theorem 6.6. *Every de Rham cohomology class has a unique harmonic representative.*

Proof. A cocycle in de Rham cohomology is a closed p-form α. Then we have by Hodge decomposition

$$\alpha = \Delta G\alpha + H\alpha = d\delta G\alpha + \delta dG\alpha + H\alpha = d\delta G\alpha + \delta Gd\alpha + H\alpha.$$

Since α is closed, $d\alpha = 0$, so

$$\alpha = d\delta G\alpha + H\alpha.$$

Coboundaries consist of exact forms, so α and $H\alpha$ belong to the same cohomology class. Thus, we have a harmonic representative. We claim that this representative is unique. If α, α' are two harmonic forms in the same class, they differ by some exact form, i.e. $\alpha' = \alpha + d\beta$. Then since α' is harmonic $d\delta d\beta = 0$. Using adjointness $\langle \delta d\beta, \delta d\beta \rangle = 0$, so $\delta d\beta = 0$. Then $\delta\alpha' - \delta\alpha = 0$. Now

$$\langle d\beta, \alpha - \alpha' \rangle = \langle \beta, \delta\alpha - \delta\alpha' \rangle = 0.$$

Now we have $0 = d\beta + (\alpha - \alpha')$ and $d\beta$ and $(\alpha - \alpha')$ are orthogonal. But then by uniqueness of orthogonal decomposition, $d\beta = 0$ which shows that $\alpha = \alpha'$. □

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