# An improved bound for the non-existence of radial solutions of the Brezis-Nirenberg problem in $\mathbb{H}^n$

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*To Pavel Exner on the occasion of his 70th birthday*

**Abstract.** Using a Rellich–Pohozaev argument and Hardy's inequality, we derive an improved bound on the nonlinear eigenvalue for the non existence of radial solutions of a Brezis–Nirenberg problem, with Dirichlet boundary conditions, on a geodesic ball of $\mathbb{H}^n$, for $2 < n < 4$.

## 1. Introduction

For a long time virial theorems have played a key role in the localization of linear and nonlinear eigenvalues. In the spectral theory of Schrödinger Operators, the virial theorem has been widely used to prove the absence of positive eigenvalues for various multiparticle quentum systems (see, e.g., [12, 10, 1]). In 1983, Brézis and Nirenberg [4] considered the existence and nonexistence of solutions of the nonlinear equation

$$-\Delta u = \lambda u + |u|^{p-1}u,$$

defined on a bounded, smooth domain of $\mathbb{R}^n$, $n > 2$, with Dirichlet boundary conditions, where $p = (n + 2)/(n - 2)$ is the critical Sobolev exponent. In particular, they used a virial theorem, namely the Pohozaev identity [8], to prove the nonexistence of regular solutions when the domain is star–shaped, for any $\lambda \leq 0$, in any $n > 2$. After the classical paper [4] of Brézis and Nirenberg, many people have considered extensions of this problem in different settings. In particular, the Brézis–Nirenberg (BN) problem has been studied on bounded, smooth, domains of the hyperbolic space $\mathbb{H}^n$ (see, e.g., [11, 2, 6, 3]), where one replaces the Laplacian by the Laplace–Beltrami operator in $\mathbb{H}^n$. Stapelkamp [11] proved the analog of the above mentioned nonexistence result of Brézis–Nirenberg in $\mathbb{H}^n$. Namely

she proved that there are no regular solutions of the BN problem for bounded, smooth, star–shaped domains in $\mathbb{H}^n$ ($n > 2$), if $\lambda \leq n(n-2)/4$. The purpose of this manuscript is to give an improved bound on $\lambda$ for the nonexistence of radial (not necessarily positive) radial, regular solutions of the BN problem on geodesic balls of $\mathbb{H}^n$ for $2 < n < 4$ (see Theorem 2.1 below). Notice that for the case of radial solutions of the BN problem on a geodesic ball one can consider noninteger values of $n$, which can be considered just as a parameter.

Consider the Brezis–Nirenberg problem

$$-\Delta_{\mathbb{H}^n} u = \lambda u + |u|^{p-1} u, \tag{1}$$

on $\Omega \subset \mathbb{H}^n$, where $\Omega$ is smooth and bounded, with Dirichlet boundary conditions, i.e., $u = 0$ in $\partial\Omega$. After expressing the Laplace Beltrami operator $\Delta_{\mathbb{H}^n}$ in terms of the conformal Laplacian, Stapelkamp [11] proved that (1) does not admit any regular solution for star-shaped domains $\Omega$ provided

$$\lambda \leq \frac{n(n-2)}{4}. \tag{2}$$

Here, we consider the BN problem (1) for radial solutions on geodesic balls of $\mathbb{H}^n$. We can prove a different bound, namely the problem for radial solutions on a geodesic ball $\Omega^*$ does not admit a solution if

$$\lambda \leq \frac{n^2(n-1)}{4(n+2)} \tag{3}$$

for $n > 2$. Our bound is better than (2) in the radial case, if $2 < n < 4$. Both bounds coincide when $n = 4$. In the rest of this manuscript we give the proof of (3).

## 2. Nonexistence of solutions of the BN problem on geodesic balls in $\mathbb{H}^n$, for $2 < n < 4$.

In the sequel we consider (not necessarily positive) radial solutions of the BN problem (1) on geodesic balls of $\mathbb{H}^n$. In *radial coordinates*, (1) can be written as

$$-u''(x) - (n-1)\coth(x)u'(x) = \lambda u(x) + |u|^{p-1}u(x), \tag{4}$$

with $u'(0) = u(R) = 0$, where $R$ is the *radius* of the geodesic ball. Here, as before, $p = (n+2)/(n-2)$. Notice that (4) makes sense also if $n$ is not an integer. For that reason henceforth we consider $n \in \mathbb{R}$, with $2 < n < 4$. Our main result is the following

**Theorem 2.1.** *The Boundary Value problem (1), with $u'(0) = u(R) = 0$, has no regular solutions if*

$$\lambda \leq \frac{n^2(n-1)}{4(n+2)},$$

*for $2 < n < 4$.*

**Remark 2.2.** Notice that our bound $n^2(n-1)/(4(n+2))$ is strictly bigger than $n(n-2)/4$ for $n < 4$. Notice, on the other hand, that Stapelkamp's bound holds for all regular solutions, while our improved bound only holds for regular radial solutions. We do not know whether our bound is optimal, i.e., we do not know if there are solutions for $\lambda > n^2(n-1)/(4(n+2))$. In view of [3], there can be no positive solutions if $\lambda < \mu(n)$ (see [3] for the definition of $\mu(n)$). It is important to notice that for $n = 4$, we have that,

$$\frac{n(n-2)}{4} = \frac{n^2(n-1)}{4(n+2)} = \mu(4),$$

so at least our result is optimal as $n \to 4$.

*Proof.* We use a Rellich–Pohozaev argument [9, 8]. Multiplying equation (4) by $u(x)\sinh^{n-1}(x)$ and integrating, we obtain

$$-\int_0^R u''(x)(u(x)\sinh^{n-1}(x))\,dx - (n-1)\int_0^R u(x)u'(x)\cosh(x)\sinh^{n-2}(x)\,dx$$
$$= \lambda\int_0^R u^2\sinh^{n-1}(x)\,dx + \int_0^R |u(x)|^{p+1}\sinh^{n-1}(x)\,dx.$$
$$(5)$$

Integrating the first term by parts, we can write this equation as

$$\int_0^R u'^2\sinh^{n-1}(x)\,dx = \lambda\int_0^R u^2\sinh^{n-1}(x)\,dx + \int_0^R |u|^{p+1}\sinh^{n-1}(x)\,dx. \quad (6)$$

Now let $G(x) = \int_0^x \sinh^{n-1}(s)\,ds$. Multiplying equation (4) by $u'G$ and integrating, we obtain

$$-\int_0^R \left(\frac{u'^2}{2}\right)' G\,dx - (n-1)\int_0^R \coth(x)u'^2 G\,dx$$
$$= \lambda\int_0^R \left(\frac{u^2}{2}\right)' G\,dx + \int_0^R \left(\frac{|u|^{p+1}}{p+1}\right)' G\,dx.$$

After integrating by parts, and since $G(0) = 0$, we obtain

$$\frac{u'^2(R)G(R)}{2} + \int_0^R u'^2\left((n-1)G\coth(x) - \frac{G'}{2}\right)dx$$
$$= \lambda\int_0^R \frac{u^2}{2}G'\,dx + \frac{1}{p+1}\int_0^R |u|^{p+1}G'\,dx.$$
$$(7)$$

Substituting equation (6) into equation (7), and since $1/2 - 1/(p+1) = 1/n$, it follows that

$$\int_0^R u'^2 \left( (n-1)G \coth(x) - \frac{G'}{2} - \frac{G'}{p+1} \right) dx + \frac{u'^2(R)G(R)}{2}$$
$$= \frac{\lambda}{n} \int_0^R u^2 \sinh^{n-1}(x) \, dx. \tag{8}$$

Notice that in equation (9) we have written $\sinh^{n-1}(x)$ as $G'(x)$. Thus, since the boundary term is positive, and since $1/2 + 1/(p+1) = (n-1)/n$, we have

$$\lambda \geq \frac{n(n-1) \int_0^R u'^2 \left( G \coth(x) - \frac{G'}{n} \right) dx}{\int_0^R u^2 G' \, dx}. \tag{9}$$

Now let $L(x) = G \coth(x) - \dfrac{G'}{n}$. Then $L \geq 0$. In fact, we can write $L(x) = m(x)/\sinh(x)$, where

$$m(x) = G \cosh(x) - \sinh^n(x)/n.$$

Then, since $G(0)$, we have $m(0) = 0$. Also,

$$m'(x) = G \sinh(x) + G' \cosh(x) - \sinh^{n-1}(x) \cosh(x) = G \sinh(x).$$

It follows that $m' \geq 0$, and therefore that $L \geq 0$.

We now use a Hardy type inequality to write the denominator integral in terms of $u'^2$. For a review on Hardy's inequa;ities see, e.g., [7, 5]. Integrating by parts, we can write

$$\int_0^R u^2 G' \, dx = -2 \int_0^R \left( u \sinh^{\frac{n-1}{2}}(x) \right) \left( \frac{G u'}{\sinh^{\frac{n-1}{2}}}(x) \right) dx.$$

Then, usiong Cauchy-Schwarz, it follows that

$$\left( \int_0^R u^2 G' \, dx \right)^2 \leq 4 \int_0^R u^2 G' \, dx \int_0^R \frac{G^2 u'^2}{G'} \, dx.$$

That is,

$$\int_0^R u^2 G' \, dx \leq 4 \int_0^R \frac{u'^2 G^2}{G'} \, dx. \tag{10}$$

Using inequality (10) in the quotient (9), we conclude that

$$\lambda \geq \frac{n(n-1) \int_0^R u'^2 \left( G \coth(x) - \frac{G'}{n} \right) dx}{4 \int_0^R \frac{u'^2 G^2}{G'} \, dx}. \tag{11}$$

In the Lemma 2.3 below, we show that $L(x) \geq c\dfrac{G^2}{G'}$, where $c = \dfrac{n}{n+2}$. With this, we conclude that

$$\lambda \geq \frac{n^2(n-1)}{4(n+2)}. \tag{12}$$

$\square$

**Lemma 2.3.** *Let* $x \geq 0$ *and let* $L(x) = G\coth(x) - \dfrac{G'}{n}$. *Then*

$$L(x) \geq \frac{n}{(n+2)}\frac{G^2}{G'}.$$

*Here, as above,* $G(x) = \displaystyle\int_0^x \sinh^{n-1}(s)\, ds$.

*Proof.* Let $f(x) = L(x)G'(x) - c\,G^2(x)$, where $c = \dfrac{n}{n+2}$. It suffices to show that $f \geq 0$.

As before, we write $L(x) = \dfrac{m(x)}{\sinh(x)}$, where $m(x) = G\cosh(x) - \dfrac{\sinh^n(x)}{n}$ and $m'(x) = G\sinh(x)$. Then,

$$f(x) = \sinh^{n-2}(x)m(x) - c\,G^2(x).$$

Notice that since $\sinh(0) = G(0) = 0$, one has that $f(0) = 0$, so it suffices to show that $f' \geq 0$. We have that

$$f'(x) = \sinh^{n-3}(x)\left((n-2)\cosh(x)m(x) + G\sinh^2(x)(1-2c)\right).$$

Let $\sigma = 2c - 1 = 1/p$, where $p = (n+2)/(n-2)$ is the critical Sobolev exponent; and let $g(x) = (n-2)\cosh(x)m(x) - \sigma G\sinh^2(x)$. It suffices to show that $g \geq 0$. Since $m(0) = 0$, then $g(0) = 0$.

Also,

$$g'(x) = 2\sinh(x)\cosh(x)G(x)\,(n-2-\sigma) - \sinh^{n+1}(x)\left(\frac{(n-2)}{n} + \sigma\right)$$

and in particular $g'(0) = 0$. Since

$$n - 2 - \sigma = \frac{(n-2)(n+1)}{(n+2)}$$

and

$$\frac{(n-2)}{n} + \sigma = \frac{2(n+1)(n-2)}{n(n+2)}$$

we can write

$$g'(x) = \frac{2(n+1)(n-2)}{n(n+2)} \sinh(x) \left[ nG \cosh(x) - \sinh^n(x) \right].$$

Finally, let $h(x) = nG \cosh(x) - \sinh^n(x)$. If we show $h(x) \geq 0$, then we will have $g' \geq 0$, which will imply $g \geq 0$, and thus, that $f \geq 0$, as desired. Notice that $h(0) = 0$. Also, since $G'(x) = \sinh^{n-1}(x)$, we have

$$h'(x) = nG \sinh(x).$$

That is, $h' \geq 0$, which concludes the proof of Lemma 2.3.                    □

**Remark 2.4.** In the proof of Lemma 2.3, the constant $\sigma = 1/p$ plays a crucial role, where $p$ is the critical Sobolev exponent. It is worth noting that for small $x$ and $g$ as in the proof above,

$$g(x) = x^{n+2} \left( \frac{1}{np} - \frac{\sigma}{n} \right) + \mathcal{O}(x^{n+4}).$$

It follows that if $\sigma \leq 1/p$, then $g$ is positive in a neighborhood of the origin. It was this observation that led us to realize that $\sigma = 1/p$ would yield the optimal estimate.

## 3. References

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