THE SOLUTION GAP OF THE BREZIS–NIRENBERG PROBLEM ON THE HYPERBOLIC SPACE

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Abstract. We consider the positive solutions of the nonlinear eigenvalue problem $-\Delta_{\mathbb{H}^n} u = \lambda u + u^p$, with $p = \frac{n+2}{n-2}$ and $u \in H_0^1(\Omega)$, where Ω is a geodesic ball of radius θ_1 on $\mathbb{H}^n$. For radial solutions, this equation can be written as an ODE having n as a parameter. In this setting, the problem can be extended to consider real values of n. We show that if $2 < n < 4$ this problem has a unique positive solution if and only if $\lambda \in (n(n-2)/4 + L^* , \lambda_1)$. Here L^* is the first positive value of $L = -\ell(\ell+1)$ for which a suitably defined associated Legendre function $P_\ell^{-\alpha}(\cosh\theta) > 0$ if $0 < \theta < \theta_1$ and $P_\ell^{-\alpha}(\cosh\theta_1) = 0$, with $\alpha = (2-n)/2$.

1. Introduction

Given a bounded domain Ω in $\mathbb{R}^n$, Brezis and Nirenberg [5] considered the problem of existence of a function $u \in H_0^1(\Omega)$ satisfying

$$\begin{aligned} -\Delta u &= \lambda u + u^p \quad \text{on } \Omega \\ u &> 0 \quad \text{on } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $p = (n+2)/(n-2)$ is the critical Sobolev exponent. If $\lambda \geq \lambda_1$, where λ_1 is the first Dirichlet eigenvalue, this problem has no solutions. Moreover, if the domain is star-shaped, there is no solution if $\lambda \leq 0$. Thus, when Ω is a ball, for any given value of n there may exist a solution only if $\lambda \in (0, \lambda_1)$. It was shown in [5] that in dimension $n \geq 4$, there exists a solution for all λ in this range. However, in dimension $n = 3$ Brezis and Nirenberg showed there is an additional interval where there is no solution, which we will refer to in this article as the *solution gap* of the Brezis-Nirenberg problem. When the domain is the unit ball, the solution gap when $n = 3$ is the interval $\left(0, \frac{\lambda_1}{4}\right]$.

The dimensions for which semilinear second order elliptic problems with a nonlinear term of critical exponent (of which (1) is an example) have a solution gap are referred to in the literature as critical dimensions. This definition was first introduced by Pucci and Serrin in [13]. In [9], Jannelli studies a general class of such problems, and the associated critical dimensions. He gives an alternative proof to the existence results obtained in [5] for problem (1). When Ω is a ball, and $n = 3$, Jannelli shows that (1) has no solution if $\lambda \leq j_{\alpha,1}^2$, where $\alpha = (2-n)/2$ and $j_{\alpha,1}$ denotes the first positive zero of the Bessel function J_α.

If u is radial, problem (1) can be written as an ordinary differential equation,

$$-u'' - \frac{(n-1)}{r} u' = \lambda u + u^p,$$

where n can be thought of as a parameter in the equation, rather than the dimension of the space. By doing so one can study the behavior of the solution gap with respect to n by taking n to be a real number instead of a natural number. Jannelli's methods in [9] can be easily extended to the case $2 < n < 4$, thus concluding that the solution gap of the Brezis-Nirenberg problem defined in the unit ball is the interval $\left(0, j_{\alpha,1}^2\right]$. In particular, it follows that $n = 4$ is the first value of n for which there is no solution gap.

The solution gap of the Brezis-Nirenberg problem can also be studied in non-Euclidean spaces. On the sphere $\mathbb{S}^n$, for a fixed n, the solution gap is the subinterval of $(-n(n-2)/4, \lambda_1)$ for which (1) has no solution. As in the Euclidean case, $n = 3$ is a critical dimension, whereas $n \geq 4$ are not. It was shown in [1] that if Ω is a geodesic cap of radius θ_1 in $\mathbb{S}^3$ the solution gap is the interval $(-n(n-2)/4, (\pi^2 - 4\theta_1^2)/4\theta_1^2]$. If u is radial, then (1) can be written as an ordinary differential equation that still makes sense when n is a real number. It was shown in [3] that if $2 < n < 4$, the solution gap is the interval $(-n(n-2)/4, \left((2\ell^* + 1)^2 - (n-1)^2\right)/4]$, where ℓ^* is the first positive value of ℓ for which the associated Legendre function $P_\ell^\alpha(\cos\theta_1)$ vanishes. Here $\alpha = (2-n)/2$.

In this article we consider the Brezis-Nirenberg problem on the hyperbolic space $\mathbb{H}^n$. That is, we consider the problem

$$\begin{aligned} -\Delta_{\mathbb{H}^n} u &= \lambda u + u^p \quad \text{on } \Omega \\ u &> 0 \qquad\qquad \text{on } \Omega \\ u &= 0 \qquad\qquad \text{on } \partial\Omega, \end{aligned} \tag{2}$$

where $p = (n+2)/(n-2)$, Ω is a geodesic ball on $\mathbb{H}^n$ of radius $\theta_1 \in (0, \infty)$, and $\Delta_{\mathbb{H}^n}$ is the Laplace-Beltrami operator.

It is not hard to show (see, e.g., page 285 in [15]) that there can be no solutions for $\lambda \notin (n(n-2)/4, \lambda_1)$. Stapelkamp [15] showed that if $n \geq 4$ there is no solution gap, that is, that there is a solution for all values of λ in this interval. When $n = 3$, however, she showed there is no solution if $\lambda \in (n(n-2)/4, \lambda^*]$. Here

$$\lambda^* = 1 + \frac{\pi^2}{16 \operatorname{arctanh}^2 R},$$

where R is the radius of the ball that results by taking the stereographic projection of the geodesic ball onto $\mathbb{R}^3$. Moreover, Stapelkamp shows that for each $\lambda \in (\lambda^*, \lambda_1)$, there exists a unique solution, and this solution is radial. A full characterization of the solutions to this problem in dimension $n \in \mathbb{N}$ (and any $p > 1$) is given in [2]. After the results of Stapelkamp and Bandle, there has been a vast literature on Brezis-Nirenberg type equations on hyperbolic spaces (see, e.g., [11], [7], [8], [4]).

For radial functions u, problem (2) can be written as an ordinary differential equation, with n now simply representing a parameter in the equation rather than the dimension of the space. Our main result is that the solution gap of the Brezis-Nirenberg problem on the hyperbolic space has width L^*, where L^* is the first positive value of $L = -\ell(\ell+1)$ for which a suitably defined (see equation (6)) associated Legendre function $P_\ell^{-\alpha}(\cosh\theta)$ is positive if $0 < \theta < \theta_1$ and $P_\ell^{-\alpha}(\cosh\theta_1) = 0$. Here, as before, $\alpha = (2-n)/2$.

More precisely, we show the following:

Theorem 1.1. *For any* $2 < n < 4$ *and* $\theta_1 \in (0, \infty)$, *the boundary value problem*

$$-u''(\theta) - (n-1)\coth\theta\, u'(\theta) = \lambda u + u^{\frac{n+2}{n-2}} \tag{3}$$

with $u \in H_0^1(\Omega)$, $u'(0) = u(\theta_1) = 0$, *and* $\theta \in [0, \theta_1]$ *has a unique positive solution if and only if*

$$\lambda \in \left(\frac{n(n-2)}{4} + L^*\, ,\, \lambda_1 \right). \tag{4}$$

In Figure 1 the graph $\lambda(n)$ illustrates the results of Theorem 1.1 when $\theta_1 = 1$. The shaded region represents the solution gap, and the region between the dotted and the solid lines corresponds to the region of existence of solutions given by (4).

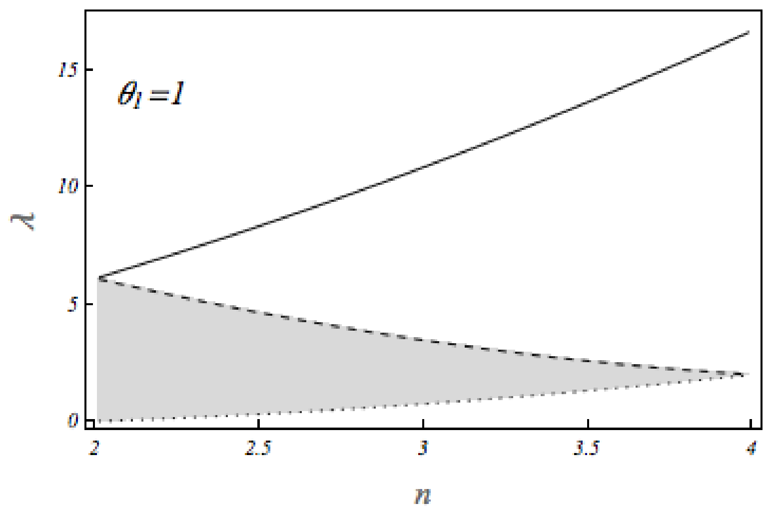

FIGURE 1. The shaded region depicts the solution gap of the Brezis-Nirenberg problem in the hyperbolic space. The solid line corresponds to λ_1, the dashed line to $\lambda = n(n-2)/4 + L^*$, and the dotted line to $\lambda = n(n-2)/4$.

In Section 2 we derive an expression for the first Dirichlet eigenvalue in terms of the parameter ℓ of an associated Legendre function, and use this expression to show that the interval of existence given by (4) is non-empty if $2 < n < 4$. In Section 3 we use a classical Lieb lemma to show the existence of solutions for λ as in (4). In Section 4 we use a Pohozaev type argument to show that if $2 < n < 4$ there is a solution gap of the Brezis-Nirenberg problem. That is, we show there are no solutions if $\lambda \in (n(n-2)/4\, ,\, n(n-2)/4 + L^*]$. Finally, in Section 5 we show that the uniqueness of solutions follows directly from [10].

2. Preliminaries

The associated Legendre functions $P_\ell^{\alpha}(\cosh\theta)$ and $P_\ell^{-\alpha}(\cosh\theta)$ are solutions of the Legendre equation

$$y''(\theta) + \coth\theta\, y'(\theta) + \left(-\ell(\ell+1) - \frac{\alpha^2}{\sinh^2\theta} \right) y(\theta) = 0. \tag{5}$$

We will adopt the following convention for the associated Legendre functions:

$$P_\ell^\alpha(\cosh\theta) = \frac{1}{\Gamma(1-\alpha)} \coth^\alpha\left(\frac{\theta}{2}\right) {}_2F_1\left[-\ell, \ell+1, 1-\alpha; -\sinh^2\left(\frac{\theta}{2}\right)\right], \tag{6}$$

where for complex numbers a, b, and c, the hypergeometric function ${}_2F_1[a,b,c;z]$ is given by

$${}_2F_1[a,b,c;z] = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}, \tag{7}$$

where $(\beta)_n := \Pi_{j=0}^{n-1}(\beta+j)$, for $\beta \in \mathbb{C}$.

Remark 2.1. *Notice that the associated Legendre functions $P_\ell^\alpha(\cosh\theta)$ depend on ℓ through the product $\ell(\ell+1)$, rather than from ℓ and $\ell+1$ independently.*

The associated Legendre functions given by (5) satisfy the following raising and lowering relations (see, e.g., [14], page 55, equations (20.11-1) and (20.11-2) with $x = \cosh\theta$):

$$\dot{P}_\ell^\alpha(\cosh\theta) = \frac{1}{\sinh\theta} P_\ell^{\alpha+1}(\cosh\theta) + \frac{\alpha\cosh\theta}{\sinh^2\theta} P_\ell^\alpha(\cosh\theta), \tag{8}$$

and

$$\dot{P}_\ell^{\alpha+1}(\cosh\theta) = \frac{\ell(\ell+1)-\alpha(\alpha+1)}{\sinh\theta} P_\ell^\alpha(\cosh\theta) - \frac{(\alpha+1)\cosh\theta}{\sinh^2\theta} P_\ell^{\alpha+1}(\cosh\theta). \tag{9}$$

Here $\dot{P}_\ell^\alpha$ means the derivative of P_ℓ^α with respect to its argument. That is,

$$\frac{d}{d\theta} P_\ell^\alpha(\cosh\theta) = \sinh\theta \dot{P}_\ell^\alpha(\cosh\theta).$$

Equations (8) and (9) are used in the proof of the non-existence result on Section 4.

Definition 1. *Let $L = -\ell(\ell+1)$. For $2 < n < 4$, $\alpha = (2-n)/2$, and $\theta_1 \in (0,\infty)$, let L_1 be the smallest positive value of L such that $P_\ell^\alpha(\cosh\theta) > 0$ if $0 < \theta < \theta_1$ and $P_\ell^\alpha(\cosh\theta_1) = 0$. Similarly, let L^* be the smallest positive value of L such that $P_\ell^{-\alpha}(\cosh\theta) > 0$ if $0 < \theta < \theta_1$ and $P_\ell^{-\alpha}(\cosh\theta_1) = 0$.*

In the next lemma we derive an expression for the first Dirichlet eigenvalue of $-\Delta_{\mathbb{H}^n} u = \lambda u$ on a geodesic ball in terms of L_1. In Lemma 2.4 we use the expression for λ_1 obtained in Lemma 2.2 to show that the interval of existence given in equation (4) is non-empty if $2 < n < 4$.

Lemma 2.2. *The first Dirichlet eigenvalue of equation*

$$-u'' - (n-1)\coth\theta u' = \lambda_1 u. \tag{10}$$

is given by

$$\lambda_1 = \frac{n(n-2)}{4} + L_1.$$

Proof. Making the change of variables $u(\theta) = \sinh^\alpha \theta v(\theta)$, we can write equation (10) as

$$v''(\theta) + (2\alpha \coth\theta + (n-1)\coth\theta)v'(\theta) + (\alpha(\alpha+n-2)\coth^2\theta + \alpha + \lambda_1)v(\theta) = 0.$$

Choosing $\alpha = \frac{2-n}{2}$, one obtains

$$v''(\theta) + \coth\theta\, v'(\theta) + (\alpha + \lambda_1 - \alpha^2 \coth^2\theta)v(\theta) = 0.$$

That is,

$$v''(\theta) + \coth\theta\, v'(\theta) + \left(\lambda_1 - \alpha(\alpha-1) - \frac{\alpha^2}{\sinh^2\theta}\right)v(\theta) = 0.$$

The solutions to this equation are $P_\ell^\alpha(\cosh\theta)$ and $P_\ell^{-\alpha}(\cosh\theta)$, where $\ell(\ell+1) = \alpha(\alpha-1) - \lambda_1$. Since α is negative if $2 < n < 4$, the regular solution of (10) is

$$u(\theta) = \sinh^\alpha\theta P_\ell^\alpha(\cosh\theta).$$

To satisfy the boundary condition $u(\theta_1) = 0$, while having $u(\theta) > 0$ in $(0, \theta_1)$, we must choose ℓ such that $-\ell(\ell+1) = L_1$. Thus,

$$\lambda_1 = \frac{n(n-2)}{4} + L_1.$$

□

Remark 2.3. *It is known by* [12] *that* $\lambda_1 \geq \frac{(n-1)^2}{4}$. *Thus,* $-L_1 \leq \frac{n(n-2)}{4} - \frac{(n-1)^2}{4} = -\frac{1}{4}$.

Lemma 2.4. *Let* L_1 *and* L^* *be as in Definition 1. Then* $L^* < L_1$.

Proof. Let $y_1(\theta) = P_{\ell_1}^\alpha(\cosh\theta)$, and $y_2(\theta) = P_{\ell^*}^{-\alpha}(\cosh\theta)$. Then y_j, $j \in \{1,2\}$, satisfy

$$y_j'' + \coth\theta y_j' + k_j y_j = 0, \tag{11}$$

where

$$k_1 = L_1 - \frac{\alpha^2}{\sinh^2\theta}.$$

and

$$k_2 = L^* - \frac{\alpha^2}{\sinh^2\theta}.$$

Let $W = y_1' y_2 - y_2' y_1$ and $W' = y_1'' y_2 - y_1 y_2''$. Then it follows from equation (11) that

$$W' + \coth\theta\, W = (k_2 - k_1) y_1 y_2.$$

Multiplying by $\sinh\theta$ and integrating one has that

$$\int_0^{\theta_1} (W \sinh\theta)'\, d\theta = [L^* - L_1] \int_0^{\theta_1} y_1 y_2 \sinh\theta\, d\theta.$$

By choice of L_1 and L^* it follows that y_1 and y_2 are positive on $[0, \theta_1)$ and vanish at θ_1, so that $\int_0^{\theta_1} y_1 y_2 \sinh\theta\, d\theta$ is positive and $W(\theta_1) = 0$. Thus, it suffices to show that $\lim_{\theta\to 0} W(\theta)\sinh\theta$ is negative.

It follows from equation (6) that the behavior of y_1 and y_2 near zero is

$$y_1 \approx \frac{1}{\Gamma(1-\alpha)} \coth^{\alpha}\left(\frac{\theta}{2}\right),$$

and

$$y_2 \approx \frac{1}{\Gamma(1+\alpha)} \coth^{-\alpha}\left(\frac{\theta}{2}\right).$$

Therefore,

$$\begin{aligned}\lim_{\theta\to 0} W(\theta)\sinh\theta &= \frac{-\alpha}{\Gamma(1-\alpha)\Gamma(1+\alpha)} \lim_{\theta\to 0}\sinh\theta\left(\frac{\tanh\left(\frac{\theta}{2}\right)}{\sinh^2\left(\frac{\theta}{2}\right)}\right)\\ &= \frac{-2\alpha}{\Gamma(1-\alpha)\Gamma(1+\alpha).}\end{aligned}$$

Finally, since $\Gamma(1+\alpha) = \alpha\Gamma(\alpha)$, $\Gamma(\alpha)\Gamma(1-\alpha) = \pi\sin^{-1}(\pi\alpha)$, and $0 < \alpha < 1$, we conclude that

$$\lim_{\theta\to 0} W(\theta)\sinh\theta = \frac{-2\sin(\pi\alpha)}{\pi} < 0.$$

□

From Lemmas 2.2 and 2.4 it follows that the interval of existence given by (4), that is, $(n(n-2)/4 + L^*, n(n-2)/4 + L_1)$, is nonempty if $2 < n < 4$.

3. Existence of solutions

In this section we present the proof of the following theorem:

Theorem 3.1. *For any $2 < n < 4$ and $\theta_1 \in (0,\infty)$, the boundary value problem*

$$-u''(\theta) - (n-1)\coth\theta\, u'(\theta) = \lambda u + u^{\frac{n+2}{n-2}} \tag{12}$$

with $u \in H_0^1(\Omega)$, $u'(0) = u(\theta_1) = 0$, and $\theta \in [0,\theta_1]$ has a positive solution if

$$\lambda \in \left(\frac{n(n-2)}{4} + L^*\,,\ \lambda_1\right).$$

Here L^* is as in Definition 1.

For natural values of n, the positive solutions of

$$-\Delta_{\mathbb{H}^n} u = \lambda u + u^p,$$

on a geodesic ball with Dirichlet boundary conditions correspond to minimizers of

$$Q_\lambda(u) = \frac{\displaystyle\int |\nabla u|^2 \rho^{n-2}\, dx - \lambda \int u^2 \rho^n\, dx}{\left(\displaystyle\int u^{\frac{2n}{n-2}} \rho^n\, dx\right)^{\frac{n-2}{n}}}.$$

Here $\rho(x) = \dfrac{2}{1-|x|^2}$ is such that $ds = \rho\, dx$.

If u is radial, we can write

$$Q_\lambda(u) = \frac{\omega_n \int_0^R u'^2 \rho^{n-2} r^{n-1}\, dr - \lambda \omega_n \int_0^R u^2 \rho^n r^{n-1}\, dr}{\left(\omega_n \int_0^R u^{\frac{2n}{n-2}} \rho^n r^{n-1}\, dr\right)^{\frac{n-2}{n}}}. \tag{13}$$

Here $r = \tanh(\theta/2)$, $R = \tanh(\theta_1/2) < 1$, and ω_n represents the surface area of the unit sphere in n-dimensions, and is explicitly given by $\omega_n = 2\pi^{\frac{n}{2}}/\Gamma(n/2)$. This quotient is well defined if n is a real number instead of a natural number.

Lemma 3.2. *There exists a function $u \in H_0^1(\Omega)$, with $u'(0) = u(\theta_1) = 0$, such that $Q_\lambda(u) < S_n$ for all $\lambda > \dfrac{n(n-2)}{4} + L^*$. Here S_n is the Sobolev constant.*

Proof. Let φ be an arbitrary cutoff function such that $\varphi(0) = 1$, $\varphi'(0) = 0$ and $\varphi(R) = 0$, and let

$$v_\epsilon(r) = \frac{\varphi(r)}{(\epsilon + r^2)^{\frac{n-2}{2}}}.$$

As in [15], let

$$u_\epsilon(r) = \rho^{\frac{2-n}{2}}(r) v_\epsilon(r).$$

With this choice of u_ϵ, and after integrating by parts, we have

$$\begin{aligned}\int_0^R u'^2 \rho^{n-2} r^{n-1}\, dr = \frac{n(n-2)}{4} \int_0^R \rho^2 v_\epsilon^2 r^{n+1}\, dr + \frac{n(n-2)}{2} \int_0^R v_\epsilon^2 \rho r^{n-1}\, dr \\ + \int_0^R v_\epsilon'^2 r^{n-1}\, dr.\end{aligned} \tag{14}$$

Using the fact that $r^2 + \dfrac{2}{\rho} = 1$ to combine the first two terms of equation (14), it follows that,

$$Q_\lambda(u_\epsilon) = \frac{\omega_n \left(\frac{n(n-2)}{4} - \lambda\right) \int_0^R v_\epsilon^2 \rho^2 r^{n-1}\, dr + \omega_n \int_0^R v_\epsilon'^2 r^{n-1}\, dr}{\left(\omega_n \int_0^R v_\epsilon^{\frac{2n}{n-2}} r^{n-1}\, dr\right)^{\frac{n-2}{2}}}. \tag{15}$$

Claim 3.3.

$$\begin{aligned}\omega_n \left(\frac{n(n-2)}{4} - \lambda\right) \int_0^R v_\epsilon^2 \rho^2 r^{n-1}\, dr = &\omega_n \left(\frac{n(n-2)}{4} - \lambda\right) \int_0^R \varphi^2 r^{3-n} \rho^2\, dr \\ &+ \mathcal{O}\left(\epsilon^{\frac{4-n}{2}}\right).\end{aligned}$$

Proof. Let

$$I(\epsilon) = \int_0^R v_\epsilon^2 \rho^2 r^{n-1}\, dr = \int_0^R \frac{\varphi^2}{(\epsilon + r^2)^{n-2}} \rho^2 r^{n-1}\, dr.$$

Then $I(0) = \int_0^R \varphi^2 \rho^2 r^{3-n}\, dr$. Thus, it suffices to show that $|I(\epsilon) - I(0)| = \mathcal{O}\left(\epsilon^{\frac{4-n}{2}}\right)$.

If $0 < r < R < 1$, then $\rho(r) = \dfrac{2}{1-r^2} < \dfrac{2}{1-R^2}$. Thus,

$$\begin{aligned}
|I(\epsilon) - I(0)| &\leq \frac{4}{(1-R^2)^2} \left| \int_0^R \varphi^2 r^{n-1} \left(\frac{1}{(\epsilon + r^2)^{n-2}} - \frac{1}{r^{2(n-2)}} \right) dr \right| \\
&= \frac{4(n-2)}{(1-R^2)^2} \left| \int_0^R \int_0^\epsilon \frac{(\varphi^2 - 1 + 1)\, r^{n-1}}{(a + r^2)^{n-1}}\, da\, dr \right|.
\end{aligned}$$

Let

$$L_1(\epsilon) = \int_0^\epsilon \left(\int_0^R \frac{r^{n-1}}{(a + r^2)^{n-1}}\, dr \right) da,$$

and

$$L_2(\epsilon) = \int_0^R (\varphi^2 - 1) r^{n-1} \int_0^\epsilon \frac{1}{(a + r^2)^{n-1}}\, da\, dr.$$

Making the change of variables $r = u\sqrt{a}$ in the inner integral of $L_1(\epsilon)$, we have

$$\int_0^R \frac{r^{n-1}}{(a + r^2)^{n-1}}\, dr = a^{\frac{2-n}{2}} \int_0^{\frac{R}{\sqrt{a}}} \frac{u^{n-1}}{(1 + u^2)^{n-1}}\, du \leq a^{\frac{2-n}{2}} \int_0^\infty \frac{u^{n-1}}{(1 + u^2)^{n-1}}\, du.$$

Since we are considering $n > 2$, this last integral converges. Thus, and since $n < 4$,

$$L_1(\epsilon) \leq C \int_0^\epsilon a^{\frac{2-n}{2}}\, da = \mathcal{O}\left(\epsilon^{\frac{4-n}{2}}\right).$$

On the other hand, since $\varphi(0) = 1$ and $\varphi'(0) = 0$, for $0 \leq r < 1$ we have that $\varphi^2 - 1 \leq Cr^2$ for some $C > 0$. Thus,

$$\begin{aligned}
L_2(\epsilon) &\leq C \int_0^R r^{n+1} \int_0^\epsilon \frac{1}{(a + r^2)^{n-1}}\, da\, dr \\
&\leq C \int_0^R r^{n+1} \int_0^\epsilon \frac{1}{r^{2n-2}}\, da\, dr = C\epsilon \int_0^R r^{3-n}\, dr.
\end{aligned}$$

Since $n < 4$, this last integral converges. Thus $L_2(\epsilon) = \mathcal{O}(\epsilon)$, and in particular $\mathcal{O}(\epsilon^{\frac{4-n}{2}})$. ◻

Claim 3.4.

$$\omega_n \int_0^R v_\epsilon'^2 r^{n-1}\, dr = \omega_n \int_0^R \varphi'(r)^2 r^{3-n}\, dr + K_1 \epsilon^{\frac{2-n}{2}} + \mathcal{O}(\epsilon^{\frac{4-n}{2}}),$$

where

$$K_1 = \frac{\pi^{\frac{n}{2}} n(n-2)\Gamma\left(\frac{n}{2}\right)}{\Gamma(n)}.$$

Proof. Let

$$J = \omega_n \int_0^R v_\epsilon'^2 r^{n-1}\, dr.$$

Then we can write

$$J = \omega_n \int_0^R r^{n-1} \left[\frac{\varphi'^2}{(\epsilon + r^2)^{n-2}} - \frac{2(n-2)r\varphi\varphi'}{(\epsilon + r^2)^{n-1}} + \frac{r^2\varphi^2(n-2)^2}{(\epsilon + r^2)^n} \right] dr.$$

Integrating by parts the second term, and since by hypothesis $\varphi(R) = 0$, we have

$$\begin{aligned} J =& \omega_n \int_0^R \frac{\varphi'^2 r^{n-1}}{(\epsilon + r^2)^{n-2}}\, dr + \omega_n n(n-2) \int_0^R \frac{\varphi^2 r^{n-1}}{(\epsilon + r^2)^{n-1}}\, dr \\ & - 2\omega_n (n-2)(n-1) \int_0^R \frac{\varphi^2 r^{n+1}}{(\epsilon + r^2)^n}\, dr + \omega_n (n-2)^2 \int_0^R \frac{\varphi^2 r^{n+1}}{(\epsilon + r^2)^n}\, dr. \end{aligned}$$

Thus, since $(n-2)^2 - 2(n-2)(n-1) = -n(n-2)$, combining the last three terms we have

$$J = \omega_n \int_0^R \frac{\varphi'^2 r^{n-1}}{(\epsilon + r^2)^{n-2}}\, dr + \omega_n n(n-2)\epsilon \int_0^R \frac{\varphi^2 r^{n-1}}{(\epsilon + r^2)^n}\, dr. \tag{16}$$

Let us now estimate

$$J_1(\epsilon) \equiv \int_0^R \varphi'(r)^2 (\epsilon + r^2)^{2-n} r^{n-1}\, dr.$$

Notice that

$$J_1(0) = \int_0^R \varphi'(r)^2 r^{3-n}\, dr.$$

In what follows we estimate the difference, i.e., $\Delta(\epsilon) \equiv J_1(\epsilon) - J_1(0)$. We write,

$$\Delta(\epsilon) = \int_0^1 \varphi'(r)^2 r^{3-n} (-A)\, dr,$$

where

$$A = 1 - (\epsilon + r^2)^{2-n}\, r^{2n-4} = 1 - (1 + \epsilon r^{-2})^{2-n} > 0,$$

since $n > 2$. Using the fact that

$$(1+x)^{-m} > 1 - mx$$

for $x = \epsilon / r^2 \geq 0$ and $m = n-2 > 0$, we conclude that

$$A < (n-2)\, \epsilon\, r^{-2}.$$

Thus,

$$|\Delta(\epsilon)| < \epsilon(n-2) \int_0^R \varphi'(r)^2 r^{1-n}\, dr. \tag{17}$$

Notice that the integral on equation (17) converges. In fact, since $\varphi(0) = 1$ and $\varphi'(0) = 0$, for $0 \leq r < 1$ one has $\varphi'(r)^2 \leq C^2 r^2$ for some positive constant C; thus $\varphi'(r)^2 r^{1-n} \leq C r^{3-n}$, which is integrable near 0 for all $2 < n < 4$. Hence $|\Delta(\epsilon)| = \mathcal{O}(\epsilon)$. Thus, from equation (16) we have

$$J = \omega_n \int_0^R \varphi'^2 r^{3-n}\, dr + \omega_n n(n-2)\epsilon \int_0^R \frac{\varphi^2 r^{n-1}}{(\epsilon + r^2)^n}\, dr + \mathcal{O}(\epsilon). \tag{18}$$

Now let

$$J_2(\epsilon) \equiv \int_0^R \frac{(\varphi^2 - 1)\, r^{n-1} + r^{n-1}}{(\epsilon + r^2)^n}\, dr.$$

Making the change of variables $r = s\sqrt{\epsilon}$, we have

$$\int_0^R \frac{r^{n-1}}{(\epsilon + r^2)^n}\, dr = \epsilon^{\frac{-n}{2}} \left(\int_0^\infty \frac{s^{n-1}}{(1+s^2)^n}\, ds - \int_{\frac{R}{\sqrt{\epsilon}}}^\infty \frac{s^{n-1}}{(1+s^2)^n}\, ds \right).$$

But

$$\int_{\frac{R}{\sqrt{\epsilon}}}^\infty \frac{s^{n-1}}{(1+s^2)^n}\, ds \leq \int_{\frac{R}{\sqrt{\epsilon}}}^\infty s^{-n-1}\, ds = \frac{\epsilon^{\frac{n}{2}}}{nR^n}.$$

Notice that making the change of variables $u = s^2$, we can write

$$\int_0^\infty \frac{s^{n-1}}{(1+s^2)^n}\, ds = \frac{1}{2} \int_0^\infty \frac{u^{\frac{n}{2}-1}}{(1+u)^n}\, du = \frac{1}{2} \frac{\Gamma\left(\frac{n}{2}\right)^2}{\Gamma(n)}.$$

Here we have used the standard integral

$$\int_0^\infty \frac{x^{k-1}}{(1+x)^{k+m}}\, dx = \frac{\Gamma(k)\Gamma(m)}{\Gamma(k+m)}$$

(see, e.g., [6], equation 856.11, page 213), which holds for all m, $k > 0$. Thus,

$$\int_0^R \frac{r^{n-1}}{(\epsilon + r^2)^n}\, dr = \frac{\Gamma\left(\frac{n}{2}\right)^2 \epsilon^{\frac{-n}{2}}}{2\Gamma(n)} + \mathcal{O}(1). \tag{19}$$

On the other hand, since $\varphi^2(r) \leq 1 + Cr^2$, and setting once more $r = s\sqrt{\epsilon}$, we have that

$$\int_0^R \frac{(\varphi^2 - 1) r^{n-1}}{(\epsilon + r^2)^n}\, dr \leq C\epsilon^{\frac{2-n}{2}} \left(\int_0^\infty \frac{s^{n+1}}{(1+s^2)^n}\, ds - \int_{\frac{R}{\sqrt{\epsilon}}}^\infty \frac{s^{n+1}}{(1+s^2)^n}\, ds \right).$$

But

$$\int_{\frac{R}{\sqrt{\epsilon}}}^\infty \frac{s^{n+1}}{(1+s^2)^n}\, ds \leq \int_{\frac{R}{\sqrt{\epsilon}}}^\infty s^{1-n}\, ds = \mathcal{O}\left(\epsilon^{\frac{n-2}{2}}\right),$$

and $\int_0^\infty \frac{s^{n+1}}{(1+s^2)^n}\, ds$ is finite. Thus, and since $2 < n < 4$,

$$\int_0^R \frac{(\varphi^2 - 1) r^{n-1}}{(\epsilon + r^2)^n}\, dr \leq C \int_0^R \frac{r^{n+1}}{(\epsilon + r^2)^n}\, dr = \mathcal{O}(\epsilon^{\frac{2-n}{2}}). \tag{20}$$

Therefore, from equations (19) and (20) it follows that

$$J_2(\epsilon) = \frac{\Gamma\left(\frac{n}{2}\right)^2 \epsilon^{\frac{-n}{2}}}{2\Gamma(n)} + \mathcal{O}(\epsilon^{\frac{2-n}{2}}).$$

Finally, from equation (18) it follows that

$$J = \omega_n \int_0^1 \varphi'^2 r^{3-n}\, dr + \omega_n n(n-2)\epsilon^{\frac{2-n}{2}} \left(\frac{\Gamma\left(\frac{n}{2}\right)^2}{2\Gamma(n)} \right) + \mathcal{O}(\epsilon^{\frac{4-n}{2}}).$$

But we are taking $\omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$. Thus,

$$J = \omega_n \int_0^1 \varphi'^2 r^{3-n}\, dr + \epsilon^{\frac{2-n}{2}} \left(\frac{n(n-2)\pi^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}{\Gamma(n)} \right) + \mathcal{O}(\epsilon^{\frac{4-n}{2}}).$$

□

Claim 3.5.

$$\left(\omega_n \int_0^R v_\epsilon^{\frac{2n}{n-2}} r^{n-1}\, dr \right)^{\frac{n-2}{n}} = \epsilon^{\frac{2-n}{2}} K_2 + \mathcal{O}(\epsilon^{\frac{4-n}{2}}),$$

where

$$K_2 = \left(\pi^{n/2} \frac{\Gamma(n/2)}{\Gamma(n)} \right)^{\frac{n-2}{n}}.$$

Proof. Let

$$H(\epsilon) \equiv \omega_n \int_0^R v_\epsilon^{\frac{2n}{n-2}} r^{n-1}\, dr = \omega_n \int_0^R \frac{\varphi(r)^{2n/(n-2)}}{(\epsilon + r^2)^n} r^{n-1} dr.$$

Since $\varphi(0) = 1$, this integral diverges when $\epsilon \to 0$. Denote by H_1 the leading behavior of $H(\epsilon)$, that is,

$$H_1(\epsilon) = \omega_n \int_0^R \frac{r^{n-1}}{(\epsilon + r^2)^n}\, dr.$$

As in equation (19), we have

$$H_1(\epsilon) = c_n \epsilon^{-n/2} + O(1), \tag{21}$$

where

$$c_n = \frac{\omega_n}{2} \frac{\Gamma(n/2)^2}{\Gamma(n)} = \pi^{n/2} \frac{\Gamma(n/2)}{\Gamma(n)}. \tag{22}$$

It suffices now to show that

$$H(\epsilon) - H_1(\epsilon) = \omega_n \int_0^R \frac{\varphi(r)^{2n/(n-2)} - 1}{(\epsilon + r^2)^n} r^{n-1} dr = \mathcal{O}\left(\epsilon^{\frac{2-n}{2}}\right).$$

But since $\varphi(r) \le 1 + C\, r^2$ for some positive constant C, then

$$|H(\epsilon) - H_1(\epsilon)| \le C_n \int_0^R \frac{r^{n+1}}{(\epsilon + r^2)^n}\, dr = \mathcal{O}\left(\epsilon^{\frac{2-n}{2}}\right), \tag{23}$$

where the last equality follows from equation (20). Thus, from (21) and (23), we conclude that

$$H(\epsilon) = \epsilon^{-n/2}[c_n + O(\epsilon)],$$

where c_n is given by (22).

□

Replacing the estimates obtained in the three previous claims in the definition of $Q_\lambda(u_\epsilon)$ given in equation (15), we obtain

$$Q_\lambda(u_\epsilon) = \frac{K_1}{K_2} + \frac{\epsilon^{\frac{n-2}{2}}\omega_n}{K_2}\left(\left(\frac{n(n-2)}{4} - \lambda\right)\int_0^R \varphi^2 r^{3-n}\rho^2\,dr + \int_0^R \varphi'^2 r^{3-n}\,dr\right) + \mathcal{O}(\epsilon).$$

Here

$$K_1 = \frac{\pi^{\frac{n}{2}} n(n-2)\Gamma\left(\frac{n}{2}\right)}{\Gamma(n)},$$

and

$$K_2 = \left(\pi^{n/2}\frac{\Gamma(n/2)}{\Gamma(n)}\right)^{\frac{n-2}{n}}.$$

But

$$\frac{K_1}{K_2} = \pi n(n-2)\left(\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma(n)}\right)^{\frac{2}{n}},$$

which is precisely the Sobolev critical constant S_n (see, e.g., [16], with $p=2$, $m=n$ and $q=\frac{2n}{n-2}$). Therefore, to conclude that $Q_\lambda(u_\epsilon) < S_n$, it suffices to show that for $\lambda > n(n-2)/4 + L^*$, there exists a choice of φ such that

$$F(\varphi) \equiv \left(\frac{n(n-2)}{4} - \lambda\right)\int_0^R \varphi^2 r^{3-n}\rho^2\,dr + \int_0^R \varphi'^2 r^{3-n}\,dr$$

is negative.

Let

$$M(\varphi) = \int_0^R \varphi'^2 r^{3-n}\,dr,$$

and let φ_1 be the minimizer of $M(\varphi)$ subject to the constraint $\int_0^R \varphi^2 r^{3-n}\rho^2\,dr = 1$. Then φ_1 satisfies the Euler equation

$$-\left(\varphi_1' r^{3-n}\right)' = \mu\varphi_1 r^{3-n}\rho^2. \tag{24}$$

Here μ is the Lagrange multiplier. Multiplying equation (24) by φ_1 and integrating this equation by parts, and since $\int_0^R \varphi_1^2 r^{3-n}\rho^2\,dr = 1$, we obtain

$$\int_0^R \varphi_1'^2 r^{3-n}\,dr = \mu. \tag{25}$$

It follows that $F(\varphi_1) = \frac{n(n-2)}{4} - \lambda + \mu$. Thus, $F(\varphi_1)$ is negative as long as $\lambda > \frac{n(n-2)}{4} + \mu$. Notice that from (25) one has that μ is positive.

It suffices now to show that $\mu = L^*$. Multiplying equation (24) by $-r^{n-3}$, we obtain

$$\varphi'' + \frac{(3-n)}{r}\varphi' + \mu\varphi\rho^2 = 0. \tag{26}$$

Making the change of variables $\varphi(r) = r^{\frac{n-2}{2}}v(r)$, and after some rearrangement of terms, we can write equation (26) as

$$v'' + \frac{v'}{r} + \left(\mu\rho^2 - \frac{(n-2)^2}{4r^2}\right)v = 0. \tag{27}$$

Changing back to geodesic coordinates, and since $r = \tanh\frac{\theta}{2}$, we can rewrite equation (27) as

$$v'' + \coth\theta v' + \left(\mu - \frac{\alpha^2}{\sinh^2\theta}\right)v = 0, \tag{28}$$

where $\alpha = \frac{2-n}{2}$. Equation (28) is a Legendre equation, whose solutions are P_ℓ^α and $P_\ell^{-\alpha}$, where $-\ell(\ell+1) = \mu$. It follows from equation (6) that the regular solution to equation (26) is

$$\varphi(\theta) = \tanh^{-\alpha}\left(\frac{\theta}{2}\right)P_\ell^{-\alpha}(\cosh\theta).$$

Since the solution must vanish at the boundary, it follows that $L = L^*$. Thus, $\mu = L^*$. This finishes the proof of Lemma 3.2.

□

The proof of Theorem 3.1 now follows easily from a result by Lieb. In fact, by Lemma 1.2 in [5], it follows that if there exists some u such that $Q_\lambda(u) < S_n$, then there exists a minimizer of Q_λ. Given any constant $\eta > 0$, the quotient $Q_\lambda(u)$ is invariant under the transformation $u \to \eta u$. In order to compute the corresponding Euler equation, we minimize the numerator of equation (13) subject to the constraint $\omega_n\int_0^R u^{\frac{2n}{n-2}}\rho^n r^{n-1}\,dr = 1$. We obtain

$$\left(u'\rho^{n-2}r^{n-1}\right)' + \lambda u\rho^n r^{n-1} + \eta u^p\rho^n r^{n-1} = 0, \tag{29}$$

where η is a Lagrange multiplier. Multiplying through by $\omega_n u$, integrating between 0 and R, and integrating by parts, we obtain

$$\begin{aligned}\eta &= \omega_n\left(\int_0^R u'^2\rho^{n-2}r^{n-1}\,dr - \lambda\int_0^R u^2\rho^n r^{n-1}\,dr\right)\\ &\geq (\lambda_1 - \lambda)\omega_n\int_0^R u^2\rho^n r^{n-1}\,dr.\end{aligned}$$

This last inequality follows from the variational characterization of λ_1. It follows that $\eta > 0$ provided that $\lambda < \lambda_1$. Setting $u = \eta^{\frac{-1}{p-1}}v$ in (29) one has that v satisfies

$$\left(u'\rho^{n-2}r^{n-1}\right)' + \lambda u\rho^n r^{n-1} + u^p\rho^n r^{n-1} = 0. \tag{30}$$

Finally, setting $r = \tanh\frac{\theta}{2}$, equation (30) becomes (12). This finishes the proof of Theorem 3.1.

4. Nonexistence of solutions

In this section we use a Pohozaev type argument to show that if $2 < n < 4$ then problem (3) has a solution gap.

Theorem 4.1. *For any $2 < n < 4$ and $\theta_1 \in (0, \infty)$, the boundary value problem*

$$-u''(\theta) - (n-1)\coth\theta\, u'(\theta) = \lambda u + u^{\frac{n+2}{n-2}} \tag{31}$$

with $u \in H_0^1(\Omega)$, $u'(0) = u(\theta_1) = 0$, and $\theta \in [0, \theta_1]$, has no solution if

$$\lambda \in \left(\frac{n(n-2)}{4}, \frac{n(n-2)}{4} + L^* \right]. \tag{32}$$

Here L^* is as in Definition 1.

Proof. Let g be a smooth nonnegative function such that $g(0) = g'(0) = 0$. Writing equation (31) as

$$\frac{-(\sinh^{n-1}\theta\, u')'}{\sinh^{n-1}\theta} = \lambda u + u^p, \tag{33}$$

multiplying through by $g(\theta)u'(\theta)\sinh^{2n-2}\theta$, and integrating, we obtain

$$-\int_0^{\theta_1} \left(\frac{(\sinh^{n-1}\theta\, u')^2}{2} \right)' g\, d\theta = \lambda \int_0^{\theta_1} \left(\frac{u^2}{2} \right)' g \sinh^{2n-2}\theta\, d\theta + \int_0^{\theta_1} \left(\frac{u^{p+1}}{p+1} \right)' g \sinh^{2n-2}\theta\, d\theta.$$

Integrating by parts, and since $u(\theta_1) = 0$, we obtain

$$\frac{1}{2}\int_0^{\theta_1} u'^2 g' \sinh^{2n-2}\, d\theta + \frac{\lambda}{2}\int_0^{\theta_1} u^2 (g\sinh^{2n-2}\theta)'\, d\theta + \int_0^{\theta_1} \frac{u^{p+1}}{p+1}(g\sinh^{2n-2}\theta)'\, d\theta = \frac{\sinh^{2n-2}\theta_1 (u'(\theta_1))^2 g(\theta_1)}{2}. \tag{34}$$

Let $f(\theta) = \frac{1}{2} g' \sinh^{n-1}\theta$. Multiplying equation (33) by $f(\theta)u(\theta)\sinh^{n-1}\theta$ and integrating, we obtain

$$-\int_0^{\theta_1} (\sinh^{n-1}\theta\, u')' f u\, d\theta = \lambda \int_0^{\theta_1} f \sinh^{n-1}\theta\, u^2\, d\theta + \int_0^{\theta_1} u^{p+1} f \sinh^{n-1}\theta\, d\theta.$$

After integrating by parts, this last equation can be written as

$$\int_0^{\theta_1} u'^2 f \sinh^{n-1}\theta\, d\theta = \int_0^{\theta_1} u^2 \left(\lambda f \sinh^{n-1}\theta + \frac{1}{2}(f'\sinh^{n-1}\theta)' \right) d\theta + \int_0^{\theta_1} u^{p+1} f \sinh^{n-1}\theta\, d\theta. \tag{35}$$

By subtracting equation (34) from equation (35) we obtain

$$\int_0^{\theta_1} A(\theta) u(\theta)^2\, d\theta + \int_0^{\theta_1} B(\theta) u(\theta)^{p+1}\, d\theta = \frac{\sinh^{2n-2}\theta_1 (u'(\theta_1))^2 g(\theta_1)}{2}, \tag{36}$$

where

$$A(\theta) \equiv \frac{1}{2}(f'(\theta)\sinh^{n-1}\theta)' + \lambda f(\theta)\sinh^{n-1}\theta + \frac{\lambda}{2}(g(\theta)\sinh^{2n-2}\theta)';$$

and

$$B(\theta) = f(\theta)\sinh^{n-1}\theta + \frac{(g(\theta)\sinh^{2n-2}(\theta))'}{p+1}.$$

Notice that the right-hand side of equation (36) is nonnegative. We will show that the left-hand side of (36) is negative, thus arriving at a contradiction.

Using the definition of f and simplifying, we can write

$$A(\theta) = \sinh^{2n-2}\theta\left[\frac{g'''}{4} + \frac{3}{4}(n-1)\coth\theta g'' + \left(\lambda + \frac{n-1}{4} + \frac{(n-1)(2n-3)}{4}\coth^2\theta\right)g' + \lambda(n-1)\coth\theta g\right].$$

Finally, making the change of variables $T(\theta) = g(\theta)\sinh^2\theta$, we obtain

$$A(\theta) = \sinh^{2n-4}\theta\left[\frac{T'''}{4} + \frac{3}{4}(n-3)\coth\theta T'' + \left(\frac{1}{4}\coth^2\theta(n-3)(2n-11) + \lambda + \frac{1}{4}(n-7)\right)T' + (n-3)\left(\coth\theta(\lambda-2) - \coth^3\theta(n-4)\right)T\right].$$

Simplifying B, we obtain

$$B(\theta) = \frac{(n-1)\sinh^{2n-2}\theta}{n}\left(g'(\theta) + (n-2)\coth\theta g\right).$$

As before, we make the change of variables $T(\theta) = g(\theta)\sinh^2\theta$, to obtain

$$B(\theta) = \frac{(n-1)}{n}\sinh^{2n-4}\theta\left(T' + (n-4)\coth\theta T\right).$$

We will show that there is a choice of T for which $A(\theta) \equiv 0$. We will then show that for this choice of T, $B(\theta)$ is negative as long as

$$\lambda \in \left(\frac{n(n-2)}{4}, \frac{n(n-2)}{4} + L^*\right]. \tag{37}$$

Lemma 4.2. *Consider the equation*

$$\frac{T'''}{4} + \frac{3}{4}(n-3)\coth\theta T'' + \left(\frac{1}{4}\coth^2\theta(n-3)(2n-11) + \lambda + \frac{1}{4}(n-7)\right)T' + (n-3)\left(\coth\theta(\lambda-2) - \coth^3\theta(n-4)\right)T = 0. \tag{38}$$

Then

$$T(\theta) = \sinh^{4-n}\theta P_\ell^\alpha(\cosh\theta)P_\ell^{-\alpha}(\cosh\theta)$$

is a solution of (38), where $\alpha = (2-n)/2$ and $\ell(\ell+1) = \alpha(\alpha-1) - \lambda$.

Proof. Let $v(\theta) = y_1(\theta)y_2(\theta)$, where $y_1(\theta) = P_\ell^\alpha(\cosh\theta)$ and $y_2(\theta) = P_\ell^{-\alpha}(\cosh\theta)$. Then y_1 and y_2 are solutions of

$$y''(\theta) + \coth\theta y'(\theta) + k(\theta)y(\theta) = 0, \tag{39}$$

where

$$k(\theta) = -\ell(\ell+1) - \frac{\nu^2}{\sinh^2\theta}.$$

It follows from equation (39) that

$$y_1''\, y_2 + y_2''\, y_1 = -\coth\theta\, v' - 2kv,$$

and from the above that

$$v'' = 2y_1'y_2' - \coth\theta v' - 2kv.$$

Similarly, we can write

$$y_1''y_2' + y_1'y_2'' = -2\coth\theta y_1'y_2' - kv',$$

from which it follows that

$$v''' = -\coth\theta v'' + \left(\frac{1}{\sinh^2\theta} - 4k\right) v' - 2k'v - 4\coth\theta y_1'y_2'.$$

Using the fact that $y_1'y_2' = \frac{1}{2}\left(v'' + \coth\theta v' + 2kv\right)$, we obtain

$$v''' + 3\coth\theta v'' + \left(2\coth^2\theta + 4k - \frac{1}{\sinh^2\theta}\right) v' + \left(2k' + 4k\coth\theta\right) v = 0. \tag{40}$$

Finally, replacing $v(\theta) = T(\theta)\sinh^{n-4}\theta$ in equation (40), using the fact that $\ell(\ell+1) = \alpha(\alpha-1) - \lambda$, and after significant simplification and rearrangement of terms, we obtain precisely equation (38).

□

It suffices now to show that for T as in the previous lemma, B is negative. We do so in the following lemma.

Lemma 4.3. *Let*

$$T(\theta) = \sinh^{4-n}\theta P_\ell^\alpha(\cosh\theta)P_\ell^{-\alpha}(\cosh\theta)$$

where $\alpha = (2-n)/2$, $\theta \in (0, \theta_1)$, *and* $L = -\ell(\ell+1) = \lambda - \alpha(\alpha-1)$. *Then*

$$B(\theta) = \frac{(n-1)}{n}\sinh^{2n-4}\theta\left(T' + (n-4)\coth\theta\, T\right) \tag{41}$$

is negative if $0 < L \le L^*$.

Proof. Notice that the condition $0 < L \le L^*$ is precisely the same as (37). Substituting $T(\theta) = \sinh^{4-n}\theta P_\ell^\alpha(\cosh\theta)P_\ell^{-\alpha}(\cosh\theta)$ in equation (41), we obtain

$$B(\theta) = \frac{(n-1)}{n}\sinh^{n+1}\theta\left(\dot{P}_\ell^\alpha P_\ell^{-\alpha} + P_\ell^\alpha \dot{P}_\ell^{-\alpha}\right).$$

Since $\sinh\theta$ is positive for $\theta > 0$, and since $P_\ell^\alpha P_\ell^{-\alpha} > 0$ if $0 < L \le L^*$, it suffices to show that

$$\frac{\dot{P}_\ell^\alpha}{P_\ell^\alpha} + \frac{\dot{P}_\ell^{-\alpha}}{P_\ell^{-\alpha}} < 0.$$

Let

$$y_\nu(\theta) = \frac{1}{\sinh\theta}\frac{P_\ell^{\nu+1}}{P_\ell^\nu} + \frac{\nu}{2\sinh^2\frac{\theta}{2}}. \tag{42}$$

Then, by the raising relation given by equation (8) it follows that

$$\frac{\dot{P}_\ell^\alpha}{P_\ell^\alpha} + \frac{\dot{P}_\ell^{-\alpha}}{P_\ell^{-\alpha}} = \frac{1}{\sinh\theta}\left(\frac{P_\ell^{\alpha+1}}{P_\ell^\alpha} + \frac{P_\ell^{-\alpha+1}}{P_\ell^{-\alpha}}\right) = y_\alpha + y_{-\alpha}.$$

We will show that for $\theta \in (0,\theta_1)$, and if $-1 < \nu < 1$, then $y_\nu(\theta) < 0$. This will imply that $y_\alpha(\theta) + y_{-\alpha}(\theta) < 0$, and therefore that B is negative.

From equations (6) and (7) it follows that

$$P_\ell^\nu = \frac{1}{\Gamma(1-\nu)}\coth^\nu\left(\frac{\theta}{2}\right)\left(1 + \frac{\ell(\ell+1)}{1-\nu}\sinh^2\left(\frac{\theta}{2}\right) + \mathcal{O}\left(\sinh^4\left(\frac{\theta}{2}\right)\right)\right).$$

Then, and since $\Gamma(1-\nu) = -\nu\Gamma(-\nu)$, we can write

$$\frac{P_\ell^{\nu+1}}{P_\ell^\nu} = -\nu\coth\left(\frac{\theta}{2}\right)\left(1 - \frac{\ell(\ell+1)}{\nu(1-\nu)}\sinh^2\left(\frac{\theta}{2}\right) + \mathcal{O}\left(\sinh^4\left(\frac{\theta}{2}\right)\right)\right).$$

Therefore, and since $\coth\left(\frac{\theta}{2}\right)/\sinh\theta = \left(2\sinh^2\left(\frac{\theta}{2}\right)\right)^{-1}$, we have

$$y_\nu = \frac{\ell(\ell+1)}{2(1-\nu)} + \mathcal{O}\left(\sinh^2\left(\frac{\theta}{2}\right)\right).$$

Thus, if $-1 < \nu < 1$, and since $\ell(\ell+1) < 0$,

$$\lim_{\theta\to 0} y_\nu(\theta) = \frac{\ell(\ell+1)}{2(1-\nu)} < 0.$$

We will show by contradiction that there is no point at which y_ν changes sign, thus concluding that $y_\nu(\theta)$ is negative for all $\theta > 0$.

Taking the derivative of equation (42), we obtain

$$y_\nu' = -\frac{\cosh\theta}{\sinh^2\theta}\frac{P_\ell^{\nu+1}}{P_\ell^\nu} + \frac{\dot{P}_\ell^{\nu+1}}{P_\ell^\nu} - \frac{\dot{P}_\ell^\nu}{P_\ell^\nu}\frac{P_\ell^{\nu+1}}{P_\ell^\nu} - \frac{\nu}{2}\frac{\cosh\left(\frac{\theta}{2}\right)}{\sinh^3\left(\frac{\theta}{2}\right)}.$$

Using the raising and lowering relations given in equations (8) and (9), we can write

$$\begin{aligned} y_\nu' = \frac{-1}{\sinh\theta}\left(\frac{P_\ell^{\nu+1}}{P_\ell^\nu}\right)^2 + \frac{(-2\nu-2)\cosh\theta}{\sinh^2\theta}\left(\frac{P_\ell^{\nu+1}}{P_\ell^\nu}\right) \\ + \frac{\ell(\ell+1) - \nu(\nu+1)}{\sinh\theta} - \frac{\nu\cosh\frac{\theta}{2}}{2\sinh^3\frac{\theta}{2}}. \end{aligned}$$

Solving for $\left(\frac{P_\ell^{\nu+1}}{P_\ell^\nu}\right)$ from equation (42), and after rearranging terms, we obtain

$$y_\nu' = -\sinh\theta y_\nu^2 + \frac{2(\nu-\cosh\theta)}{\sinh\theta}y_\nu + \frac{\ell(\ell+1)}{\sinh\theta}. \tag{43}$$

Now suppose there was a point θ^* at which $y_\nu(\theta^*)$ crossed the θ-axis. At this point, we would have $y_\nu(\theta^*) = 0$ and $y_\nu'(\theta^*) > 0$. But evaluating equation (43) at θ^*, we obtain

$$y_\nu'(\theta^*) = \frac{\ell(\ell+1)}{\sinh\theta^*} < 0,$$

arriving at a contradiction.

□

This completes the proof of Theorem 4.1.

□

5. Uniqueness

Lemma 5.1. *The problem*

$$u''(\theta) + (n-1)\coth(\theta)u'(\theta) + \lambda u(\theta) + u(\theta)^p = 0 \tag{44}$$

with $u'(0) = u(\theta_1) = 0,\ 2 < n < 4,$ *and* $\lambda > \frac{n(n-2)}{4}$*, has at most one positive solution.*

Proof. The proof of this lemma follows directly from [10]. In fact, making the change of variables $u \to v$ given by $u(\theta) = \sinh^\alpha(\theta)v(\theta)$, where $\alpha = \frac{2-n}{2}$, equation (44) can be written as

$$\sinh^2(\theta)v''(\theta) + \sinh\theta\cosh\theta v'(\theta) + G_\lambda(\theta)v(\theta) + v(\theta)^p = 0, \tag{45}$$

where

$$G_\lambda(\theta) = -\alpha^2 + \left[\lambda - \frac{n(n-2)}{4}\right]\sinh^2\theta.$$

We define the energy function

$$E[v] \equiv \sinh^2\theta v'(\theta)^2 + \frac{2}{p+1}v(\theta)^{p+1} + G_\lambda(\theta)v(\theta)^2 = 0.$$

Then if $v(\theta)$ is a solution of (45),

$$\frac{dE}{d\theta} = G_\lambda'(\theta)v(\theta)^2.$$

The function $G_\lambda(\theta)$ is increasing as long as $\lambda > \frac{n(n-2)}{4}$. That is, $G_\lambda(\theta)$ is a $\Lambda - function$ and it follows from [10] that v (and therefore u) is unique. □

Remark 5.2. *Uniqueness of solutions to this problem for* $\lambda \in (n(n-2)/4, (n-1)^2/4]$ *was obtained by Mancini and Sandeep (see Proposition 4.4 in* [11]*). Notice that* $\lambda = (n-1)^2/4$ *corresponds to the first eigenvalue in the limiting case* $\theta_1 = \infty$*. The interval considered in* [11] *is a strict subinterval of the interval we consider here.*

References

[1] Bandle, C., Benguria, R.: The Brézis-Nirenberg problem on $\mathbb{S}^3$. J. Differential Equations **178**(1), 264–279 (2002)

[2] Bandle, C., Kabeya, Y.: On the positive, “radial” solutions of a semilinear elliptic equation in $\mathbb{H}^N$. Adv. Nonlinear Anal. **1**(1), 1–25 (2012)

[3] Benguria, R., Benguria, S.: The Brezis-Nirenberg problem on $\mathbb{S}^n$, in spaces of fractional dimension. arXiv:1503.06347 (2015)

[4] Bonforte, M., Gazzola, F., Grillo, G., Vázquez, J.L.: Classification of radial solutions to the Emden-Fowler equation on the hyperbolic space. Calc. Var. Partial Differential Equations **46**(1-2), 375–401 (2013)

[5] Brézis, H., Nirenberg, L.: Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. Comm. Pure Appl. Math. **36**(4), 437–477 (1983)

[6] Dwight, H.B.: Tables of integrals and other mathematical data. 4th ed. The Macmillan Company, New York (1961)

[7] Ganguly, D., Sandeep, K.: Sign changing solutions of the Brezis-Nirenberg problem in the hyperbolic space. Calc. Var. Partial Differential Equations **50**(1-2), 69–91 (2014)

[8] Ganguly, D., Sandeep, K.: Nondegeneracy of positive solutions of semilinear elliptic problems in the hyperbolic space. Commun. Contemp. Math. **17**(1), 1450,019, 13 (2015)

[9] Jannelli, E.: The role played by space dimension in elliptic critical problems. J. Differential Equations **156**(2), 407–426 (1999)

[10] Kwong, M.K., Li, Y.: Uniqueness of radial solutions of semilinear elliptic equations. Trans. Amer. Math. Soc. **333**(1), 339–363 (1992)

[11] Mancini, G., Sandeep, K.: On a semilinear elliptic equation in $\mathbb{H}^n$. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **7**(4), 635–671 (2008)

[12] McKean, H.P.: An upper bound to the spectrum of Δ on a manifold of negative curvature. J. Differential Geometry **4**, 359–366 (1970)

[13] Pucci, P., Serrin, J.: Critical exponents and critical dimensions for polyharmonic operators. J. Math. Pures Appl. (9) **69**(1), 55–83 (1990)

[14] Richtmyer, R.D.: Principles of advanced mathematical physics. Vol. II. Springer-Verlag, New York-Berlin (1981). Texts and Monographs in Physics

[15] Stapelkamp, S.: The Brézis-Nirenberg problem on $\mathbb{H}^n$. Existence and uniqueness of solutions. In: Elliptic and parabolic problems (Rolduc/Gaeta, 2001), pp. 283–290. World Sci. Publ., River Edge, NJ (2002)

[16] Talenti, G.: Best constant in Sobolev inequality. Ann. Mat. Pura Appl. (4) **110**, 353–372 (1976)

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