Vaught's conjecture for theories of one unary operation

by

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Abstract. Vaught's conjecture [11] is that for any countable first order theory T $\omega(T) \leqslant \aleph_0$ or $\omega(T) = 2^{\aleph_0}$, where $\omega(T)$ is the number of nonisomorphic countable models of T. It is shown that Vaught's conjecture is true for any first order theory T in the language of one unary operation. Also an example is given of a pseudo $(L_{\omega_1\omega})$ elementary class in the language of une unary operation with exactly $\aleph_1$ nonisomorphic countable models.

For $\mathfrak{A} = (A, R)$ where R is a binary relation on A define:

(1) For $a, b \in A$ $\delta(a, b)$ is the least $n < \omega$ such that there are $x_i \in A$ for $i \leqslant n$ with $a = x_0$ and $b = x_n$, and for all $i < n$ $\big(R(x_i, x_{i+1})$ or $R(x_{i+1}, x_i)\big)$. If no n exists let $\delta(a, b) = \infty$.

(2) For $a, b \in A$ a is connected to b iff $\delta(a, b) = n$ for some $n < \omega$.

(3) A loop is a set of distinct elements of A, $\{x_0, \ldots, x_n\}$ with $n > 1$ and such that for any $i < n$ $\big(R(x_i, x_{i+1})$ or $R(x_{i+1}, x_i)\big)$ and $\big(R(x_n, x_0)$ or $R(x_0, x_n)\big)$.

(4) A component is a maximal connected subset of A.

(5) $\omega(\mathfrak{A})$ is the number of nonisomorphic elementary substructures of $\mathfrak{A}$.

THEOREM A. *If $\mathfrak{A} = \langle A, R\rangle$ is countable and every component of $\mathfrak{A}$ contains only finitely many loops then $\omega(\mathfrak{A}) \leqslant \aleph_0$ or $\omega(\mathfrak{A}) = 2^{\aleph_0}$.*

Remarks.

(1) If R is a partial function on A $\big(\forall x, y, z \in A\big(R(x, z)$ and $R(y, z)\big) \Rightarrow x = y\big)$ then each component contains at most one loop.

(2) Theorem A can be generalized to show that if $\mathfrak{A}$ is expanded by adding countably many unary predicates or constants, then $\omega(\mathfrak{A}) \leqslant \aleph_0$ or $\omega(\mathfrak{A}) = 2^{\aleph_0}$.

THEOREM B. *If T is a complete theory in the language of one binary relation and every countable model of T has the property that every component contains only finitely many loops, then $\omega(T) = 1$, $\aleph_0$ or $2^{\aleph_0}$.*

Theorem B was proved by Leo Marcus [6] and myself independently. Later M. Rubin pointed out that the fact $\big(\omega(T) > 1 \rightarrow \omega(T) \geqslant \aleph_0\big)$ can be obtained as a corollary of a theorem of Lachlan [5].

THEOREM C. *There is a θ a* PC($L_{\omega_1\omega}$) *sentence in one unary operation such that* $\omega(\theta) = \aleph_1$.

This disproves Theorem 1 of S. Burris [1], since it implies that the quantifier ranks of the Scott sentences of countable unary operations are arbitrarily high. J. Steel [10] has proved Vaught's conjecture for $L_{\omega_1\omega}$ sentences in one unary operation. M. Rubin proved Vaught's conjecture for theories of a linear order [8] and more recently for $L_{\omega_1\omega}$ sentences of a linear order [9].

In my abstract [7] I mistakenly stated Theorem C for PC($L_{\omega\omega}$). Does there exist a PC($L_{\omega\omega}$) sentence θ in one unary operation with $\omega(\theta) = \aleph_1$?

The proof of Theorem A. We only prove Theorem A for $\mathfrak{A} = (A, R, \bar{a})$ where R is binary, symmetric, and irreflexive; and $\bar{a}$ is finitely many constants, since it is easy to generalize.

DEFINITIONS. (1) For $\mathfrak{A}$ having a distinguished constant $\underline{O}$ let

$$\mathfrak{A}_n = \{a \in A \colon \delta(x, \underline{O}) \leqslant n\}.$$

(2) $\mathfrak{A} \equiv_n \mathfrak{B}$ iff Player II has a winning strategy in the Ehrenfeucht game of length n [2].

Our main lemma is the following.

LEMMA 1. *If $\mathfrak{A}$ and $\mathfrak{B}$ are connected with distinguished constants then*

$$(\forall n<\omega\, \mathfrak{A}_n \equiv_n \mathfrak{B}_n) \Rightarrow \mathfrak{A} \equiv \mathfrak{B}.$$

PROPOSITION 1. *If $\mathfrak{A}$ is connected with distinguished constant then*

$$\forall n<\omega\, \forall\varphi(\bar{x}, \bar{y})\, \exists N\geqslant n\; N<\omega\, \exists\Gamma$$

finite

$$\forall \bar{a} \in \mathfrak{A}-\mathfrak{A}_N \exists\varphi^*(\bar{y}) \in \Gamma \forall \bar{b} \in \mathfrak{A}_n(\mathfrak{A} \models \varphi(\bar{a}, \bar{b}) \text{ iff } \mathfrak{A}_N \models \varphi^*(\bar{b})).$$

Proof. The proof is by induction on the logical complexity of $\varphi(\bar{x}, \bar{y})$. For the atomic case put $N = n+2$ and $\Gamma = \{T, F, x_1 = x_2, R(x_1, x_2)\}$. On the induction step "$\neg$" and "$\wedge$" are both easy. We do the case of $\exists z\varphi(\bar{x}, z, \bar{y})$. By induction $\exists\Gamma_1 \exists N_1 \geqslant n$ such that

$$\forall a\bar{a} \in \mathfrak{A}-\mathfrak{A}_{N_1} \exists\sigma(\bar{y}) \in \Gamma_1 \forall\bar{b} \in \mathfrak{A}_n(\mathfrak{A} \models \varphi(\bar{a}, a, \bar{b}) \text{ iff } \mathfrak{A}_{N_1} \models \sigma(\bar{b})).$$

Also by induction $\exists\Gamma_2\exists N_2\geqslant N_1$ such that

$$\forall\bar{a} \in \mathfrak{A}-\mathfrak{A}_{N_2}\exists\tau(z, \bar{y}) \in \Gamma_2 \forall b\bar{b} \in \mathfrak{A}_{N_1}(\mathfrak{A} \models \varphi(\bar{a}, b, \bar{b}) \text{ iff } \mathfrak{A}_{N_2} \models \tau(b, \bar{b})).$$

Let $N = N_2$ and

$$\Gamma = \{\bigvee\!\!\bigvee_{\sigma\in F} \sigma^{N_1}(y) \vee \exists z \in \mathfrak{A}_{N_1}\tau(z, \bar{y})\colon F \subseteq \Gamma_1, \tau \in \Gamma_2\},$$

and where σ^{N_1} is the relativization of σ to $\mathfrak{A}_{N_1}$. These work since given $\bar{a} \in \mathfrak{A}-\mathfrak{A}_{N_2}$ let

$$F = \{\sigma(\bar{y}) \in \Gamma_1 \colon \exists a \in \mathfrak{A}-\mathfrak{A}_{N_1}\forall\bar{b} \in \mathfrak{A}_n(\mathfrak{A} \models \varphi(\bar{a}, a, \bar{b}) \leftrightarrow \mathfrak{A}_{N_1} \models \sigma(\bar{b}))\}$$

and $\tau(z, \bar{y})$ so

$$\forall b\bar{b} \in \mathfrak{A}_{N_1}(\mathfrak{A} \models \varphi(\bar{a}, b, \bar{b}) \leftrightarrow \mathfrak{A}_{N_2} \models \tau(b, \bar{b})).$$

Let

$$\varphi^*(\bar{y}) = \bigvee\!\!\bigvee_{\sigma\in F} \sigma^{N_1}(\bar{y}) \vee \exists z \in \mathfrak{A}_{N_1}\tau(z, \bar{y}). \quad ■$$

Remark. Proposition 1 was motivated by the main lemma in Feferman–Vaught [3].

PROPOSITION 2. *If $\mathfrak{A}$ is connected with a distinguished constant then*

$$\forall\varphi(x, \bar{y})\forall n<\omega\exists N<\omega\forall\bar{b} \in \mathfrak{A}_n$$

if $\mathfrak{A} \models \exists x\varphi(x, \bar{b})$ *then*

$$\exists a \in \mathfrak{A}_N\, \mathfrak{A} \models \varphi(a, \bar{b}).$$

Proof. Let N_1, Γ be from Proposition 1 for $\varphi(x, \bar{y})$ and n. Define: $\varphi^*(y) \in \Gamma$ is a testing formula for $a \in \mathfrak{A}-\mathfrak{A}_{N_1}$ if

$$\forall\bar{b} \in \mathfrak{A}_n(\mathfrak{A} \models \varphi(a, \bar{b}) \leftrightarrow \mathfrak{A}_{N_1} \models \varphi^*(\bar{b})).$$

Choose $N\geqslant N_1$, $N<\omega$ so that $\forall a \in \mathfrak{A}-\mathfrak{A}_{N_1}$ if $\varphi^*(\bar{y}) \in \Gamma$ is a testing formula for a then there exists $a' \in \mathfrak{A}_N$ so that $\varphi^*(\bar{y})$ is a testing formula for a'. This N works because

$$\mathfrak{A} \models \varphi(a, b) \leftrightarrow \mathfrak{A}_{N_1} \models \varphi^*(b) \leftrightarrow \mathfrak{A} \models \varphi(a', \bar{b})$$

some $a' \in \mathfrak{A}_N$ with same testing formula $\varphi^*(\bar{y})$ as a. ■

PROPOSITION 3. *If $\mathfrak{A}$ is connected with a distinguished constant and $\mathfrak{A} \equiv \mathfrak{B}$ then $\bigcup\{\mathfrak{B}_n\colon n<\omega\}$ is an elementary substructure of $\mathfrak{B}$.*

Proof. If $b \in \mathfrak{B}_n$ and $\varphi(x, \bar{y})$ are given then taking $N<\omega$ from Proposition 2,

$$\mathfrak{A} \models \text{“}\forall\bar{y} \in \mathfrak{A}_n(\exists x\varphi(x, \bar{y}) \leftrightarrow \exists x \in \mathfrak{A}_N\, \varphi(x, \bar{y}))\text{”}.$$

So if $\mathfrak{B} \models \exists x\varphi(x, \bar{b})$ then $\exists b \in \mathfrak{B}_N\, \mathfrak{B} \models \varphi(b, \bar{b})$. By Tarski's criterion we are done. ■

The proof of Lemma 1. HC is the set of hereditarily countable sets. Let M be an elementary extension of (HC, $\in$) such that ω^M is nonstandard. We assume $\mathfrak{A}, \mathfrak{B} \in$ HC. Let $\mathfrak{A}^*$ be the structure determined by M corresponding to $\mathfrak{A}$ and $\mathfrak{A}^*_{st} = \bigcup\{\mathfrak{A}^*_n\colon n<\omega\}$. Let $n^* \in \omega^M-\omega$ and $M \models$ "s is a strategy for player II in the Ehrenfeucht game of length n^* played between $\mathfrak{A}^*_{n^*}$ and $\mathfrak{B}^*_{n^*}$". Since n^* is nonstandard the strategy s gives a back and forth property to show $\mathfrak{A}^*_{st} \equiv \mathfrak{B}^*_{st}$ (if player I plays $a \in \mathfrak{A}^*_{st}$ then s must respond with $b \in \mathfrak{B}^*_{st}$). By Proposition 3 $\mathfrak{A}^*_{st} \prec \mathfrak{A}^*$ and $\mathfrak{B}^*_{st} \prec \mathfrak{B}^*$ and also $\mathfrak{A} \prec \mathfrak{A}^*$ and $\mathfrak{B} \prec \mathfrak{B}^*$ so $\mathfrak{A} \equiv \mathfrak{B}$. ■

LEMMA 2. *If for every component $\mathscr{C}$ of $\mathfrak{A}$ $\omega(\mathscr{C}) \leqslant \aleph_0$ or $\omega(\mathscr{C}) = 2^{\aleph_0}$, then $\omega(\mathfrak{A}) \leqslant \aleph_0$ or $\omega(\mathfrak{A}) = 2^{\aleph_0}$.*

Proof. Note that from Lemma 1 if $\mathfrak{B} \prec \mathfrak{A}$ then the components of $\mathfrak{B}$ are elementary substructures of the corresponding components of $\mathfrak{A}$. If $\omega(\mathscr{C}) = 2^{\aleph_0}$ for some $\mathscr{C}$ which is a component of $\mathfrak{A}$, then using Ehrenfeucht games we see that $\omega(\mathfrak{A}) = 2^{\aleph_0}$. Otherwise let $\{\mathscr{C}_n\colon n<\omega\}$ be pairwise nonisomorphic so that for any $\mathscr{C}$ an elementary substructure of some component of $\mathfrak{A}$ there is $n<\omega$ such that $\mathscr{C}\simeq\mathscr{C}_n$. For $k\colon \omega\to\omega+1$ let $\mathfrak{A}_k$ be a structure (obtained continuously from k) with exactly $k(n)$ copies of $\mathscr{C}_n$ for each n and universe a subset of ω. Let $X = \{k\in(\omega+1)^\omega\colon \mathfrak{A}_k$ can be elementarily embedded into $\mathfrak{A}\}$, then X is a Σ_1^1 set and $|X| = \omega(\mathfrak{A})$, so by a classical theorem of descriptive set theory [4] $\omega(\mathfrak{A})\leqslant\aleph_0$ or $\omega(\mathfrak{A}) = 2^{\aleph_0}$. ■

Note that if for all $a\in A$ $\omega((\mathfrak{A}, a))\leqslant\aleph_0$, then $\omega(\mathfrak{A})\leqslant\aleph_0$; and also if there exist $a\in A$ with $\omega((\mathfrak{A}, a)) = 2^{\aleph_0}$ then $\omega(\mathfrak{A}) = 2^{\aleph_0}$. If $\mathfrak{A}$ is connected and $Y\subseteq A$ is finite and includes all of $\mathfrak{A}$'s loops then define $\mathfrak{A}\{y\}$ for $y\in Y$ as follows. $\mathfrak{A}\{y\} = \{a\in A\colon a$ is connected to y by a path which only intersects Y at $y\}$. By Lemma 1 note that for $\mathfrak{B}\subseteq\mathfrak{A}$ $(\langle\mathfrak{B}, y\rangle_{y\in Y}\prec\langle\mathfrak{A}, y\rangle_{y\in Y})$ iff $(\mathfrak{B}\{y\}\prec\mathfrak{A}\{y\}$ for all $y\in Y)$. Hence it is enough to count the number of elementary substructures of a tree. Define $\mathfrak{A}$ is a tree iff $\mathfrak{A}$ is countable, connected, has no loops, and has a distinguished constant O. For the rest of the proof of Theorem A we will assume all structures are trees.

DEFINITIONS. (1) a is *below* b iff b lies on the unique shortest path connecting a to $\underline{O}$.

(2) $\mathfrak{A}(a)$ is the *tree* with universe $\{b\in A\colon b$ is below $a\}$ and distinguished constant a.

(3) $P(\mathfrak{A}) = \{a\in A\colon \delta(a, \underline{O}) = 1\}$ and for $a\in A$ $P(a) = P(\mathfrak{A}(a))$.

(4) For $X\subseteq P(\mathfrak{A})$ $\mathfrak{A}[X]$ is the *tree* with universe the elements of A below things in X and with distinguished constant O.

(5) Given $x_n\in P(\mathfrak{A})$ for $n<\omega$ $[x_n\to y$ iff for all $n\neq m$ $x_n\neq x_m$ and the type of x_n in $\mathfrak{A}$ converges to the type of y in $\mathfrak{A}$, i.e. for all $\Psi(v)$ first order there is $N<\omega$ such that for all $n\geqslant N$ $(\mathfrak{A}\models\Psi(x_n)$ iff $\mathfrak{A}\models\Psi(y))]$.

LEMMA 3. *If $X\cup Y = P(\mathfrak{A})$, X and Y are disjoint, and for every $y\in Y$ $\exists x_n\in X$ for $n<\omega$ such that $x_n\to y$, then $\mathfrak{A}[X]$ is an elementary substructure of $\mathfrak{A}$.*

Proof. It is easy to find $X_y = \{x_n^y\colon n<\omega\}$ included in X for $y\in Y$, so that $X_y\cap X_{y'} = \varnothing$ *for* $y\neq y'$ and for each $y\in Y$ $x_n^y\to y$.

CLAIM. *For every $n_0<\omega$ and $y\in Y$ $\mathfrak{A}_{n_0}[X_y]$ is an elementary substructure of $\mathfrak{A}_{n_0}[X_y\cup\{y\}]$.*

Proof. Let $\mathfrak{B} = \mathfrak{A}_{n_0}$ and $X_y = \{x_n\colon n<\omega\}$. Clearly $x_n\to y$ in the sense of $\mathfrak{B}$, hence we know from the basic lemma on Ehrenfeucht games ([2]) that

$$\forall n<\omega\,\exists N<\omega\,\forall m>N\,\mathfrak{B}(x_m)\equiv_n\mathfrak{B}(y)\,.$$

Given $\bar{a}\in\mathfrak{B}[X_y]$ and $n_1<\omega$, choose N sufficiently large so that $\bar{a}\in\mathfrak{B}[\{x_n\colon n<N\}]$ and for $m>N$ $\mathfrak{B}(x_m)\equiv_{n_1}\mathfrak{B}(y)$. Now patch together appropriate strategies for Player II by letting $\mathfrak{B}(x_i)$ correspond to $\mathfrak{B}(x_i)$ for $i\leqslant N$ (and play the identity), letting $\mathfrak{B}(x_{N+1})$ correspond to $\mathfrak{B}(y)$, and letting $\mathfrak{B}(x_{N+1})$ correspond to $\mathfrak{B}(x_{N+i-1})$ for $i>1$. ■

From Lemma 1 and the claim, $\mathfrak{A}[X_y]$ is an elementary substructure of $\mathfrak{A}[X_y\cup\{y\}]$ for each $y\in Y$, hence by an easy Ehrenfeucht game argument $\mathfrak{A}[X]\prec\mathfrak{A}$. ■

DEFINITION. $\mathfrak{A}$ is *simple* iff for every $a\in A$ only finitely many nonprincipal types in $\mathrm{Th}(\mathfrak{A}(a))$ are realized in $P(a)$.

Note. By using Lemma 3 if $\mathfrak{A}$ is not simple then $\omega(\mathfrak{A}) = 2^{\aleph_0}$.

DEFINITION. Given $(\mathfrak{B}_a\colon a\in A)$ such that $\mathfrak{B}_a\subseteq\mathfrak{A}(a)$ for each a the fusion of $(\mathfrak{B}_a\colon a\in A)$ is the *tree* $\mathfrak{B}$ with $\underline{O}^{\mathfrak{B}} = \underline{O}^{\mathfrak{A}}$ and universe $\{b\colon$ for all a between $\underline{O}$ and b, $b\in|\mathfrak{B}_a|\}$.

LEMMA 4. *Given $(\mathfrak{B}_a\colon a\in A)$ with $\mathfrak{B}_a\prec\mathfrak{A}(a)$ for all $a\in A$, then the fusion $\mathfrak{B}$ is an elementary substructure of $\mathfrak{A}$.*

Proof. By Lemma 1 we may assume $\mathfrak{A} = \mathfrak{A}_n$ for some $n<\omega$. Now prove it by induction on n. Thus $\mathfrak{B}(b)\prec\mathfrak{A}(b)$ for all $b\in P(\mathfrak{A})$, hence $\mathfrak{B}(b)\prec\mathfrak{B}_{\underline{O}}(b)$ $\forall b\in P(\mathfrak{B}_{\underline{O}})$ and by an easy Ehrenfeucht game argument $\mathfrak{B}\prec\mathfrak{B}_{\underline{O}}\prec\mathfrak{A}$. ■

DEFINITION. If $\mathfrak{A}$ is simple let $\mathfrak{B}_a^p = \mathfrak{A}(a)[\{x\colon \mathrm{tp}(x, \mathfrak{A}(a))$ is principal$\}]$ for each $a\in A$, and $\mathfrak{A}^p$ be the fusion of $\langle\mathfrak{B}_a^p\colon a\in A\rangle$. By Lemma 3 $\mathfrak{B}_a^p\prec\mathfrak{A}(a)$ and by Lemma 4 $\mathfrak{A}^p$ is an elementary substructure of $\mathfrak{A}$.

LEMMA 5. *If $\mathfrak{A}^p = \mathfrak{A}$ then $\omega(\mathfrak{A}) = 1$.*

The proof is straightforward and left to the reader. ■

DEFINITIONS. For a fixed tree $\mathfrak{A} = \langle A, R\rangle$ let

(1) $N(a) = \{x\in P(\mathfrak{A}(a))\colon$ the type of x in $\mathfrak{A}(a)$ is nonprincipal$\}$,

(2) $L = \{a\in A\colon N(a)\neq\varnothing\}$,

(3) $T = \{b\in A\colon \exists a\in L$ b lies on the unique shortest path connecting a to $\underline{O}\}$.

LEMMA 6. *If $L = \{a_n\colon n<\omega\}$ and for every n $(N(a_n) = \{b_n\}$ and $a_{n+1}\in\mathfrak{A}(b_n))$, then $\omega(\mathfrak{A})\leqslant\aleph_0$.*

Proof. Let for each $n<\omega$ $\mathfrak{B}_n = \mathfrak{A}-\mathfrak{A}(b_n)$, then these are all the nonisomorphic elementary substructures of $\mathfrak{A}$. ■

DEFINITIONS. (1) $[T]$ is the set of infinite branches of T.

(2) $a\in A$ isolates $f\in[T]$ iff $\mathfrak{A}(a)$ is as in the hypothesis of Lemma 6 with $a\in f$.

LEMMA 7. *If $\mathfrak{A}$ is simple and $\exists f\in[T]$ such that no $a\in A$ isolates f then $\omega(\mathfrak{A}) = 2^{\aleph_0}$.*

Proof. Choose $a_n\in L$ and $b_n\in N(a_n)$ for $n<\omega$ as follows: Having chosen them for $m<n$, let c be any element of f lower than any of the a_m and b_m for $m<n$. Since c does not isolate f there is $a_n\in\mathfrak{A}(c)\cap L$ and $b_n\in N(a_n)$ such that $b_n\notin f$. Let $B = \{c\colon c$ is between some b_n and $\underline{O}\}$. For every $a\in\mathfrak{A}$ let $\mathfrak{B}_a = \mathfrak{A}(a)[X_a]$ where $X_a = P(a)\cap(\{x\colon$ the type of x in $\mathfrak{A}(a)$ is principal$\}\cup B)$. If $\mathscr{C}$ is the fusion of the $\mathfrak{B}_a$'s then $\mathscr{C}\prec\mathfrak{A}$. For any $n<\omega$ note that there are at most two $x\in C$ such that $\delta(x, \underline{O}) = \delta(a_n, \underline{O})$ and $N(x)^{\mathscr{C}}\neq\varnothing$. For any $X\subseteq\omega$ let $\mathscr{C}_X\prec\mathscr{C}$ be gotten by fusion so that for all $n<\omega[b_n\in|\mathscr{C}_X|$ iff $n\in X]$. Thus if $X\neq X'$ then $\mathscr{C}_X\not\cong\mathscr{C}_{X'}$. ■

LEMMA 8. *If for every $a\in P(\mathfrak{A})$ $\omega(\mathfrak{A}(a))\leqslant\aleph_0$ or $\omega(\mathfrak{A}(a)) = 2^{\aleph_0}$, then $\omega(\mathfrak{A})\leqslant\aleph_0$ or $\omega(\mathfrak{A}) = 2^{\aleph_0}$.*

The proof of this is similar to the proof of Lemma 2. ■

LEMMA 9. *If $\mathfrak{A}$ is a tree then $\omega(\mathfrak{A})\leqslant\aleph_0$ or $\omega(\mathfrak{A}) = 2^{\aleph_0}$.*

Proof. If $\mathfrak{A}$ is not simple then $\omega(\mathfrak{A}) = 2^{\aleph_0}$ by using Lemma 3. Define $D(T) = \{x \in T: x$ does not isolate any $f \in [T]\}$. By Lemma 7 if $D(T)$ is not well founded ($[D(T)] \neq \emptyset$) then $\omega(\mathfrak{A}) = 2^{\aleph_0}$. If $D(T) = \emptyset$ then by Lemmas 5 or 6 $\omega(\mathfrak{A}) \leqslant \aleph_0$. Hence we may assume $D(T)$ is well-founded and then the lemma is proved by induction on the rank of $D(T)$ by using Lemma 8. ■

The proof of Theorem B. If a countable theory T fails to have an ω-saturated countable model then $\omega(T) = 2^{\aleph_0}$, hence by Theorem A we have only to show that if $\omega(T) < \aleph_0$ then $\omega(T) = 1$. This follows immediately from Lachlan's Theorem [5], since T is superstable. We need to show that for any $\mathfrak{A} \models T$ with $|A| \geqslant 2^{\aleph_0}$ that $\mathrm{Th}((\mathfrak{A}, a\colon a \in A))$ has at most $|A|$ types. So let $\mathfrak{B}$ be any elementary extension of $\mathfrak{A}$ and $b \in B - A$.

Case 1. For all $a \in A$ $\delta(a, b) = \infty$ (the δ which is defined in $\mathfrak{B}$).

In this case for any $c \in B$, $(\mathfrak{B}, b, a\colon a \in A) \equiv (\mathfrak{B}, c, a\colon a \in A)$ iff $(\mathfrak{B}, b) \equiv (\mathfrak{B}, c)$ and for all $a \in A$ $\delta(a, c) = \infty$. To prove this note that by Lemma 1 the component of $\mathfrak{B}$ containing c is elementarily equivalent to the component of $\mathfrak{B}$ containing b, so patch together Ehrenfeucht game strategies.

Case 2. There is $n < \omega$ and $a \in A$ such that $\delta(a, b) = n$.

Choose $Y \subseteq A$ finite, connected in itself, and including all the loops of the component of $\mathfrak{B}$ containing b. Let $a_0 \in Y$ so that $\forall a \in A\, \delta(a, b) \geqslant \delta(a_0, b)$. Let $A^1 = A \cap |\mathfrak{B}\{a_0\}|$, then by Lemma 1 and Ehrenfeucht games for any $c \in B$, $(\mathfrak{B}, b, a\colon a \in A) \equiv (\mathfrak{B}, c, a\colon a \in A)$ iff $(\mathfrak{B}\{a_0\}, b, a\colon a \in A') \equiv (\mathfrak{B}\{a_0\}, c, a\colon a \in A')$. Now suppose $\mathfrak{A}$ and $\mathfrak{B}$ are trees with distinguished constant $\underline{O}$, and $b \in B - A$. Choose $a_0 \in A$ so that $b \in \mathfrak{B}(a_0)$ and if x is the unique element of $P(\mathfrak{B}(a_0))$ such that $b \in \mathfrak{B}(x)$, then $x \notin A$. For any $c \in |\mathfrak{B}(a_0)| - \{a_0\}$ if y is the unique element of $P(\mathfrak{B}(a_0))$ such that $c \in \mathfrak{B}(y)$, then $(\mathfrak{B}, b, a\colon a \in A) \equiv (\mathfrak{B}, c, a\colon a \in A)$ iff $(\mathfrak{B}, b, a_0) \equiv (\mathfrak{B}, c, a_0)$ and $y \notin A$. To see this note that by the right hand side and Lemma 1 $(\mathfrak{B}(x), b) \equiv (\mathfrak{B}(y), c)$, so it is easy to patch together Ehrenfeucht game strategies.

Thus we see that there are at most $2^{\aleph_0} \cdot |A| = |A|$ 1-types in $\mathrm{Th}((\mathfrak{B}, a\colon a \in A))$. Similar arguments show that for any $n < \omega$ there are at most $|A|$ n-types, so T is superstable.

The proof of Theorem C. For any $(L, <)$ a linear order define the following unary operation (U_L, F_L), where $U_L = \{(a_0, \ldots, a_{n-1})\colon n < \omega,\ a_0 > a_1 > a_2 > \ldots > a_{n-1}$ and for $i < n$ $a_i \in L\}$, $F_L(\langle\ \rangle) = \langle\ \rangle$ ($\langle\ \rangle$ is the empty sequence), and

$$F_L(\langle a_0, \ldots, a_n\rangle) = \langle a_0, \ldots, a_{n-1}\rangle .$$

CLAIM. *If $L = L_1 + L_2$ and $\bar{L} = \bar{L}_1 + \bar{L}_2$ are countable linear orders, L_1 and $\bar{L}_1$ are isomorphic well orders, and either L_2 and $\bar{L}_2$ are both empty or they are both nonempty and have no least element then (U_L, F_L) is isomorphic to $(U_{\bar{L}}, F_{\bar{L}})$. Thus $\theta = \{(U, F)\colon$ there is a countable linear order $(L, <)$ such that $(U, F) \simeq (U_L, F_L)\}$ is $\mathrm{PC}(L_{\omega_1\omega})$ and $\omega(\theta) = \aleph_1$.* ■

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