Universal Functions

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Abstract[2]

A function of two variables $F(x,y)$ is universal if for every function $G(x,y)$ there exists functions $h(x)$ and $k(y)$ such that

$$G(x,y) = F(h(x), k(y))$$

for all x, y. Sierpiński showed that assuming the Continuum Hypothesis there exists a Borel function $F(x,y)$ which is universal. Assuming Martin's Axiom there is a universal function of Baire class 2. A universal function cannot be of Baire class 1. Here we show that it is consistent that for each α with $2 \leq \alpha < \omega_1$ there is a universal function of class α but none of class $\beta < \alpha$. We show that it is consistent with ZFC that there is no universal function (Borel or not) on the reals, and we show that it is consistent that there is a universal function but no Borel universal function. We also prove some results concerning higher-arity

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universal functions. For example, the existence of an F such that for every G there are h_1, h_2, h_3 such that for all x, y, z

$$G(x, y, z) = F(h_1(x), h_2(y), h_3(z))$$

is equivalent to the existence of a binary universal F, however the existence of an F such that for every G there are h_1, h_2, h_3 such that for all x, y, z

$$G(x, y, z) = F(h_1(x, y), h_2(x, z), h_3(y, z))$$

follows from a binary universal F but is strictly weaker.

1. Introduction

A function $F : X \times X \to X$ is said to be *universal* if for any

$$G : X \times X \to X$$

there is $g : X \to X$ such that for all $x, y \in X$

$$G(x, y) = F(g(x), g(y)).$$

In the Scottish book (problem 132, see Mauldin [15]) Sierpiński asked if there always is a Borel function which is universal, when X is the real line. He had shown that there is a Borel universal function assuming the Continuum Hypothesis (Sierpiński [29]). This notion of universal function is also studied in Rado [21] (see Theorem 6 there).

Remark 1.1. *Without loss of generality we may use different functions on the x and y coordinates, i.e., $G(x, y) = F(g_0(x), g_1(y))$ in the definition of universal function F. To see this suppose we are given F^* such that for every G we may find g_0, g_1 with $G(x, y) = F^*(g_0(x), g_1(y))$ for all x, y. Then we can construct a universal F which uses only a single g. Take a bijection, i.e., a pairing function, between $X \times X$ and X, which write as: $(x_0, x_1) \mapsto \langle x_0, x_1\rangle$. Define*

$$F(\langle x_0, x_1\rangle, \langle y_0, y_1\rangle) = F^*(x_0, y_1).$$

Given any g_0, g_1 define $g(u) = \langle g_0(u), g_1(u)\rangle$ and note that

$$F(g(x), g(y)) = F^*(g_0(x), g_1(y))$$

for every x, y.

In the case $X = 2^\omega$ there is a pairing function which is a homeomorphism and hence the Borel complexity of F and F^* are the same. For abstract universal F a pairing function exists for any infinite X. For finite sets X, universal functions exist if and only if $|X| \leq 1$.

Remark 1.2. *The definition of universal function is not changed by requiring the function g to be injective, as, given a function $\pi\colon X \to X$ for which $|\pi^{-1}(x)| = |X|$ for all $x \in X$, we can replace a given $F(x,y)$ in the original sense with $F(\pi(x), \pi(y))$.*

The notion of universal function naturally generalizes to functions of the form $f\colon X \times Y \to Z$, as follows.

Definition 1.3. *Given sets X, Y and Z, a function $F : X \times Y \to Z$ is* universal *if for each function*

$$G : X \times Y \to Z$$

there exist functions $h : X \to X$ and $k\colon Y \to Y$ such that for all $(x,y) \in X \times Y$,

$$G(x,y) = F(h(x), k(y)).$$

We record a few simple observations about functions of this type.

Remark 1.4. *If $f\colon X \times Y \to Z$ is universal, $Z' \subseteq Z$ and $z_0 \in Z'$, then the function $f'\colon X \times Y \to Z'$ defined by setting*

$$f'(x,y) = \begin{cases} f(x,y) & \text{if } f(x,y) \in Z' \\ z_0 & \text{otherwise} \end{cases}$$

is also universal.

The following observation shows that the existence of a universal function from $2^\omega \times 2^\omega$ to 2^ω is equivalent to the existence of a universal function from $2^\omega \times 2^\omega \to 2$, even when one asks for a universal function in a particular complexity class. Similarly, for all infinite sets X and Y, and any $n \in \omega$, the existence of a universal function from $X \times Y$ to Z implies the existence of a universal function from $X \times Y$ to Z^n.

Proposition 1.5. *If κ is a cardinal, $f\colon X \times Y \to Z$ is a universal function, $|X^\kappa| = |X|$ and $|Y^\kappa| = |Y|$, then there is a universal function $F\colon X \times Y \to Z^\kappa$.*

Proof
Fix bijections $\pi\colon X^\kappa \to X$ and $\nu\colon Y^\kappa \to Y$. For each $(x,y) \in X \times Y$, let $F(x,y)$ be

$$\langle f(\pi^{-1}(x)(\alpha), \nu^{-1}(y)(\alpha)) : \alpha < \kappa \rangle.$$

To see that F is universal, fix $G\colon X \times Y \to Z^\kappa$. For each $\alpha < \kappa$, define $g_\alpha\colon X \times Y \to Z$ by setting $g_\alpha(x,y) = G(x,y)(\alpha)$. By the universality of f, there exist functions $h_\alpha\colon X \to X$ and $k_\alpha\colon Y \to Y$ $(\alpha < \kappa)$ such that for all $\alpha < \kappa$ and all $(x, y) \in X \times Y$, $g_\alpha(x, y) = f(h_\alpha(x), k_\alpha(y))$. Define $H\colon X \to X$ and $K\colon Y \to Y$ by setting

$$H(x) = \pi(\langle h_\alpha(x) : \alpha < \kappa\rangle)$$

and $K(y) = \nu(\langle k_\alpha(y) : \alpha < \kappa\rangle)$. Then for all $(x, y) \in X \times Y$, $F(H(x), K(y)) = \langle f(h_\alpha(x), k_\alpha(y)) : \alpha < \kappa\rangle = \langle g_\alpha(x, y) : \alpha < \kappa\rangle = G(x, y)$.
QED

In Section 2 we show that the existence of a Borel universal function is equivalent under a weak cardinality assumption to the statement that every subset of the plane is in the σ-algebra generated by the abstract rectangles. We also show that a universal function cannot be of Baire class 1.

In Section 3 we prove some results concerning Martin's axiom and universal functions. We show that although MA implies that there is a universal function of Baire class 2 it is consistent to have $\mathrm{MA}_{\aleph_1}$ hold but no analytic universal functions.

In Section 4 we consider universal functions of special kinds, for example, $F(x, y) = k(x + y)$. We also discuss special versions due to Todorcevic and Davies.

In Section 5 we consider abstract universal functions, i.e., those defined on a cardinal κ with no notion of definability, Borel or otherwise. We show that if $2^{<\kappa} = \kappa$, then they exist. We also show that it is consistent that none exist for $\kappa = 2^{\aleph_0}$, and we construct some weak abstract versions of universal functions from the assumption $\mathrm{MA}_{\aleph_1}$.

In Section 6 we take up the problem of universal functions of higher arity. We show that there is a natural hierarchy of such notions and we show that this hierarchy is strictly descending.

In Section 7 we compare the notion of universal function with the notional of universality from model theory.

1.1. **Cardinal characteristics.** The following definitions show up at various points in this paper. We let $\mathfrak{c}$ denote the cardinality of the continuum, i.e., $2^{\aleph_0}$. The cardinal $\mathfrak{p}$ is the psuedo-intersection number, the smallest cardinality of a collection of infinite subsets of ω having the finite intersection property (i.e., all finite subcollections have nonempty intersection) but no pseudo-intersection (i.e., no infinite subset of ω is contained mod-finite in each member of the collection). Equivalently, it is the smallest cardinal for which Martin's Axiom for σ-centered posets fails. This equivalence is due to Bell [1]; for the proof see also Weiss [33]. The tower number $\mathfrak{t}$ is smallest cardinality of a collection of infinite subsets of ω linearly-ordered by mod-finite containment but having no pseudo-intersection. Evidently, $\mathfrak{p} \leq \mathfrak{t}$, but a recent result of Malliaris and Shelah [14] shows that $\mathfrak{p} = \mathfrak{t}$ (in light of this fact, the hypotheses of Propositions 6.16 and 7.14 are each equivalent to $\mathfrak{p} = \mathfrak{c}$). The cardinal $\mathfrak{b}$ is the smallest cardinality of a set $X \subseteq \omega^\omega$ such that for every $f\colon \omega \to \omega$ there exist a $g \in X$ with $\{n \in \omega \mid g(n) \geq f(n)\}$ infinite. See pages 426-427 of [2] for a proof that $\mathfrak{t} \leq \mathfrak{b}$.

The cardinal $\mathfrak{q}$ is the smallest cardinality of a set $X \subseteq 2^\omega$ which is not a Q-set, i.e., for which there exists a set $Y \subseteq X$ such that $Z \cap X \neq Y$, for every G_δ set $Z \subseteq 2^\omega$. The inequality $\mathfrak{p} \leq \mathfrak{q}$ can be proved in ZFC. This is due to Silver; see Section 5 of [20]. The cardinal characteristic $\mathfrak{ap}$ is defined to be the least cardinal κ such that there exist an almost disjoint family $\{x_\alpha : \alpha < \kappa\}$ (i.e., each x_α is an infinite subset of ω, and for each distinct pair $\alpha, \beta < \kappa$, $x_\alpha \cap x_\beta$ is finite) and a set $A \subseteq \kappa$ such that for no $y \subseteq \omega$ does it hold for all $\alpha < \kappa$ that $\alpha \in A$ if and only if $y \cap x_\alpha$ is infinite. Standard arguments show that $\mathfrak{p} \leq \mathfrak{ap} \leq \mathfrak{q}$.

For any cardinal κ, MA_κ implies that $\mathfrak{p} \geq \kappa$, which means that Martin's Axiom implies that $\mathfrak{p} = \mathfrak{b} = \mathfrak{ap} = \mathfrak{q} = \mathfrak{c}$. See [2] for more on cardinal characteristics of the continuum.

2. Borel universal functions

Definition 2.1. *We let $\mathcal{R}$ denote the family of abstract rectangles,*

$$\mathcal{R} = \{A \times B \ : \ A, B \subseteq 2^\omega\}.$$

Definition 2.2. *For $\alpha < \omega_1$, $\mathbf{\Sigma}^0_\alpha(\mathcal{R})$ and $\mathbf{\Pi}^0_\alpha(\mathcal{R})$ are inductively defined by:*

- $\mathbf{\Sigma}^0_0(\mathcal{R}) = \mathbf{\Pi}^0_0(\mathcal{R}) =$ *the set of finite boolean combinations of sets from $\mathcal{R}$,*
- $\mathbf{\Sigma}^0_\alpha(\mathcal{R})$ *is the set of countable unions of sets from*
$$\mathbf{\Pi}^0_{<\alpha}(\mathcal{R}) = \bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta(\mathcal{R}),$$
and
- $\mathbf{\Pi}^0_\alpha(\mathcal{R})$ *is the set of countable intersections of sets from $\mathbf{\Sigma}^0_{<\alpha}(\mathcal{R})$.*

Definition 2.3. *A Borel function $F : 2^\omega \times 2^\omega \to 2^\omega$ is at the α-level if for any $n \in \omega$ the set $\{(u, v) \ : \ F(u, v)(n) = 1\}$ is $\mathbf{\Sigma}^0_\alpha$.*

We write "a function of level α" for "a function which is at the α-th level". A Borel function at level α is in Baire class α, but the converse does not hold. In the context of 2^ω, a function is of Baire class α if the preimage of every clopen set is $\mathbf{\Delta}_{\alpha+1}$. For more on the classical theory of Baire class α, see Kechris [8], p. 190.

Proposition 2.4. *A universal function cannot be of Baire class 1.*

Proof
Suppose toward a contradiction that $F\colon 2^\omega \times 2^\omega \to 2^\omega$ is a universal function of Baire class 1. Let $\{h_\xi\}_{\xi\in\mathfrak{c}}$ enumerate all functions with domain a countable subset of 2^ω and range dense in itself. Let $\{r_\xi\}_{\xi\in\mathfrak{c}}$ enumerate all of 2^ω. For each $\xi \in \mathfrak{c}$, partition the domain of h_ξ into A_ξ and B_ξ such that $\overline{h_\xi[A_\xi]} = \overline{h_\xi[B_\xi]}$. Let $G : (2^\omega)^2 \to 2^\omega$ be any function such that for each $\xi \in \mathfrak{c}$ and $r \in 2^\omega$, $G(r_\xi, r) = 1$ if $r \in A_\xi$ and $G(r_\xi, r) = 0$ if $r \in B_\xi$.

Now suppose that $h : 2^\omega \to 2^\omega$ witnesses the universality of F with respect to the function G. The range of h must be uncountable; otherwise there would be a countable collection $\{\{C_i, D_i\} : i < \omega\}$ of partitions of 2^ω such that for each $\xi \in \mathfrak{c}$ there exists an $i \in \omega$ such that $\overline{h_\xi[C_i \cap \mathrm{dom}(h_\xi)]} = \overline{h_\xi[D_i \cap \mathrm{dom}(h_\xi)]}$, and it not hard to build a counterexample to this. Hence, there is ξ such that $h_\xi \subseteq h$, and for all $r \in A_\xi \cup B_\xi$, $G(r_\xi, r) = F(h(r_\xi), h_\xi(r))$.

If f is the function defined by setting $f(y) = F(h(r_\xi), y)$, then f must be Baire class 1 and, in particular, letting $C = \overline{h_\xi[A_\xi]}$ (which is equal to $\overline{h_\xi[B_\xi]}$), it follows that $f \restriction C$ is Baire class 1. However,

$$f(h_\xi(r)) = F(h(r_\xi), h_\xi(r)) = G(r_\xi, r) = 1 \text{ for } r \in A_\xi.$$

Similarly $f(h_\xi(r)) = 0$ for $r \in B_\xi$. This is impossible for any Baire class 1 function on the perfect set C.
QED

Theorem 2.5. *If $2^{<\mathfrak{c}} = \mathfrak{c}$, then the following are equivalent.*

1. *There is a Borel function $F : 2^\omega \times 2^\omega \to 2^\omega$ which is universal.*
2. *Every subset of the plane $2^\omega \times 2^\omega$ is in the σ-algebra generated by the abstract rectangles, $\mathcal{R}$.*

Furthermore, for any ordinal α, $\mathcal{P}(2^\omega \times 2^\omega) = \mathbf{\Sigma}^0_\alpha(\mathcal{R})$ if and only if there is a universal function from $2^\omega \times 2^\omega$ to 2^ω at the α-level.

Proof
(1) $\to$ (2).
Suppose that there is a Borel universal $F : 2^\omega \times 2^\omega \to 2$. Let $A \subseteq 2^\omega \times 2^\omega$ be arbitrary and suppose that $g : 2^\omega \to 2^\omega$ has the property that

$$\forall x, y \quad (x, y) \in A \Leftrightarrow F(g(x), g(y)) = 1.$$

Let B be the Borel set $F^{-1}[\{1\}]$. Then for all $(x, y) \in 2^\omega \times 2^\omega$, $(x, y) \in A$ if and only if $(g(x), g(y)) \in B$.

The set B is generated by countable unions and intersections from sets of the form $C \times D$, for C, D clopen subsets 2^ω. Define h on $2^\omega \times 2^\omega$ by setting $h(x,y) = (g(x), g(y))$, and note that

$$h^{-1}[C \times D] = g^{-1}[C] \times g^{-1}[D]$$

for all sets $C, D \subseteq 2^\omega$. Since preimages pass over countable unions and intersections, for each $\alpha < \omega_1$, the h-preimage of each $\mathbf{\Sigma}^0_\alpha$ set is in $\mathbf{\Sigma}^0_\alpha(\mathcal{R})$. In particular, if $\alpha < \omega_1$ is such that B is $\mathbf{\Sigma}^0_\alpha$, then $A = h^{-1}[B]$ is $\mathbf{\Sigma}^0_\alpha(\mathcal{R})$.

(2) $\rightarrow$ (1).
We show first that there exists an $X \subseteq 2^\omega$ of cardinality $\mathfrak{c}$ which has the property that every $Y \subseteq X$ of cardinality strictly smaller than $\mathfrak{c}$ is Borel relative to X, i.e., is the intersection of a Borel set with X. The following argument is modeled after the one in Bing, Bledsoe, and Mauldin [3]. Let $A \subseteq \mathfrak{c} \times \mathfrak{c}$ be such that for every $B \in [\mathfrak{c}]^{<\mathfrak{c}}$ there exists a $\delta < \mathfrak{c}$ such that

$$B = A_\delta =^{def} \{\gamma < \mathfrak{c} \ : \ (\delta, \gamma) \in A\}.$$

This is possible, as $2^{<\mathfrak{c}} = \mathfrak{c}$. Fix $\alpha < \omega_1$ such that A is in $\mathbf{\Sigma}^0_\alpha(\mathcal{R})$, and fix a sequence $\langle B_n : n \in \omega \rangle$ of subsets of $\mathfrak{c}$ such that A is generated in α many steps from the sets

$$\{B_n \times B_m \ : \ n, m < \omega\}.$$

Let $f : \mathfrak{c} \rightarrow 2^\omega$ be the Marczewski characteristic function for the sequence $\langle B_n : n < \omega \rangle$, i.e.,

$$f(\delta)(n) = \begin{cases} 0 & \text{if } \delta \notin B_n \\ 1 & \text{if } \delta \in B_n \end{cases}$$

Define the function $f^2 \colon \mathfrak{c} \times \mathfrak{c} \rightarrow 2^\omega \times 2^\omega$ by setting $f^2(\alpha, \beta) = (f(\alpha), f(\beta))$. Each set of the form $B_n \times B_m$ is the f^2-preimage of the clopen set

$$\{x \in 2^\omega \mid n \in x\} \times \{x \in 2^\omega \mid m \in x\}.$$

Again using the fact that preimages pass over countable unions and intersections, we can find a $\mathbf{\Sigma}^0_\alpha$ set $Z \subseteq 2^\omega \times 2^\omega$ whose f^2-preimage is A.

Let $X = f[\mathfrak{c}]$. Let us check that X has the required property. Let Y be a subset of X of cardinality less than $\mathfrak{c}$, and let B be a subset of $\mathfrak{c}$ of cardinality less than $\mathfrak{c}$ such that $Y = f[B]$. Then Y will be a section of Z, intersected with X, i.e.,

$$Y = f[A_\delta] = \{(x, y) \in Z \mid x = f(\delta), y \in X\},$$

where $\delta < \mathfrak{c}$ is such that $B = A_\delta$. It follows then that Y is $\mathbf{\Sigma}^0_\alpha$ relative to X.

Now let $U \subseteq 2^\omega \times 2^\omega$ be a universal $\mathbf{\Sigma}^0_\alpha$ set. Define $G : 2^\omega \times 2^\omega \to 2^\omega$ by setting $G(x, y)(n) = 1 \Leftrightarrow (x_n, y) \in U$, where $x \overset{\Phi}{\mapsto} \langle x_n : n < \omega\rangle \in (2^\omega)^\omega$ is a homeomorphism. Let $k \colon \mathfrak{c} \to X$ be a bijection.

Let $f_1 : \mathfrak{c} \times \mathfrak{c} \to 2^\omega$ be an arbitrary function with the property that if $\gamma < \delta < \mathfrak{c}$, then $f_1(\delta, \gamma) = \vec{0}$ (the identically zero map). We claim that there exists a function $h_1 \colon \mathfrak{c} \to 2^\omega$ such that

$$f_1(\delta, \gamma) = G(h_1(\gamma), k(\delta)) \text{ for all } (\delta, \gamma) \in \mathfrak{c} \times \mathfrak{c}.$$

To see this, note that for each $n < \omega$ and each $\gamma < \mathfrak{c}$ the set

$$Y_n =^{def} \{k(\delta) : f_1(\delta, \gamma)(n) = 1\}$$

is a subset of X of cardinality less that $\mathfrak{c}$, so there exists a $y_n \in 2^\omega$ such that $Y_n = X \cap U_{y_n}$. Let $h_1(\gamma) = y$ be chosen such that the homeomorphism Φ sends y to the sequence $\langle y_n : n < \omega\rangle$.

By an analogous argument, if $f_2 : \mathfrak{c} \times \mathfrak{c} \to 2^\omega$ is an arbitrary map with the property that $\mathfrak{c} > \gamma > \delta$ implies $f_2(\delta, \gamma) = \vec{0}$, then there exists a functions $h_2 : \mathfrak{c} \to 2^\omega$ such that

$$f_2(\delta, \gamma) = G(h_2(\delta), k(\gamma)) \text{ for all } (\delta, \gamma) \in \mathfrak{c} \times \mathfrak{c}.$$

Now define $F : 2^\omega \times 2^\omega \to 2^\omega$ by letting $\langle x, y\rangle$ be a pairing function (a homeomorphism) from $2^\omega \times 2^\omega$ to 2^ω and setting

$$F(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle) = \max(G(x_2, x_1), G(y_1, y_2)),$$

where $\max : 2^\omega \times 2^\omega \to 2^\omega$ is the pointwise maximum, i.e., $\max(u, v) = w$, where $w(n)$ is the maximum of $u(n)$ and $v(n)$ for each $n < \omega$. Then $F(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle)(n) = 1$ if and only if $1 \in \{G(x_2, x_1)(n), G(y_1, y_2)(n)\}$.

We show that F is universal. Given an arbitrary $f : \mathfrak{c} \times \mathfrak{c} \to 2^\omega$ we can find f_1 and f_2 as above so that

$$f(\delta, \gamma) = \max(f_1(\delta, \gamma), f_2(\delta, \gamma))$$

for all $(\delta, \gamma) \in \mathfrak{c} \times \mathfrak{c}$. For each $\delta, \gamma < \mathfrak{c}$, set $l_1(\delta) = \langle k(\delta), h_2(\delta)\rangle$ and $l_2(\gamma) = \langle h_1(\gamma), k(\gamma)\rangle$. Then, for all $\delta, \gamma < \mathfrak{c}$, $f(\delta, \gamma) = F(l_1(\delta), l_2(\gamma))$.

Also, F is at the α-level, i.e., for any n the set

$$\{(u, v) \ : \ F(u, v)(n) = 1\}$$

is in $\mathbf{\Sigma}^0_\alpha$.
QED

Remark 2.6. *By Proposition 1.5, part (1) of Theorem 2.5 is equivalent to the alternate version where the range of F is 2 instead of 2^ω. This variation allows for an alternate, possibly simpler, proof of the reverse direction of Theorem 2.5.*

Corollary 2.7. *For each α with $2 \le \alpha < \omega_1$ there is a c.c.c. forcing extension in which there is a universal function of level α but none of level $\beta < \alpha$. There is a c.c.c. forcing extension in which there is a universal function but no Borel universal function.*

Proof
The first part follows from the corresponding results about the σ-algebra of abstract rectangles, see Miller [17], Theorems 37 and 52 ($\mathfrak{c}^{<\mathfrak{c}} = \mathfrak{c}$ in the models from these theorems). For the second, the existence of an abstract universal function follows from $\mathfrak{c}^{<\mathfrak{c}} = \mathfrak{c}$ by Theorem 5.1, and this holds in many models in which not every subset of the plane is in the σ-algebra generated by the abstract rectangles. For example, Kunen in his Ph.D. thesis [9] showed this is true after a finite support iteration of Cohen forcing of length ω_2 over a model of GCH. QED

Remark 2.8. *Theorem 4.8 and Proposition 6.16 each show that if $\mathfrak{p} = \mathfrak{c}$, then there is a universal function at level 2.*

Question 2.9. *Suppose that there is a universal function of Baire class α. Then is there a universal function of level α?*

The techniques of Miller [18] can be used to produce models with an analytic universal function (that is, a universal function which is analytic), but no Borel universal function.

3. Universal functions and Martin's Axiom

Proposition 6.16 below shows that if Martin's Axiom holds then there are universal functions on the reals of Baire class 2. Here we show that the axiom $\mathrm{MA}_{\aleph_1}$ (the restriction of MA to collections of $\aleph_1$ many dense sets) is not strong enough for this result.

The following lemma will be our tool for showing that a given function is not universal.

Lemma 3.1. *Let $F\colon 2^\omega \times 2^\omega \to 2^\omega$ be a function, and suppose that there exist $S_{y,z} \subseteq 2^\omega$ $(y, z \in 2^\omega)$ such that*

1. *each $S_{y,z}$ is a subset of 2^ω containing $\{y, z\}$ and closed under F;*
2. *no $S_{y,z}$ contains 2^ω;*

(3) *for each function* $h\colon 2^\omega \to 2^\omega$ *there exist* $y, z \in 2^\omega$ *such that* $\{h(y), h(z)\} \subseteq S_{y,z}$.

Then F *is not universal.*

Proof

Let $G\colon 2^\omega \times 2^\omega \to 2^\omega$ be such that each value $G(y,z)$ is an element of $2^\omega \setminus S_{y,z}$. Then for each $h\colon 2^\omega \to 2^\omega$ it is possible to find reals y and z such that $\{h(y), h(z)\} \subseteq S_{y,z}$. Since $S_{y,z}$ is closed under F, $F(h(y), h(z)) \in S_{y,z}$. Since $G(y,z) \notin S_{y,z}$, it follows that $F(h(y), h(z)) \neq G(y,z)$, so F is not universal.

QED

Combined with Lemma 3.1, Theorem 3.3 below shows that if there is a model of set theory then there is a model of set theory in which there is no analytic universal function on the reals. First we note a general combinatorial fact, which is a generalization of one of Sierpiński's characterizations of the failure of the Continuum Hypothesis (see [30]). In our first application of the lemma, δ will be ω; in the second it will be an arbitrary uncountable cardinal. We let $\mathcal{P}_\kappa(\lambda)$ denote the collection of subsets of λ of cardinality less than κ.

Lemma 3.2. *Suppose that* δ *and* κ *are cardinals with* $\kappa > \delta^+$, *and let* $f\colon \kappa \times \kappa \to \kappa$ *be injective. Then for each function* $H\colon \kappa \to \mathcal{P}_{\delta^+}(\kappa)$ *there exist* $\alpha < \delta^+$ *and* $\beta \in \kappa$ *such that* $f(\alpha, \beta) \notin H(\alpha) \cup H(\beta)$.

Proof

Choose $\beta \in \kappa$ such that, for all $\alpha < \delta^+$, $f(\alpha, \beta) \notin H(\alpha)$. Now choose $\alpha < \delta^+$ such that $f(\alpha, \beta) \notin H(\beta)$.

QED

The proofs of following theorems apply to any class of functions with the property that for each F in the class there exists a set of ordinals x of cardinality less than κ with the property that every inner model with x as a member is closed under F.

Theorem 3.3. *Suppose that* $\kappa > \omega_1$ *is a cardinal of uncountable cofinality. Then there is no analytic universal function on* 2^ω *in any model obtained by forcing with a finite support product of* κ *many nontrivial c.c.c. partial orders.*

Proof

Let $\mathbb{P}_\alpha$ be a c.c.c. partial order for each $\alpha \in \kappa$ and suppose that

$$G \subseteq \prod_{\alpha \in \kappa} \mathbb{P}_\alpha$$

is generic over V. Since infinite finite-support products of nontrivial partial orders add reals, by grouping together products of countably many $\mathbb{P}_\alpha$'s we may assume that each $\mathbb{P}_\alpha$ adds a real. We work in $V[G]$. Let $F\colon 2^\omega \times 2^\omega \to 2^\omega$ be analytic, and let $x \in 2^\omega$ be a code for F.

For each $\beta \in \kappa$, let G^*_β denote the restriction of G to $\prod_{\alpha\in\kappa\setminus\{\beta\}} \mathbb{P}_\alpha$. Since $\prod_{\alpha\in\kappa} \mathbb{P}_\alpha$ is c.c.c., each real is in $V[G^*_\beta]$ for all but countably many $\beta \in \kappa$. Fix $X \subseteq 2^\omega$ with $|X| = \kappa$, and let $f\colon X \times X \to \kappa$ be injective with the property that $\{x, y, z\} \subseteq V[G^*_{f(y,z)}]$ for each pair $(y, z) \in X \times X$. Define $S_{y,z} \subseteq 2^\omega$ for each $y, z \in 2^\omega$ by setting

$$S_{y,z} = 2^\omega \cap V[G^*_{f(x,y)}]$$

whenever $(y, z) \in X \times X$, and letting $S_{y,z} = \{y, z\}$ otherwise. Then item (1) of Lemma 3.1 clearly holds, and item (2) follows from the fact that each $\mathbb{P}_\alpha$ adds a real.

To see that (3) holds, fix a function $h\colon 2^\omega \to 2^\omega$. Applying the c.c.c. of $\prod_{\alpha\in\kappa} \mathbb{P}_\alpha$ we can find a function $H\colon X \to \mathcal{P}_{\aleph_1}(\kappa)$ such that, for each $y \in X$, $h(y) \in V[G \restriction \prod_{\alpha\in H(y)} \mathbb{P}_\alpha]$. Applying Lemma 3.2, we can find $y, z \in X$ such that $f(y, z) \notin H(y) \cup H(z)$, which means that $\{h(y), h(z)\} \subseteq S_{y,z}$.
QED

Theorem 3.4. *Suppose that λ and κ are uncountable cardinals such that $\lambda^+ < \kappa$, $\kappa^\lambda = \kappa$ and κ has uncountable cofinality. Then there is a c.c.c. forcing extension in which $\mathfrak{c} = \kappa$, MA_λ holds and there is no analytic universal function.*

Proof
Let $\mathbb{P}$ be a finite support product of c.c.c. partial orders $\mathbb{P}_\alpha$ $(\alpha < \kappa)$, such that each $\mathbb{P}_\alpha$ has cardinality at most λ and adds a real. Let $G \subseteq \mathbb{P}$ be a V-generic filter, and, for each $X \subseteq \kappa$, let G_X be the restriction of G to $\prod_{\alpha\in X} \mathbb{P}_\alpha$. For each $\alpha < \kappa$, let G^*_α denote $G_{\kappa\setminus\{\alpha\}}$, and let a_α be an element of $(2^\omega \cap V[G]) \setminus V[G^*_\alpha]$.

Working in $V[G]$, let $\mathbb{Q}$ be the direct limit of a finite support iteration $\langle \mathbb{Q}_\alpha, \dot{R}_\alpha : \alpha < \kappa\rangle$ of c.c.c. partial orders on λ, such that $\mathbb{Q}$ forces MA_λ. For each $X \subseteq \kappa$, let $\mathbb{Q}_X$ be the subiteration of $\mathbb{Q}$ consisting of those $\dot{R}_\alpha$'s which depend only on $\prod_{\alpha\in X} \mathbb{P}_\alpha$ (as opposed to all of $\mathbb{P}$) and the initial segment of $\mathbb{Q}_X$ before stage α. Since $\mathbb{P} * \mathbb{Q}$ is in V, each $\mathbb{Q}_X$ is in $V[G_X]$ and regularly embeds into $\mathbb{Q}$. Furthermore, each $\dot{R}_\alpha$ (and each countable set of $\dot{R}_\alpha$'s) is part of $\mathbb{Q}_X$ for some $X \subseteq \kappa$ of cardinality λ.

Let K be $V[G]$-generic filter for $\mathbb{Q}$, and for each $X \subseteq \kappa$, let K_X be the restriction of K to $\mathbb{Q}_X$. For each $\alpha \in \kappa$, let K^*_α denote $K_{\kappa\setminus\{\alpha\}}$.

Then every element of 2^ω in $V[G][K]$ is in $V[G_X][K_X]$ for some $X \subseteq \kappa$ of cardinality λ. By mutual genericity, no a_α is in $V[G^*_\alpha][K^*_\alpha]$.

Now suppose that F is an analytic function in $V[G][K]$, coded by some $x \in 2^\omega$. Fix $X \subseteq \kappa$ of cardinality λ such that x is in $V[G_X][K_X]$. Let

$$i\colon \{a_\alpha : \alpha \in \kappa\}^2 \to \{a_\alpha : \kappa \setminus X\}$$

be an injection such that $\{a_\alpha, a_\beta\} \subseteq V[G^*_{i(a_\alpha,a_\beta)}][K^*_{i(a_\alpha,a_\beta)}]$ for all $\alpha, \beta \in \kappa$. Let $S_{y,z} = 2^\omega \cap V[G^*_{i(y,z)}][K^*_{i(y,z)}]$, for each pair $(y, z) \in \{a_\alpha : \alpha \in \kappa\}^2$, and let $S_{y,z} = \{y, z\}$ for all other pairs (y, z) from $2^\omega \times 2^\omega$. Then items (1) and (2) of Lemma 3.1 are clearly satisfied.

Fix a function $h\colon 2^\omega \to 2^\omega$. For each $\alpha \in \kappa$, we have a set $H(\alpha) \subseteq \kappa$ of size λ, containing X, such that $h(\alpha)$ is in $V[G_{H(\alpha)}][K_{H(\alpha)}]$. By Lemma 3.2, there are α and β in κ such that $i(a_\alpha, a_\beta)$ is not equal to a_γ for any $\gamma \in H(\alpha) \cup H(\beta)$. It follows that $\{h(a_\alpha), h(a_\beta)\} \subseteq S_{a_\alpha,a_\beta}$. Then item (3) of Lemma 3.1 is satisfied, showing that F is not universal in $V[G][K]$.

QED

4. Universal functions of special kinds

Elementary functions in the calculus of two variables can be obtained from addition, the elementary functions of one variable and closing under composition. For example, $xy = \frac{1}{2}((x+y)^2 - x^2 - y^2)$. We might ask if there could be a universal function which uses addition.

Proposition 4.1. *Suppose that $U : 2^\omega \times 2^\omega \to 2^\omega$ is a universal function. Then there is a universal function $F(x, y) = k(x + y)$, where $k : 2^\omega \to 2^\omega$ has the same Borel complexity as U and $x + y$ refers to pointwise addition in 2^ω.*

Proof

Given any $u \in 2^\omega$ let u_0 be u shifted onto the even coordinates, i.e, $u_0(2n) = u(n)$ and $u_0(2n + 1) = 0$. Similarly for $v \in 2^\omega$ let v_1 be v shifted onto the odd coordinates. Note that (u, v) is easily recovered from $u_0 + v_1$. Hence we can define k by $k(w) = U(u, v)$ where $w = u_0 + v_1$. Then, given $H : 2^\omega \times 2^\omega \to 2^\omega$ there is $g : 2^\omega \to 2^\omega$ such that $H(u, v) = U(g(u), g(v))$. Let $g_0(x) = (g(x))_0$ and $g_1(x) = (g(x))_1$. Then $H(u, v) = U(g(u), g(v)) = k((g(u))_0 + (g(v))_1) = F(g_0(u), g_1(v))$. Now apply Remark 1.1.

QED

Proposition 4.3 gives a generalization of the result above which applies, for example, to any Borel subgroup of a Polish group or even a

Borel subsemigroup of a Polish cancellation[3] semigroup. First we prove a general result about Borel binary operations. We say that a binary operation $*$ on a set B is *separately one-to-one* if for every $x, y, z \in B$, if $x * y = x * z$ or $y * x = z * x$ then $y = z$.

Lemma 4.2. *Suppose that $*$ is a Borel binary operation on an uncountable Borel $B \subseteq 2^\omega$, and that $*$ is separately one-to-one. Then there exist perfect subsets P_1, P_2 of B such that $*$ is one-to-one and continuous on $P_1 \times P_2$.*

Proof
Let $Q \subseteq B$ be a perfect set. Let M be the transitive collapse of a countable elementary substructure X of $H(\mathfrak{c}^+)$ which contains reals coding Q and $(B, *)$. Let $T \subseteq 2^{<\omega}$ be the tree whose infinite branches are the elements of Q. Forcing with T is equivalent to forcing with the poset $2^{<\omega}$. It is well-known (see [4], where it is credited to Folklore) that there is a countable partial order forcing a perfect set $P \subseteq Q$ with the property that every finite sequence of distinct elements $(x_1, \dots, x_n)$ of P is T^n-generic. Let P be such a generic set over M, and let P_1 and P_2 be any pair of disjoint perfect subsets of P.

To see that $*$ is continuous on $P_1 \times P_2$ suppose that $(x_1 * x_2)\restriction n = s$. Then there must be m such that $(x_1\restriction m, x_2\restriction m)$ forces in T^2 that $(g_1 * g_2)\restriction n = s$ where g_1 and g_2 are the generic reals added by T^2. It follows that $(x_1' * x_2')\restriction n = s$ for all (x_1', x_2') in $P_1 \times P_2$ which agree with (x_1, x_2) up to m.

To see that $*$ is one-to-one on $P_1 \times P_2$, suppose that

$$z = x_1 * x_2 = x_1' * x_2'.$$

Since P_1 and P_2 are disjoint and $*$ is separately one-to-one, either (x_1, x_2) and (x_1', x_2') are the same pair or all four reals are distinct. If all four are distinct, then $((x_1, x_2), (x_1', x_2'))$ is $T^2 \times T^2$-generic over M. A well-known lemma on product forcing (see page 13 of Solovay [31]) gives that in this case,

$$M[x_1, x_2] \cap M[x_1', x_2'] = M,$$

so $z \in M$. Then there exists an $n \in \omega$ such that $(x_1\restriction n, x_2\restriction n)$ forces in T^2 that $g_1 * g_2 = z$. Since P_2 is perfect, there exists a $y \in P_2 \setminus \{x_2\}$ for which $y\restriction n = x_2\restriction n$. This contradicts our assumption that $*$ is separately one-to-one.
QED

[3] A semigroup is a cancelation semigroup if it satisfies that for all a, b and c, if $ac = bc$ then $a = b$.

Proposition 4.3. *Suppose that $U : 2^\omega \times 2^\omega \to 2^\omega$ is a universal function, and $(B, *)$ consists of an uncountable Borel set with a Borel binary operation * on B which is separately one-to-one. Then there exists a function $F : B \to 2^\omega$ such that for each $g : \mathfrak{c} \times \mathfrak{c} \to 2^\omega$ there exist h, k mapping $\mathfrak{c}$ to B with*

$$g(\alpha, \beta) = F(h(\alpha) * k(\beta)) \quad \textit{for all } \alpha, \beta \in \mathfrak{c}$$

Furthermore, if U is Borel then F can be taken to have the same Borel rank as U.

Proof
Fix P_1 and P_2 as in the conclusion of Lemma 4.2. Let R be the range of the binary operation $*$, so that $* : P_1 \times P_2 \to R$ is a homeomorphism. Let f_1, f_2 be the continuous functions with domain R such that $f_1(x * y) = x$ and $f_2(x*y) = y$. Since P_1 and P_2 are each homeomorphic to 2^ω we may assume without loss of generality that $U : P_1 \times P_2 \to 2^\omega$. Define $F\colon B \to 2^\omega$ by setting $F(z) = U(f_1(z), f_2(z))$ if $z \in R$ and $F(z) = 0$ otherwise. Then $F(x*y) = U(x, y)$ for each $(x, y) \in P_1 \times P_2$. To verify that F is universal, fix an arbitrary function $g : \mathfrak{c} \times \mathfrak{c} \to 2^\omega$. Since U is universal there are $h : \mathfrak{c} \to P_1$ and $k : \mathfrak{c} \to P_2$ such that $g(\alpha, \beta) = U(h(\alpha), k(\beta))$ for all $\alpha, \beta \in \mathfrak{c}$. Then $F(h(\alpha) * k(\beta)) = U(h(\alpha), k(\beta)) = g(\alpha, \beta)$ for all such α, β.
QED

The following proposition shows that functions which are universal with respect to symmetric functions can have a simpler form.

Proposition 4.4. *Suppose that $F : 2^\omega \times 2^\omega \to 2^\omega$ is a universal function. Then there exists a function $f : 2^\omega \to 2^\omega$ such that for each symmetric function $H : 2^\omega \times 2^\omega \to 2^\omega$ there exists a function $g : 2^\omega \to 2^\omega$ such that $H(x, y) = f(g(x) + g(y))$ for every two distinct $x, y \in 2^\omega$. Furthermore if F is Borel, then f can be taken to be Borel.*

Proof
Let $P_s \subseteq \omega$ for $s \in 2^{<\omega}$ partition ω into infinite sets. Given $s \in 2^{<\omega}$, we say that $y : P_s \to 2$ codes $x : \omega \to 2$ if $y(a_n) = x(n)$ where $a_0 < a_1 < a_2 < \dots$ is the increasing listing of P_s.

Define $q\colon 2^\omega \to 2^\omega$ by letting $q(x)$ be such that $q(x)\restriction P_{x\restriction n}$ codes x for every $n < \omega$ and $q(x)\restriction P_s$ is identically 0 for any s which is not an initial segment of x.

Notice that it is possible to recover u and v from $q(u) + q(v)$ as long as neither u nor v is the constant 0 function. To see this let

$$\Sigma = \{s \in 2^{<\omega} : \exists n \in P_s \ \ (q(u) + q(v))(n) \neq 0\}.$$

Suppose that $u(m) \neq v(m)$ and recall that neither u nor v is identically zero. Then for every $s \in \Sigma$ of length greater than m, exactly one of the following must hold:

- $q(v)\restriction P_s$ is identically zero and $(q(u)+q(v))\restriction P_s$ codes u;
- $q(u)\restriction P_s$ is identically zero and $(q(u)+q(v))\restriction P_s$ codes v.

In other words, for sufficiently large values of s the uncoding of

$$(q(u)+q(v))\restriction P_s$$

will take on only one of two possible values and these two values will be u and v.

Using this, $f(w)$ is defined as follows. If there are distinct, non-zero, u and v such that $w = q(u)+q(v)$ then these are unique. In this case define $f(w) = F(u,v)$ where $u < v$ (for the sake of avoiding arbitrary choices in the definition). Otherwise, define $f(w)$ to be the constant 0 function. If F is Borel, then f can be defined in a Borel way, although its rank might increase.

To see that this definition works, let a symmetric function

$$H : 2^\omega \times 2^\omega \to 2^\omega$$

be given. By assumption there exists h such that

$$H(x,y) = F(h(x),h(y))$$

for all $x, y \in 2^\omega$. Without loss of generality we may assume that h is one-to-one and $h(x)$ is not equal to the constant 0 function for any x. To see this, note that we may replace any h with $\hat{h}(x) = \langle 1^\frown x, h(x)\rangle$ and then alter F so that it ignores the first coordinate of $\hat{h}(x)$; that is, define

$$F_0(\langle u_0,u_1\rangle, \langle v_0,v_1\rangle) = F(u_1,v_1).$$

Now let $g = q \circ h$. Then $f(g(x)+g(y)) = f(q(h(x))+q(h(y)))$ and keep in mind that $h(x)$ and $h(y)$ are non-zero and distinct. Hence $f(q(h(x))+q(h(y))) = F(h(x),h(y)) = H(x,y)$ as required. Note that the symmetry of H allowed the arbitrary choice of ordering of u and v in the definition of f.
QED

The proof of the following result is similar to Mansfield and Rao's proof [12, 13, 23] that the universal analytic set in the plane is not in the σ-algebra generated by rectangles with measurable sides. See also Miller [19].

Proposition 4.5. *There does not exist a Borel function*

$$F : 2^\omega \times 2^\omega \to 2^\omega$$

such that for every Borel $G : 2^\omega \times 2^\omega \to 2^\omega$ there exist functions h and k from 2^ω to 2^ω such that k is Borel and

$$G(x, y) = F(h(x), k(y))$$

for all $x, y \in 2^\omega$.

Proof
Let $F\colon 2^\omega \times 2^\omega \to 2^\omega$ be a Baire class α function, let $U \subseteq 2^\omega \times 2^\omega$ be a universal $\mathbf{\Sigma}^0_{\alpha+1}$ set and let G be the characteristic function of U. Suppose that h and k are functions from 2^ω to 2^ω such that k is Borel and

$$G(x, y) = F(h(x), k(y))$$

for all $x, y \in 2^\omega$. Let $P \subseteq 2^\omega$ be a perfect set on which k is continuous, and fix x_0 so that $U_{x_0} \subseteq P$ and U_{x_0} is not $\mathbf{\Delta}^0_{\alpha+1}$. If we define $q : P \to 2^\omega$ by setting $q(y) = F(h(x_0), k(y))$, then then q is Baire class α and $U_{x_0} = q^{-1}(1)$, giving a contradiction.
QED

Remark 4.6. *The second author has recently shown that consistently the Borel subsets of the plane are not in contained any bounded level of the σ-algebra generated by the abstract rectangles. The proof of Theorem 2.5 shows that in this situation, there does not exist a Borel function*

$$F : 2^\omega \times 2^\omega \to 2^\omega$$

such that for every Borel $H : 2^\omega \times 2^\omega \to 2^\omega$ there exist functions g and h from 2^ω to 2^ω such that

$$H(x, y) = F(g(x), h(y))$$

for all $x, y \in 2^\omega$. The proof is a modification of arguments in [17].

The following type of universal function was introduced by Stevo Todorcevic.

Definition 4.7. *Given a cardinal κ and $k \in \omega$, a sequence of continuous functions*

$$F_n\colon (2^\omega)^k \to 2^\omega \ (n < \omega)$$

is κ limit-universal *if for each $X \subseteq 2^\omega$ of cardinality at most κ and each function $G\colon X^k \to 2^\omega$ there exists an injective function $h\colon X \to 2^\omega$ such that for all $x_1, \ldots, x_k \in X$,*

$$G(x_1, \ldots, x_k) = \lim_{n\to\infty} F_n(h(x_1), \ldots, h(x_k)).$$

Theorem 4.8 (Todorcevic [32])**.** *For each $k \in \omega$, there exists a $\mathfrak{p}$ limit-universal sequence of functions $F_n\colon (2^\omega)^k \to 2^\omega$, for $n < \omega$.*

Remark 4.9. *In the model from Theorem 3.3,* $\mathfrak{p} = \aleph_2$ *and* $\mathfrak{c} = \aleph_3$.

Recall that a function $F\colon 2^\omega \times 2^\omega \to 2^\omega$ being of level 2 means that for every $n \in \omega$ the set $\{(x, y) \,:\, F(x, y)(n) = 1\}$ is F_σ. The following proposition shows that the existence of a $\mathfrak{c}$ limit-universal sequence of functions is equivalent to the existence of a level 2 universal function.

Proposition 4.10. *For any cardinal* κ *the following are equivalent:*

(1) There exists continuous functions $F_n : 2^\omega \times 2^\omega \to 2^\omega$ $(n < \omega)$ *with the property that for each function* $G : \kappa \times \kappa \to 2^\omega$ *there exists a function* $h : \kappa \to 2^\omega$ *such that*

$$G(\alpha, \beta) = \lim_{n\to\infty} F_n(h(\alpha), h(\beta))$$

for all $\alpha, \beta \in \kappa$.

(2) There exists a level 2 function $F : 2^\omega \times 2^\omega \to 2^\omega$ *with the property that for each function* $G : \kappa \times \kappa \to 2^\omega$ *there exists a function* $h : \kappa \to 2^\omega$ *such that*

$$G(\alpha, \beta) = F(h(\alpha), h(\beta))$$

for all $\alpha, \beta \in \kappa$.

Proof
(1) $\to$ (2). Given the sequence of continuous functions $F_n : 2^\omega \times 2^\omega \to 2^\omega$ $(n < \omega)$ define $F : 2^\omega \times 2^\omega \to 2^\omega$ by setting $F(x, y)(k) = 1$ if and only if $F_n(x, y)(k) = 1$ for all but finitely many $n < \omega$.

(2) $\to$ (1). Using a continuous pairing function $\langle \cdot, \cdot \rangle$ on $2^\omega \times 2^\omega$, define for each $k \in \omega$ the pair of (nondisjoint) F_σ subsets of 2^ω, P^0_k and P^1_k, by setting

$$(\langle u_0, u_1\rangle, \langle v_0, v_1\rangle) \in P^i_k$$

if and only if $F(u_i, v_i)(k) = 1$. By the reduction property for F_σ sets, for each $k \in \omega$ there exist disjoint F_σ sets Q^0_k, Q^1_k with $Q^i_k \subseteq P^i_k$ and $Q^0_k \cup Q^1_k = P^0_k \cup P^1_k$. Write each Q^i_k as an increasing sequence of closed sets $Q^i_k = \bigcup_{n\in\omega} C^i_{k,n}$. For each $n, k \in \omega$, $C^0_{k,n}$ and $C^1_{k,n}$ are disjoint closed sets, so there exists a clopen set $D_{k,n}$ with $C^0_{k,n} \subseteq D_{k,n}$ and $C^1_{k,n}$ disjoint from $D_{k,n}$.

Define the continuous map $F_n\colon 2^\omega \times 2^\omega \to 2^\omega$ by setting

$$F_n(u, v)(k) = 1$$

if and only if $(u, v) \in D_{k,n}$. Now we verify that this works. Given $G\colon \kappa \times \kappa \to 2^\omega$, let G_0 be G and define $G_1\colon \kappa \times \kappa \to 2^\omega$ by setting $G_1(\alpha, \beta)(n) = 1 - G_0(\alpha, \beta)(n)$ (that is, we switch 0 and 1 on every coordinate of the output). Let h_0 and h_1 be the results of applying (2) to G_0 and G_1, respectively. We have then that for every $k \in \omega$ and all $\alpha, \beta < \kappa$,

- $G(\alpha,\beta)(k) = 1$ implies
  $F(h_0(\alpha), h_0(\beta))(k) = 1$ and $F(h_1(\alpha), h_1(\beta))(k) = 0$
- $G(\alpha,\beta)(k) = 0$ implies
  $F(h_0(\alpha), h_0(\beta))(k) = 0$ and $F(h_1(\alpha), h_1(\beta))(k) = 1$

Define $h\colon 2^\omega \to 2^\omega$ by setting $h(\gamma) = \langle h_0(\gamma), h_1(\gamma)\rangle$. Then for all $\alpha, \beta < \kappa$ and all $k \in \omega$ the following hold.

- If $G(\alpha,\beta)(k) = 1$, then $(h(\alpha), h(\beta)) \in P^0_k \setminus P^1_k$ so $(h(\alpha), h(\beta)) \in Q^0_k$ and $F_n((h(\alpha), h(\beta)))(k) = 1$ for all but finitely many n.
- If $G(\alpha,\beta)(k) = 0$, then $(h(\alpha), h(\beta)) \in P^1_k \setminus P^0_k$ so $(h(\alpha), h(\beta)) \in Q^1_k$ and $F_n((h(\alpha), h(\beta)))(k) = 0$ for all but finitely many n.

QED

Davies [6] showed that the Continuum Hypothesis is equivalent to the assertion that the function

$$F(\vec{x}, \vec{y}) = \sum_{n<\omega} x_n y_n$$

has the following universal property: for every $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ there are functions f_n, g_n for $n < \omega$ such that

$$H(x, y) = \sum_{n<\omega} f_n(x) g_n(y)$$

for all $x, y \in \mathbb{R}$. Moreover, the functions f_n and g_n $(n < \omega)$ can be taken so that the sum $\sum_{n<\omega} f_n(x)g_n(y)$ has only finitely many nonzero terms. If this requirement is relaxed to ordinary convergence of the infinite sum, then the function $F(\vec{x}, \vec{y}) = \sum_{n<\omega} x_n y_n$ is still universal under the assumption $\mathfrak{p} = \mathfrak{c}$ (Shelah [27]). However, it is not universal after adding $\aleph_2$ many Cohen reals with finite support ([27]). These considerations suggest the following context for studying universal functions. Let $(\mathbb{B}, \cdot)$ be a Banach algebra and $\psi \in \mathbb{B}^*$. The linear functional ψ can be said to be *κ-universal* if for every $f : \kappa \times \kappa \to \mathbb{R}$ there are $h : \kappa \to \mathbb{B}$ and $g : \kappa \to \mathbb{B}$ such that $f(\alpha, \beta) = \psi(h(\alpha) \cdot g(\beta))$. This will be examined elsewhere.

5. Abstract universal functions

This section considers the question of universal functions without regard to any definability properties. The notion of universal function naturally generalizes to functions of the form $F : \alpha \times \beta \to \gamma$.

Theorem 5.1. *If α and κ are cardinals such that $\alpha^{<\kappa} = \kappa$, then there is a universal function from $\kappa \times \kappa$ to α.*

Proof

Let $\mathcal{F}$ be the set of functions $f\colon \kappa \to \alpha$ for which $\{\gamma < \kappa \mid f(\gamma) \neq 0\}$ is a bounded subset of κ. Then $|\mathcal{F}| = \alpha^{<\kappa} = \kappa$. Define $U\colon (\kappa \times \mathcal{F})^2 \to \alpha$ by setting

$$U((\gamma, f_1), (\beta, f_2)) = \begin{cases} f_1(\beta) & \text{if } \beta < \gamma \\ f_2(\gamma) & \text{if } \gamma \leq \beta \end{cases}$$

By Remark 1.1, it is enough to show that for each $g\colon \kappa \times \kappa \to \alpha$ there exist $h\colon \kappa \to (\kappa \times \mathcal{F})$ and $k\colon \kappa \to (\kappa \times \mathcal{F})$ such that

$$U(h(\gamma), k(\beta)) = g(\gamma, \beta)$$

for all γ, β in κ. Fix $g\colon \kappa \times \kappa \to \alpha$. Define $h\colon \kappa \to (\kappa \times \mathcal{F})$ by setting $h(\gamma) = (\gamma, f_{1,\gamma})$, where $f_{1,\gamma}\colon \kappa \to \alpha$ is such that $f_{1,\gamma}(\beta)$ is $g(\gamma, \beta)$ for all $\beta < \gamma$ and 0 for all $\beta \geq \gamma$. Define $k\colon \kappa \to (\kappa \times \mathcal{F})$ by setting $k(\beta) = (\beta, f_{2,\beta})$, where $f_{2,\beta}\colon \kappa \to \alpha$ is such that $f_{2,\beta}(\gamma)$ is $g(\gamma, \beta)$ for all $\gamma \leq \beta$ and 0 for all $\gamma > \beta$.

Now fix γ, β in κ. If $\gamma > \beta$, then $U(h(\gamma), k(\beta)) = f_{1,\gamma}(\beta) = g(\gamma, \beta)$. If $\gamma \leq \beta$, then $U(h(\gamma), k(\beta)) = f_{2,\beta}(\gamma) = g(\gamma, \beta)$.
QED

It follows from Theorem 5.1 that if $\kappa^{<\kappa} = \kappa$, then there is a universal function $U : \kappa \times \kappa \to \kappa$. So, for example, there is a universal function from $\omega \times \omega$ to ω. If $\kappa^{<\kappa} = \kappa$, then κ must be a regular cardinal. Theorem 5.1 implies that if κ is strong limit cardinal, then for every $\alpha < \kappa$ there is a universal function from $\kappa \times \kappa$ to α. However we don't know the answer to the following question:

Question 5.2. *If κ is a singular strong limit cardinal, does there exist a universal function from $\kappa \times \kappa$ to κ?*

Proposition 5.3. *Suppose that κ is a singular strong limit cardinal and that $\alpha < \kappa$. Then there is a universal function $U : \kappa \times \alpha \to \kappa$ if and only if α is less than the cofinality of κ.*

Proof

Let τ be the cofinality of κ. If $\alpha < \tau$, then there are only κ many maps from α into κ, so U just needs to list all of them as a cross section $U_\beta(\cdot) = U(\beta, \cdot)$ for $\beta < \kappa$.

If $\alpha \geq \tau$, we can diagonalize against any U by eventually avoiding the range of any cross section. To see this suppose $U : \kappa \times \alpha \to \kappa$ is any map. Let κ_δ for $\delta < \tau$ be increasing and cofinal in κ. Construct a map $d : \tau \to \kappa$ so that

$$d(\delta) \in \kappa \setminus \{\, U(\beta, \gamma) \;:\; \beta < \kappa_\delta,\ \gamma < \alpha\}.$$

The map $f : \kappa \times \tau \to \kappa$ defined by $f(\beta, \delta) = d(\delta)$ witnesses that U is not universal.
QED

The following proposition, which also applies to singular cardinals, shows that a negative answer to Question 5.2 for $\kappa = \aleph_\omega$ must use maps with domain at least $\omega_1 \times \omega$. Its proof is similar to the proof of Theorem 5.1.

Proposition 5.4. *For each infinite cardinal κ there exists a function $U : \kappa \times \kappa \to \kappa$ such that for each $G : \omega \times \omega \to \kappa$ there exists a function $h : \omega \to \kappa$ such that $G(n, m) = U(h(n), h(m))$ for all $n, m \in \omega$.*

Proof
Let

$$\mathcal{F} = \{F : \omega \to \kappa \ : \ \forall^\infty n \ F(n) = 0\}$$

Define $U : (\omega \times \mathcal{F})^2 \to \kappa$ by

$$U((n, F_1), (m, F_2)) = \begin{cases} F_1(m) & \text{if } n > m \\ F_2(n) & \text{if } n \leq m \end{cases}$$

Given any G define $h(n) = (n, F_n)$ where $F_n \in \mathcal{F}$ has the property that $F_n(m) = G(n, m)$ whenever $m \leq n$. Define $k(m) = (m, F'_m)$ where $F'_m \in \mathcal{F}$ has the property that $F_m(n) = G(n, m)$ whenever $n \leq m$. Then

- for any $n \leq m$ $U(h(n), k(m)) = F'_m(n) = G(n, m)$ and
- for any $n > m$ $U(h(n), k(m)) = F_n(m) = G(n, m)$.

As usual we may encode distinct h, k into a single map.
QED

The following theorem shows that it is relatively consistent with ZFC that there is no universal function $F \colon \mathfrak{c} \times \mathfrak{c} \to 2$. Given sets X, Y and a cardinal κ, the partial order $\mathrm{Fn}(X, Y, \kappa)$ consists of partial functions from X to Y of cardinality less than κ, ordered by inclusion.

Theorem 5.5. *If $\mathfrak{c} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$, then the partial order*

$$\mathrm{Fn}(\omega_3, 2, \omega_1) \times \mathrm{Fn}(\omega_2, 2, \omega)$$

forces that $\mathfrak{c} = \aleph_2$ and that there is no $F \colon \omega_2 \times \omega_2 \to 2$ with the property that for every $f : \omega_2 \times \omega_1 \to 2$ there exists $g_1 : \omega_2 \to \omega_2$ and $g_2 : \omega_1 \to \omega_2$ such that $f(\alpha, \beta) = F(g_1(\alpha), g_2(\beta))$ for every $\alpha < \omega_2$ and $\beta < \omega_1$.

Proof
Suppose that $\mathfrak{c} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$. Force with $\mathrm{Fn}(\omega_3, 2, \omega_1)$ followed by $\mathrm{Fn}(\omega_2, 2, \omega)$. Let G be $\mathrm{Fn}(\omega_3, 2, \omega_1)$-generic over V and H

be $\mathrm{Fn}(\omega_2, 2, \omega)$-generic over $V[G]$. We will show there is no such F as above in the model $V[G][H]$.

By standard arguments[4] involving iteration and product forcing we may regard $V[G][H]$ as being obtained by forcing with $\mathrm{Fn}(\omega_3, 2, \omega_1)^V$ over the ground model $V[H]$. Of course, in $V[H]$ the poset $\mathrm{Fn}(\omega_3, 2, \omega_1)^V$ is not countably closed but it still must have the ω_2-cc. Hence for any $F : \omega_2 \times \omega_2 \to 2$ in $V[G][H]$ we may find $\gamma < \omega_3$ such that $F \in V[H][G \restriction \gamma]$.

Use G above γ to define $f : \omega_2 \times \omega_1 \to 2$, i.e.,

$$f(\alpha, \beta) = G(\gamma + \omega_1 \cdot \alpha + \beta).$$

Suppose towards a contradiction that in $V[G][H]$ there were functions $g_1 : \omega_2 \to \omega_2$ and $g_2 : \omega_1 \to \omega_2$ such that $f(\alpha, \beta) = F(g_1(\alpha), g_2(\beta))$ for every $\alpha < \omega_2$ and $\beta < \omega_1$. Using the ω_2-chain condition of $\mathrm{Fn}(\omega_3, 2, \omega_1) \times \mathrm{Fn}(\omega_2, 2, \omega)$, there would be an $I \subseteq \omega_3$ in V of size ω_1 such that $g_2 \in V[H][G \restriction (\gamma \cup I)]$. Choose $\alpha_0 < \omega_2$ so that $\gamma \cup I$ is disjoint from

$$D = \{\gamma + \omega_1 \cdot \alpha_0 + \beta \ : \ \beta < \omega_1\}.$$

It easy to see by a density argument that the function $G \restriction D$ is not in $V[H][G \restriction (\gamma \cup I)]$. But this is a contradiction, since $G \restriction D$ is easily defined from the function $f(\alpha_0, \cdot)$, $f(\alpha_0, \beta) = F(g_1(\alpha_0), g_2(\beta))$ for all β, and F, g_2 are in $V[H][G \restriction (\gamma \cup I)]$.
QED

Question 5.6. *Is it consistent with $2^{<\mathfrak{c}} > \mathfrak{c}$ to have a universal function $F : 2^\omega \times 2^\omega \to 2^\omega$? What about a Borel F?*

The two following propositions illustrate two cases where a universal function with range κ can be lifted to one with range κ^+.

Proposition 5.7. *For any infinite cardinal κ, there is a universal*

$$F : \kappa^+ \times \kappa^+ \to \kappa^+$$

if and only if there is a universal function $G : \kappa^+ \times \kappa^+ \to \kappa$.

Proof
The forward direction follows from Remark 1.4. For the reverse direction, suppose that we are given a universal $G : \kappa^+ \times \kappa^+ \to \kappa$. For each $\alpha < \kappa^+$ let $b_\alpha \colon \kappa \to \alpha$ be a bijection. Define $F : \kappa^+ \times \kappa^+ \to \kappa^+$ by setting

$$F(\langle \alpha_1, \alpha_2, \alpha_3 \rangle, \langle \beta_1, \beta_2, \beta_3 \rangle) = \begin{cases} b_{\beta_3}(G(\alpha_2, \beta_2)) & \text{if } \alpha_1 \leq \beta_1 \\ b_{\alpha_3}(G(\alpha_2, \beta_2)) & \text{if } \alpha_1 > \beta_1 \end{cases}$$

[4] Kunen[10] p.253, Solovay [31] p.10

where $\langle a, b, c\rangle$ represents a bijection between an infinite set and its triples, or, equivalently, $\langle a, b, c\rangle$ is defined to be $\langle\langle a, b\rangle, c\rangle$. To see that F is universal, fix a function $f : \kappa^+ \times \kappa^+ \to \kappa^+$. Let $j : \kappa^+ \to \kappa^+$ be such that $f(\alpha, \beta) < j(\max(\alpha, \beta))$ for all $\alpha, \beta < \kappa^+$. Define $f^* : \kappa^+ \times \kappa^+ \to \kappa$ by setting $f^*(\alpha, \beta) = \xi$, where $\xi < \kappa$ is such that $f(\alpha, \beta) = b_{j(\max(\alpha,\beta))}(\xi)$. As G is universal, there exists a function $h \colon \kappa^+ \to \kappa^+$ such that

$$f^*(\alpha, \beta) = G(h(\alpha), h(\beta))$$

for all $\alpha, \beta < \kappa^+$. This means that for all $\alpha, \beta < \kappa^+$,

$$f(\alpha, \beta) = b_{j(\max(\alpha,\beta))}(G(h(\alpha), h(\beta))).$$

It follows that for all $\alpha, \beta < \kappa^+$,

$$f(\alpha, \beta) = F(\langle\alpha, h(\alpha), j(\alpha)\rangle,\ \langle\beta, h(\beta), j(\beta)\rangle).$$

QED

Proposition 5.8. *For any pair of infinite cardinals $\kappa > \lambda$, there is a universal function from $\kappa \times \lambda$ to λ if and only if there is one from $\kappa \times \lambda$ to λ^+.*

Proof
Again, the reverse direction follows from Remark 1.4. For the forward direction, let $F : \kappa \times \lambda \to \lambda$ be a universal function and fix bijections $j_\alpha : \lambda \to \alpha$, for each α in the interval $[\lambda, \lambda^+)$. Construct F' with the property that for each pair $\alpha \in \kappa, \beta \in \lambda^+$ there exists a $\gamma < \kappa$ such that $F'_\gamma = j_\beta \circ F_\alpha$, i.e.,

$$F'(\gamma, \delta) = j_\beta(F(\alpha, \delta)) \text{ for all } \delta < \lambda.$$

Now we verify that F' is universal. Let $f' : \kappa \times \lambda \to \lambda^+$ be arbitrary. For each $\alpha < \kappa$, let

$$\iota_\alpha = \lambda + \sup\{f'(\alpha, \delta) + 1\ :\ \delta < \lambda\}.$$

Define f into λ by $f(\alpha, \delta) = j^{-1}_{\iota_\alpha}(f'(\alpha, \delta))$. Since F is universal there exist g, h with

$$F(g(\alpha), h(\delta)) = f(\alpha, \delta) = j^{-1}_{\iota_\alpha}(f'(\alpha, \delta))$$

for all $(\alpha, \delta) \in \kappa \times \lambda$. By our definition of F' we may construct g' so that

$$F'(g'(\alpha), h(\delta)) = j_{\iota_\alpha}(F(g(\alpha), h(\delta)))$$

for all $(\alpha, \delta) \in \kappa \times \lambda$. Then we are done, since

$$j_{\iota_\alpha}(F(g(\alpha), h(\delta))) = f'(\alpha, \delta)$$

for all $(\alpha, \delta) \in \kappa \times \lambda$.

QED

The following theorem connects the existence of universal functions on $\kappa \times \kappa$ with finite range to the existence of universal graphs.

Theorem 5.9. *For any infinite cardinal κ the following are equivalent:*

(1) *For each $n \in \mathbb{N}$ there is a universal function from $\kappa \times \kappa$ to n.*
(2) *For some $n \in \mathbb{N}$ with $n \geq 2$ there is a universal function from $\kappa \times \kappa$ to n.*
(3) *There is a symmetric, irreflexive function from $\kappa \times \kappa$ to 2 universal for all symmetric, irreflexive functions from $\kappa \times \kappa$ to 2; in other words, there is a symmetric, irreflexive function $U : \kappa \times \kappa \to 2$ such that for any symmetric, irreflexive function $f : \kappa \times \kappa \to 2$ there is $h : \kappa \to \kappa$ such that $f(\xi, \eta) = U(h(\xi), h(\eta))$ for all ξ and η.*
(4) *There is a universal graph on κ; in other words, there is a graph G whose vertex set is κ such that for any other graph G^* with vertex set κ there is a graph embedding of G^* into G.*

Proof
Statement (1) clearly implies statement (2); (2) implies (1) by Remark 1.4 and Proposition 1.5 (with finite exponent). To get from (2) to (3), given a universal function $U : \kappa \times \kappa \to n$, define

$$V(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha \neq \beta \text{ and either } U(\alpha, \beta) = 1 \text{ or } U(\beta, \alpha) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Then V is universal in the sense of (3).
Statements (3) and (4) are equivalent since the characteristic function of a graph is symmetric and irreflexive and Remark 1.2 is still in effect.

It remains only to show that one of (3) and (4) implies one of (1) and (2). We give a proof of (1) from (3). Fix $n \in \omega$ with $n \geq 2$, and let $\Psi : \kappa \to [\kappa]^n$ be a bijection. If $U : \kappa \times \kappa \to 2$ is a function universal for symmetric functions, define $U^* : \kappa \times \kappa \to n$ by letting $U^*(\alpha, \beta)$ be

$$\left(\max_{\xi \in \Psi(\alpha)} \sum_{\eta \in \Psi(\beta)} U(\xi, \eta) \right) - 1$$

if this quantity is nonnegative, and 0 otherwise. To see that U^* is universal for functions from $\kappa \times \kappa$ to n, fix $F \colon \kappa \times \kappa \to n$. Let $\{E_\xi\}_{\xi \in \kappa}$ be a partition of κ into sets of size n. We construct a symmetric function F^* from $\kappa \times \kappa$ to 2 so that for each $\alpha, \beta \in \kappa$

$$F(\alpha, \beta) = \left(\max_{\xi \in E_\alpha} \sum_{\eta \in E_\beta} F^*(\xi, \eta) \right) - 1.$$

We can realize F^* as the characteristic function of a graph. For each $\alpha \in \kappa$, let α^* denote $\min(E_\alpha)$. Construct the graph on each of the pairwise disjoint pieces $E_\alpha \times E_\beta$ by connecting α^* to $F(\alpha, \beta)+1$ elements of E_β including β^* and connecting β^* to exactly $F(\beta, \alpha) + 1$ elements of E_α. The "plus one" is so that the two minimum elements can always be connected.

Let $g : \kappa \to \kappa$ be one-to-one (see Remark 1.2) such that

$$U(g(\alpha), g(\beta)) = F^*(\alpha, \beta)$$

for all α, β in κ. Define $H : \kappa \to \kappa$ by setting each $H(\xi) = \Psi^{-1} g[E_\xi]$. Then for each α, β in κ,

$$F(\alpha, \beta) = (\max_{\xi \in E_\alpha} \sum_{\eta \in E_\beta} F^*(\xi, \eta)) - 1 = (\max_{\xi \in E_\alpha} \sum_{\eta \in E_\beta} U(g(\xi), g(\eta))) - 1,$$

which, as g is one-to-one, is equal to $(\max_{\xi \in g[E_\alpha]} \sum_{\eta \in g[E_\beta]} U(\xi, \eta)) - 1$. Since this last term is nonnegative, it is equal to $U^*(H(\alpha), H(\beta))$. QED

Question 5.10. *Does the existence of a universal $F : \omega_1 \times \omega_1 \to 2$ imply that there is a universal function $G : \omega_1 \times \omega_1 \to \omega$?*

The rest of this section concerns universal functions on $\omega_1 \times \omega_1$.

Remark 5.11. *Shelah* [24, 25, 26] *proved that it is consistent with $\mathfrak{c} > \aleph_1$ that there is a universal graph on ω_1. By Theorem 5.9, in his model there are universal functions from $\omega_1 \times \omega_1$ to n for each $n < \omega$.*

Shelah's result was generalized in Mekler ([16], Theorem 2). In Mekler's terminology, a 3-*amalgamation class* K is a class of models of a universal theory in a relational language which satisfies the following amalgamation property: if $\{M_a : a \in \mathcal{P}^-(3)\}$ are structures in K for which $M_a \cap M_b = M_{a \cap b}$ for all $a, b \in \mathcal{P}^-(3)$, then there is an $M \in K$ such that $M_a \subseteq M$ for each $a \in \mathcal{P}^-(3)$, where $\mathcal{P}^-(3)$ denotes the set $\mathcal{P}(3) \setminus \{3\}$.

Theorem 5.12 (Mekler [16]). *If $2^{\aleph_1} = \aleph_2$, then there is a c.c.c. partial order forcing that that $\mathfrak{c} = \aleph_2$ and that for every 3-amalgamation class K having only countably many finite models up to isomorphism, there is a model M in K of cardinality $\aleph_1$ such that every model in K of cardinality $\aleph_1$ is isomorphic to a substructure of M.*

The following theorem is a corollary of Mekler's result.

Theorem 5.13. *It is consistent that $2^{\aleph_0} > \aleph_1$ and there is a universal function from $\omega_1 \times \omega_1$ to ω_1.*

Proof
By Proposition 5.7 it is enough to find a universal function from $\omega_1 \times \omega_1$ to ω. Let L be the language with countably many binary predicate symbols $R_n(x, y)$. Let T be the theory with countably many axioms:

$$\forall x, y \ \ (R_n(x, y) \rightarrow \neg R_m(x, y))$$

for each $n \neq m$. Note that T has only countably many finite models up to isomorphism and is axiomatized by universal sentences.

To verify that the class of models of T satisfies the amalgamation property of 3-amalgamation classes, note that if M_a $(a \in \mathcal{P}^-(3))$ are models of T such that $M_a \cap M_b = M_{a\cap b}$ for all $a, b \in \mathcal{P}^-(3)$, then $\bigcup\{M_a : a \in \mathcal{P}^-(3)\} \models T$.

Suppose now that $(\omega_1, \{R_n\}_{n<\omega})$ is a universal model of T. Define a function $U : \omega_1^2 \rightarrow \omega$ by

$$U(\alpha, \beta) = \begin{cases} n & \text{if } R_n(\alpha, \beta) \\ 0 & \text{if } \forall n \ \neg R_n(\alpha, \beta) \end{cases}$$

Now given any $g : \omega_1^2 \rightarrow \omega$ define $R_n^g(\alpha, \beta)$ if and only if $g(\alpha, \beta) = n$. The structure $(\omega_1, \{R_n^g\}_{n<\omega})$ is a model of T. An embedding of this structure into our universal model gives a map $h : \omega_1 \rightarrow \omega_1$ such that $g(\alpha, \beta) = U(h(\alpha), h(\beta))$ for all $\alpha, \beta < \omega_1$.
QED

We take up the question of model-theoretic universality in section 7. Next we consider the problem of universal functions on ω_1 assuming Martin's Axiom.

Theorem 5.14. *Assume MA_{ω_1}. Then there exists $F : \omega_1 \times \omega \rightarrow \omega_1$ which is universal.*

Proof
By Proposition 5.8 we need only produce a universal $F : \omega_1 \times \omega \rightarrow \omega$.

Standard arguments show that there exists a family $h_\alpha : \omega \rightarrow \omega$ for $\alpha < \omega_1$ of independent functions, i.e., for any n, $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \omega_1$ and $s : \{1, \ldots, n\} \rightarrow \omega$ there are infinitely many $k < \omega$ such that

$$h_{\alpha_1}(k) = s(1)$$

$$h_{\alpha_2}(k) = s(2)$$

$$\vdots$$

$$h_{\alpha_n}(k) = s(n).$$

Define $H : \omega_1 \times \omega \to \omega$ by $H(\alpha, n) = h_\alpha(n)$. We show that H is universal mod finite, in sense which will be made clear. Given any $f : \omega_1 \times \omega \to \omega$ define the following poset $\mathbb{P}$. A condition $p = (s, F)$ is a pair such that $s \in \omega^{<\omega}$ is one-to-one and $F \in [\omega_1]^{<\omega}$. We define $p \leq q$ if and only if

(1) $s_q \subseteq s_p$,
(2) $F_q \subseteq F_p$, and
(3) $f(\alpha, n) = h_\alpha(s_p(n))$ for every $\alpha \in F_q$ and $n \in \mathrm{dom}(s_p) \backslash \mathrm{dom}(s_q)$.

The poset $\mathbb{P}$ is c.c.c., and in fact σ-centered, since any two conditions with the same s are compatible. Since the family $\{h_\alpha : \alpha < \omega_1\}$ is independent, for any $p \in \mathbb{P}$ there are extensions of p with arbitrarily long s part. It follows from MA_{ω_1} that there exists $h : \omega \to \omega$ with the property that for every $\alpha < \omega_1$ for all but finitely many n that $f(\alpha, n) = h_\alpha(h(n))$.

To get a universal map $F : \omega_1 \times \omega \to \omega$, simply take any F with the property that for every $\alpha < \omega_1$ and any $h' =^* h_\alpha$ (equal mod finite) there is β such that $F(\beta, n) = h'(n)$ for every n. Since the function h is one-to-one, it easy to find $k : \omega_1 \to \omega_1$ such that $F(k(\alpha), h(n)) = f(\alpha, n)$ for all α and n.
QED

Theorem 5.17 below shows that the existence of a universal function for ω_1 does not follow from Martin's Axiom. First we have the following lemmas, the first of which follows from a simple modification of the Sierpiński partition sending pairs of reals to some rational between.

Lemma 5.15. *There exists $S : [\omega_1]^2 \to \omega$ such that*

- *for all uncountable $X \subseteq \omega_1$ and $j \in \omega$ there is an uncountable $Z \subseteq X$ such that $S(p) > j$ for all $p \in [Z]^2$*
- *for all ξ the restriction of the mapping $\eta \mapsto S(\{\xi, \eta\})$ to ξ is one-to-one.*

Proof
Let $\{r_\xi\}_{\xi \in \omega_1}$ enumerate any uncountable set of reals and let $\mathbb{Q} = \{q_n\}_{n \in \omega}$. Any function $S : [\omega_1]^2 \to \omega$ satisfying $q_{S(\xi,\eta)}$ falls between r_ξ and r_η will satisfy the first requirement because, given j and X there is a $\subseteq$-minimal interval J with endpoints in $\{q_i, q_2, \ldots q_j\}$ such that $Z = J \cap X$ is uncountable. It is immediate that $S(p) > j$ for all $p \in [Z]^2$. Any easy inductive argument then yields an S satisfying both requirements.
QED

Lemma 5.16. *For each $r \in 2^\omega$, let $G_r\colon \omega_1 \times \omega_1 \to \omega$ be defined by setting $G_r(\eta, \xi) = r(S(\{\eta, \xi\}))$. Fix $U : \omega_1 \times \omega_1 \to \omega$, and let $H \in 2^\omega$ be Hechler generic over a model V containing U. Then, in $V[H]$, any partial order $\mathbb{P}$ such that*

$$1 \Vdash_{\mathbb{P}} \text{“}(\exists h : \omega_1 \to \omega_1)(\forall \eta)(\forall \xi \neq \eta)\ U(h(\eta), h(\xi)) = G_H(\eta, \xi)\text{”}$$

contains an uncountable antichain.

Proof

Let h be a name such that

$$1 \Vdash_{\mathbb{P}} \text{“}(\forall \eta)(\forall \xi \neq \eta)\ U(h(\eta), h(\xi)) = G_H(\eta, \xi)\text{”}.$$

For each $\xi \in \omega_1$, choose a condition (t_ξ, F_ξ) in Hechler forcing $\mathbb{H}$, a name p_ξ for an element of $\mathbb{P}$ and an ordinal α_ξ such that

$$((t_\xi, F_\xi), p_\xi) \Vdash_{\mathbb{H}\star\mathbb{P}} \text{“}h(\check{\xi}) = \check{\alpha}_\xi\text{”}.$$

Let $t : n \to \omega$ be such that there is an uncountable set $X \subseteq \omega_1$ such that $t_\xi = t$ for all $\xi \in X$. Let $Z \subseteq X$ be uncountable such that $S(w) > n$ for all $w \in [Z]^2$.

For each $\xi \in \omega_1$ let D_ξ be the partial function from ω to ω with domain $\{S(\{\xi, \eta\}) : \eta < \xi\}$ defined by setting

$$D_\xi(S(\{\xi, \eta\})) = U(\alpha_\xi, \alpha_\eta) + 1.$$

This is well defined by the one-to-one property of S. By extending the second coordinate of $(t_\xi, F_\xi) = (t, F_\xi)$ it may be assumed that $F_\xi(j) \geq D_\xi(j)$ for all $j \in \mathrm{dom}(D_\xi) \cap \mathrm{dom}(F_\xi)$. Now if ξ and η belong to Z and $\eta < \xi$ then $((t, F_\xi), p_\xi)$ and $((t, F_\eta), p_\eta)$ are incompatible.

To see this suppose that a condition $((s, F), p)$ were stronger than both $((t, F_\xi), p_\xi)$ and $((t, F_\eta), p_\eta)$. By extending (s, F), it may be assumed that $S(\{\xi, \eta\}) \in \mathrm{dom}(s)$. Since $\{\xi, \eta\} \subseteq Z$, each value $S(\{\xi, \eta\}) > n$, so $S(\{\xi, \eta\}) \in \mathrm{dom}(s) \setminus \mathrm{dom}(t)$. Hence, $((s, F), p)$ as an element of $\mathbb{H} \star \mathbb{P}$ forces the following

$$\begin{aligned} s(S(\{\xi,\eta\})) &\geq F_\xi(S(\{\xi,\eta\})) \geq D_\xi(S(\{\xi,\eta\})) > U(\alpha_\xi, \alpha_\eta) \\ &= U(h(\xi), h(\eta)) = G_H(\xi,\eta) = H(S(\{\xi,\eta\})) = s(S(\{\xi,\eta\})) \end{aligned}$$

yielding a contradiction. Since $\mathbb{H}$ has the c.c.c. it follows that $\mathbb{P}$ does not.

QED

Lemma 5.16 implies the following.

Theorem 5.17. *In the standard model of MA obtained by forcing over a model of GCH with a finite support iteration of length ω_2 of c.c.c. posets, there is no universal function from $\omega_1 \times \omega_1$ to ω.*

5.1. **Property R.** We conclude this section by connecting the existence of universal functions from $\omega_1 \times \omega_1$ to ω with the existence of certain functions on pairs from ω_1.

Definition 5.18. *A function* $\Phi\colon [\omega_1]^2 \to \omega$ *has* Property R *if*

- *whenever* $k \in \omega$ *and* $\{\{a_\xi, b_\xi\} : \xi \in \omega_1\}$ *is a family of disjoint pairs from* ω_1 *with each* $a_\xi \leq b_\xi$*, there are distinct* ξ *and* η *such that* $\Phi(\{a_\xi, a_\eta\}) \geq \Phi(\{b_\xi, b_\eta\}) \geq k$*;*
- *for each* $\xi \in \omega_1$ *and* $k \in \omega$ *there are only finitely many* $\eta \in \xi$ *such that* $\Phi(\{\xi, \eta\}) = k$*.*

Functions with similar properties appear in Theorem 6 of [28].

Given a sequence $\langle \sigma_\alpha : \alpha < \omega_1 \setminus \omega \rangle$ such that each σ_α is an infinite set of pairwise disjoint pairs from α, one can recursively define a function $\Phi\colon [\omega_1]^2 \to \omega$ with the property that for each $\eta < \omega_1$,

- for all distinct $\zeta, \rho < \eta$, $\Phi(\{\zeta, \eta\}) \neq \Phi(\{\rho, \eta\})$;
- for all $\zeta \in [\omega, \eta)$, all $\alpha \in [\omega, \zeta]$ and all $k \in \omega$, there exist $\delta < \gamma$ such that $\{\delta, \gamma\} \in \sigma_\alpha$ and $\Phi(\{\delta, \zeta\}) = \Phi(\{\gamma, \eta\}) \geq k$.

It follows that if $\diamondsuit$ holds, and is exemplified by $\langle \sigma_\alpha : \alpha < \omega_1 \setminus \omega \rangle$, then functions with Property R exist. It can be verified in a straightforward manner that functions with Property R are preserved by forcing by partial orders satisfying Knaster's condition (i.e., for which every uncountable set of conditions has an uncountable pairwise compatible subset). The existence of a function with Property R is then consistent with the statement $\mathfrak{b} > \aleph_1$.

Proposition 5.19. *If* $\mathfrak{b} > \aleph_1$ *and there exists a function* $\Phi\colon [\omega_1]^2 \to \omega$ *with Property R then there is no universal function from* $\omega_1 \times \omega_1$ *to* ω*.*

Proof

Let $U : \omega_1 \times \omega_1 \to \omega$ be given. Define $F_\xi : \omega \to \omega$ for each $\xi < \omega_1$ by setting $F_\xi(m)$ to be the largest member of the finite set

$$\{U(\xi, \eta) + 1 \mid \eta < \xi \wedge \Phi(\{\xi, \eta\}) = m\} \cup \{0\}.$$

Let $F : \omega \to \omega$ be a non-decreasing function such that $F \geq^* F_\xi$ for all ξ. Define $G : \omega_1 \times \omega_1 \to \omega$ by setting $G(\xi, \eta) = F(\Phi(\{\xi, \eta\}))$. Fixing an injection $h\colon \omega_1 \to \omega_1$, we will find ξ and η in ω_1 such that $G(\xi, \eta) \neq U(h(\xi), h(\eta))$.

Let $Z \in [\omega_1]^{\aleph_1}$ be such that $h(\xi) \geq \xi$ for all $\xi \in Z$. Choose k and $X \in [Z]^{\aleph_1}$ such that $F(j) \geq F_{h(\xi)}(j)$ for all $\xi \in X$ and $j \geq k$. Since $h(\xi) \geq \xi$ for all $\xi \in X$, it is possible to choose $\xi > \eta$ in X such that $h(\xi) > h(\eta)$ and $k \leq \Phi(\{h(\xi), h(\eta)\}) \leq \Phi(\{\xi, \eta\})$. It follows that

$$G(\xi, \eta) = F(\Phi(\{\xi, \eta\})) \geq F(\Phi(\{h(\xi), h(\eta)\})),$$

and that

$$F(\Phi(\{h(\xi),h(\eta)\})) \geq F_{h(\xi)}(\Phi(\{h(\xi),h(\eta)\}) > U(h(\xi),h(\eta)),$$

contradicting that h is an embedding.
QED

The argument of Proposition 5.19 shows that if $\mathfrak{b} > \aleph_1$ and Property R holds then there are no universal functions from $\omega_1 \times \omega_1 \to \omega$, but it does not rule out the existence of a universal functions from $\omega_1 \times \omega_1 \to 2$. The following result of Saharon Shelah addresses this question.

Definition 5.20. *A graph (V,E) is said to be universal (for $\aleph_1$) if given any graph (U,F) such that $|U| = \aleph_1$ there is a function $\Phi : U \to V$ such that $\{x,y\} \in F$ if and only if $\{\Phi(x),\Phi(y)\} \in E$. The function Φ will be called an embedding in this case.*

Theorem 5.21 (Shelah)**.** *Assuming the following two hypotheses:*

(1) *For every $\mathcal{F} \subseteq [\omega_1^{\omega_1}]^{2^{\aleph_0}}$ there exist two functions f and g in $\mathcal{F}$ such that $\{\xi \in \omega_1 \mid f(\xi) = g(\xi)\}$ is stationary.*
(2) *There exist f_ξ for every limit ordinal $\xi \in \omega_1$ such that*
   - *$f_\xi : \omega \to \xi$ is increasing and cofinal in ξ*
   - *for every club $C \subseteq \omega_1$ there is a club X such that for each $\xi \in X$ there is some n such that $f_\xi(k) \in C$ for all $k \geq n$.*

there is no universal graph on ω_1.

Proof
Suppose that U is a universal graph on ω_1. For any $r : \omega \to 2$ define the graph G_r to consist of all edges $\{\xi,\eta\}$ such that there exists $n \in \omega$ such that $r(n) = 1$ and $f_\eta(n) \leq \xi < f_\eta(n+1)$. Since U is universal there exist embeddings $h_r : \omega_1 \to \omega_1$ of G_r into U.

Now let $A \subseteq 2^\omega$ be any set of size $2^{\aleph_0}$ consisting of reals any two of which differ on an infinite set. The first hypothesis yields r and s in A such that that $E = \{\xi \in \omega_1 \mid h_r(\xi) = h_s(\xi)\}$ is stationary. Then let C be any club such that if α and β are in C and $\alpha \in \beta$ then $E \cap [\alpha,\beta) \neq \emptyset$.

The second hypothesis then yields $\xi \in E$ and $n \in \omega$ such hat $f_\xi(k) \in C$ for all $k \geq n$. Choose $k \geq n$ such that $r(k) = 1 \neq s(k)$ and let $\eta \in E$ be such that $f_\xi(k) \leq \eta < f_\xi(k+1)$. Then $\{\eta,\xi\}$ is an edge of G_r but not of G_s contradicting that $h_s(\eta) = h_r(\eta)$ and $h_s(\xi) = h_r(\xi)$.

Corollary 5.22. *It is consistent with MA that there is no universal graph on ω_1.*

Proof
Begin with model of $\diamondsuit$ and GCH and force with ccc partial order $\mathbb{P}$ of cardinality $\aleph_4$ to obtain a model of MA and $2^{\aleph_0} = \aleph_4$. The second

hypothesis of Theorem 5.21 is true because it holds in the ground model satisfying $\diamondsuit$ and clubs in the forcing extension contain clubs in the ground model.

To see that the first hypothesis is true, let $\{\dot{f}_\mu\}_{\mu\in\omega_4}$ be $\mathbb{P}$-names for functions from ω_1 to ω_1. For each $\mu \in \omega_4$ choose a function $w_\mu : \omega_1 \to \omega_1$ and conditions $p_{\mu,\xi} \in \mathbb{P}$ such that

$$p_{\mu,\xi} \Vdash_{\mathbb{P}} \text{“}\dot{f}_\mu(\xi) = w_\mu(\xi)\text{”}$$

for all $\xi \in \omega_1$. For each pair $\mu \neq \theta$ let $\dot{C}_{\mu,\theta}$ be a $\mathbb{P}$-name for a club such that $1 \Vdash_{\mathbb{P}}$ “$(\forall\xi \in \dot{C}_{\mu,\theta})\dot{f}_\mu(\xi) \neq \dot{f}_\theta(\xi)$”. Because $\mathbb{P}$ is ccc there is a club $D_{\mu,\theta}$ in the ground model such that $1 \Vdash_{\mathbb{P}}$ “$D_{\mu,\theta} \subseteq \dot{C}_{\mu,\theta}$”.

First let $E \subseteq \omega_4$ be of cardinality $\aleph_4$ such that there is a function w such that $w_\mu = w$ for all $\mu \in E$. Since the ground model satisfies $\aleph_4 \to [\aleph_1]^2_{\aleph_2}$ it follows that there is an uncountable set $B \subseteq E$ and a club D such that $D_{\mu,\theta} = D$ for $\{\mu,\theta\} \in [B]^2$. Let $\delta \in D$. Then since $\mathbb{P}$ is ccc there are distinct μ and θ in B such that there is $p \in \mathbb{P}$ such that $p \leq p_{\mu,\delta}$ and $p \leq p_{\theta,\delta}$. This contradicts that $\delta \in D$ and $p \Vdash_{\mathbb{P}}$ “$w(\xi) = w_\mu(\xi) = \dot{f}_\mu(\xi) \neq \dot{f}_\theta(\xi) = w_\theta(\xi) = w(\xi)$”.

Remark 5.23. *Justin Moore has shown that under the Proper Forcing Axiom there are no functions with property R — his argument is included in the appendix to this article. Justin Moore and Stevo Todorcevic have independently indicated to the authors that the existence of a function with Property R follows from the assumption that $\mathfrak{b} = \aleph_1$.*

6. Higher dimensional universal functions

Definition 6.1. *Given $k \in \omega$ and sets X_i $(i < k)$ and Z, a function $F\colon \prod_{i<k} X_i \to Z$ is* universal *if for each function $G\colon \prod_{i<k} X_i \to Z$ there exist functions $h_i\colon X_i \to X_i$ $(i < k)$ such that*

$$G(x_0,\ldots,x_{k-1}) = F(h_0(x_0),\ldots,h_{k-1}(x_{k-1}))$$

for all $(x_0,\ldots,x_{k-1}) \in \prod_{i<k} X_i$.

As in Remark 1.1, in the case where the X_i's are all the same set X, the existence of a universal function is not changed by requiring that the functions h_i are all the same.

Given a set X and a $k \in \omega$, we call a universal function $F\colon X^k \to X$ a *k-dimensional universal function on X*. The following proposition shows that the existence of a 2-dimensional universal function on an infinite set X is equivalent to the existence of a k-dimensional universal function, for any $k > 1$. Note however that the Baire complexity of $F(F(x,y),z)$ can be higher than that of F.

Proposition 6.2. *Let X, Y be sets such that $|X \times Y| = |X|$. If*

$$F\colon X \times Y \to X$$

is a universal function, then the function $F'\colon X \times Y \times Y \to X$ defined by setting $F'(x, y, z) = F(F(x, y), z)$ is a universal function.

Proof
Fix functions $\pi_0\colon X \to X$ and $\pi_1\colon X \to Y$ such that the function $\pi\colon X \to X \times Y$ defined by setting $\pi(x) = (\pi_0(x), \pi_1(x))$ is a bijection. Given $G\colon X \times Y \times Y \to X$, define $G_0\colon X \times Y \to X$ by setting $G_0(x, z) = G(\pi_0(x), \pi_1(x), z)$. By the universality of F there exist functions $g\colon X \to X$ and $h\colon Y \to Y$ such that $G_0(u, z) = F(g(u), h(z))$ for all $(u, z) \in X \times Y$. Again by the universality of F there are functions $g_0\colon X \to X$ and $g_1\colon Y \to Y$ such that

$$g(\pi^{-1}(x, y)) = F(g_0(x), g_1(y))$$

for all $(x, y) \in X \times Y$. Then for all $(x, y, z) \in X \times Y \times Y$,

$$G(x, y, z) = G_0(\pi^{-1}(x, y), z) = F(g(\pi^{-1}(x, y)), h(z)),$$

which is equal to $F(F(g_0(x), g_1(y)), h(z))$.
QED

One may also consider universal functions F where the parameterizing functions take in more than one variable, for example, a function $F\colon X^3 \to Y$ such that for all $G\colon X^3 \to Y$ there exist functions g, h and k from X^2 to X such that $G(x, y, z) = F(g(x, y), h(y, z), k(x, z))$ for all x, y, z in X. A 3-dimensional universal function is universal in this sense, since g, h and k can be chosen to each depend on only one variable. However, we do not know if a universal function in this sense implies the existence of a 3-dimensional universal function.

The reader will easily be able to imagine many variants. For example,

- $G(x, y, z) = F(g(x, y), h(y, z))$;
- $G(x_1, x_2, x_3, x_4) = F(g_1(x_1, x_2), g_2(x_2, x_3), g_3(x_3, x_4), g_4(x_4, x_1))$;

where we have omitted quantifiers for clarity. These two variants are each equivalent to the existence of 2-dimensional universal function. To see this in the first example put $y = 0$ and get

$$G(x, z) = F(g(x, 0), h(0, z)).$$

In the second example put $x_2 = x_4 = 0$ and get

$$G(x_1, x_3) = F(g_1(x_1, 0), g_2(0, x_3), g_3(x_3, 0), g_4(0, x_1)).$$

More generally, suppose that F and the $\vec{x_k}$'s have the property that for every G there are g_k's such that for all $\vec{x}$

$$G(\vec{x}) = F(g_1(\vec{x}_1), \dots, g_n(\vec{x}_n)).$$

Suppose that there are two variables x and y from $\vec{x}$ which do not simultaneously belong to any $\vec{x}_k$. Then we get a universal 2-dimensional function simply by setting all of the other variables equal to zero.

For the rest of this section we will often leave implicit the domains of our universal functions, for notional ease. When we talk of the complexity of universal functions, however, the underlying domain space is taken to be 2^ω.

Definition 6.3. *Given $n \in \omega \setminus \{0,1\}$, an $(n,2)$-dimensional universal function is an $\binom{n}{2}$-ary function F such that for every n-ary function G there is a binary function h with*

$$G(x_1, x_2, \dots, x_n) = F(\langle h(x_i, x_j) : 1 \leq i < j \leq n \rangle)$$

for all $x_1, x_2, \dots, x_n$.

Proposition 6.4. *If there is a $(3,2)$-dimensional universal function, then for every $n > 3$ there is a $(n,2)$-dimensional universal function F. Conversely, if there is a $(n+1,2)$-dimensional universal function for some $n \geq 3$, then there is an $(n,2)$-dimensional universal function.*

Proof
Suppose that F is a $(3,2)$-dimensional universal function and F' is an $(n,2)$-dimensional universal function, for some $n \geq 3$. Given an $(n+1)$-ary function $G(x_1, \dots, x_{n+1})$, for each fixed w we get a function $h_w(x_1, \dots, x_n)$ with

$$G(x_1, \dots, x_n, w) = F'(\langle h_w(x_i, x_j) : 1 \leq i < j \leq n \rangle)$$

for all $x_1, \dots, x_n$. Now, considering $h(y_1, y_2, y_3) = h_{y_3}(y_1, y_2)$ we get a function $k(s,t)$ with $h(y_1, y_2, y_3) = F(\langle k(y_i, y_j) : 1 \leq i < j \leq 3 \rangle)$ for all y_1, y_2, y_3. Then, for all $x_1, \dots, x_{n+1}$, $G(x_1, \dots, x_{n+1}) =$

$$F'(\langle F(k(x_i, x_j), k(x_i, x_{n+1}), k(x_j, x_{n+1})) : 1 \leq i < j \leq n \rangle).$$

From this one gets an $(n+1, s)$-dimensional universal function, with k playing the role of h in the definition.

For the converse, suppose that F is an $(n+1,2)$-dimensional universal function. Consider F as a function of variables $p_{i,j}$ $(1 \leq i < j \leq n+1)$, and, fixing a tripling function $\langle \cdot, \cdot, \cdot \rangle$, let u_1, u_2 and u_3 be functions so that $u_i(\langle x_1, x_2, x_3\rangle) = x_i$ for all x_1, x_2, x_3 and $i \in \{1, 2, 3\}$. Let K be the function which takes in a sequence

$$\langle q_{i,j} : 1 \le i < j \le n\rangle$$

and returns the sequence

$$\langle p_{i,j} : 1 \le i < j \le n+1\rangle$$

for which $p_{i,j}$ is

- $u_1(q_{i,j})$ if $j \le n$;
- $u_2(q_{i,i+1})$ if $i < n$ and $j = n+1$;
- $u_3(q_{n-1,n})$ if $i = n$ and $j = n+1$.

Define F' by setting

$$F'(\langle q_{i,j} : 1 \le i < j \le n\rangle) = F(K(\langle q_{i,j} : 1 \le i < j \le n\rangle)).$$

Given an n-ary function G, there exists a binary function h with

$$G(x_1, \ldots, x_n) = F(\langle h(x_i, x_j) : 1 \le i < j \le n+1\rangle)$$

for all $x_1, \ldots, x_{n+1}$. Fix a domain element w, and define a function h' by setting $h'(x, y) = \langle h(x, y), h(x, w), h(y, w)\rangle$. Then

$$G(x_1, \ldots, x_n) = F'(\langle h'(x_i, x_j) : 1 \le i < j \le n\rangle)$$

for all $x_1, \ldots, x_n$.
QED

Next we state a generalization of these ideas.

Definition 6.5. *Let X be a set, and n an element of ω. Suppose that $\Sigma \subseteq \mathcal{P}(\{0, 1, 2, \ldots, n-1\}) = \mathcal{P}(n)$ (the power set of n). We let $U(X, n, \Sigma)$ be the assertion that there exists a function $F : X^\Sigma \to X$ such that for every $G : X^n \to X$ there are $h_Q : X^Q \to X$ for $Q \in \Sigma$ such that*

$$G(x_0, x_1, \ldots, x_{n-1}) = F\left(\langle h_Q(\langle x_j : j \in Q\rangle) : Q \in \Sigma\rangle\right)$$

for all $x_0, \ldots, x_{n-1} \in X$.

Propositions 6.2 and 6.4 can be generalized as follows. The proofs of the first two parts of Proposition 6.6 are similar to the corresponding proofs from Proposition 6.4.

Proposition 6.6. *For any infinite set X and any positive integer n,*

1. *$U(X, n+1, [n+1]^n)$ implies $\forall m > n \ \ U(X, m, [m]^n)$;*
2. *$\exists m > n \ \ U(X, m, [m]^n)$ implies $U(X, n+1, [n+1]^n)$;*
3. *$U(X, n+1, [n+1]^n)$ implies $U(X, n+2, [n+2]^{n+1})$.*

Proof

For the first part, we follow the proof of the first part of Proposition 6.4, inducting on m. Fix $m > n$, and suppose that F_0 witnesses the statement $U(X, n+1, [n+1]^n)$ and F_1 witnesses $U(X, m, [m]^n)$. Given $G\colon X^{m+1} \to X$, we can find for each $w \in X$ functions h^w_Q ($Q \in [m]^n$) such that, for all $x_0, \ldots, x_{m-1} \in X$,

$$G(x_0, \ldots, x_{m-1}, w) = F_1\left(\langle h^w_Q(\langle x_j : j \in Q\rangle) : Q \in [m]^n\rangle\right).$$

Furthermore, there are functions k_R ($R \in [n+1]^n$) such that (abusing notation slightly on the left side of the equality)

$$h^{x_n}_Q(\langle x_i : i < n\rangle) = F_0(\langle k_R(\langle x_i : i \in R\rangle) : R \in [n+1]^n\rangle)$$

for all $x_0, \ldots, x_n \in X$. The functions k_R witness in this instance then that the function $F_1(\langle F_0(\langle y_R : R \in [Q \cup \{m\}]^n\rangle) : Q \in [m]^n\rangle)$ witnesses $U(X, m+1, [m+1]^n)$.

For the second part, we follow the proof of the second part of Proposition 6.4. Suppose that F witnesses $U(X, m, [m]^n)$. Fix a bijection $\pi\colon X^{([m]^n)} \to X$, and let p_R ($R \in [m]^n$) be the functions from X to X such that $\pi^{-1}(x) = \langle p_R(x) : R \in [m]^n\rangle$. Let $H\colon [m]^n \to [n+1]^n$ be such that $R \cap (n+1) \subseteq H(R)$ for all $R \in [m]^n$. Let K be the function which takes in a sequence

$$\langle y_Q : Q \in [n+1]^n\rangle$$

from X and returns the sequence

$$\langle p_R(y_{H(R)}) : R \in [m]^n\rangle.$$

Define $F'\colon X^{([n+1]^n)} \to X$ by setting

$$F'(\langle y_Q : Q \in [n+1]^n\rangle) = F(K(\langle y_Q : Q \in [n+1]^n\rangle)).$$

To see that this works, fix $G\colon X^{n+1} \to X$. Since F witnesses $U(X, m, [m]^n)$, there exist functions $h_R\colon X^R \to X$ ($R \in [m]^n$) such that

$$G(x_0, \ldots, x_n) = F(\langle h_R(\langle x_i : i \in R\rangle) : R \in [m]^n\rangle)$$

for all $x_0, \ldots, x_{m-1} \in X$. Fix $w \in X$. For each $Q \in [n+1]^n$, let $k_Q\colon X^Q \to X$ be defined by setting $k_Q(\langle x_i : i \in Q\rangle)$ to be

$$\pi(\langle h_R(\langle t_i : i \in R\rangle) : R \in [m]^n\rangle),$$

where each t_i is x_i if $i \in Q$, and w otherwise. To check that the functions k_Q witness that F' is as desired, it suffices to see that for all $x_0, \ldots, x_{m-1}$ in X for which $x_i = w$ for all $i \in \{n+1, \ldots, m-1\}$, and all $R \in [m]^n$, $h_R(\langle x_i : i \in R\rangle)$ is equal to $p_R(k_{H(R)}(\langle x_i : i \in H(R)\rangle))$. Now, $p_R(k_{H(R)}(\langle x_i : i \in H(R)\rangle))$ is $h_R(\langle t_i : i \in R\rangle)$, where each t_i is x_i

if $i \in H(R)$ and w otherwise. Since $R \cap (n+1) \subseteq H(R)$, $\langle t_i : i \in R\rangle = \langle x_i : i \in R\rangle$, as desired.

For the third part, we follow (loosely) the proof of Proposition 6.2. Suppose that F witnesses $U(X, n{+}1, [n{+}1]^n)$. Let $F' \colon X^{([n+2]^{n+1})} \to X$ be such that

$$F'(\langle z_Q : Q \in [n+2]^{n+1}\rangle) = F(\langle y_R : R \in [n+1]^n\rangle),$$

where each $y_R = z_{R\cup\{n+1\}}$. Fix functions $\pi_0 \colon X \to X$ and $\pi_1 \colon X \to X$ such that the function $\pi \colon X \to X \times X$ defined by setting

$$\pi(x) = (\pi_0(x), \pi_1(x))$$

is a bijection. Given $G \colon X^{n+2} \to X$, define $G_0 \colon X^{n+1} \to X$ by setting

$$G_0(x_0, \ldots, x_n) = G(x_0, \ldots, x_{n-1}, \pi_0(x_n), \pi_1(x_n)).$$

By the universality of F there exist functions h_R ($R \in [n+1]^n$) such that

$$G_0(x_0, \ldots, x_n) = F(\langle h_R(\langle x_i : i \in R\rangle) : R \in [n+1]^n\rangle)$$

for all $x_0, \ldots, x_n$ from X. Let k_{n+1} be any function from X^{n+1} to X, and let $k_{n\cup\{n+1\}} \colon X^{n+1} \to X$ be such that

$$k_{n\cup\{n+1\}}(x_0, \ldots, x_n) = h_n(x_0, \ldots, x_{n-1})$$

for all $x_0, \ldots, x_n$ from X. For each $Q \in [n{+}2]^{n+1}$ containing $\{n, n{+}1\}$, let $k_Q \colon X^{n+1} \to X$ be the function defined by setting

$$k_Q(x_0, \ldots, x_n) = h_{Q\cap(n+1)}(x_0, \ldots, x_{n-2}, \pi^{-1}(x_{n-1}, x_n)).$$

Then for all $x_0, \ldots, x_{n+1}$,

$$G(x_0, \ldots, x_{n+1}) = G_0(x_0, \ldots, x_{n-1}, \pi^{-1}(x_n, x_{n+1})),$$

which is equal to $F(\langle h_R(\langle y_i : y \in R\rangle) : R \in [n+1]^n\rangle)$, where $y_i = x_i$ for all $i < n$, and $y_n = \pi^{-1}(x_n, x_{n+1})$. Furthermore,

$$F(\langle h_R(\langle y_i : i \in R\rangle) : R \in [n+1]^n\rangle)$$

is equal to

$$F'(\langle k_Q(\langle x_i : i \in Q\rangle) : Q \in [n+2]^{n+1}\rangle),$$

as $h_R(\langle y_i : i \in R\rangle) = k_{R\cup\{n+1\}}(\langle x_i : i \in R \cup \{n+1\}\rangle)$ for each $R \in [n+1]^n$.
QED

In the following definition, n is the arity of the inside parameter functions. The arity of the universal function is less important.

Definition 6.7. *For any infinite set X, and any $n \in \omega$, we define $U(X, n)$ to be any of the equivalent propositions $U(X, m, [m]^n)$ for m in $\omega \setminus (n+1)$.*

Proposition 6.8 and Theorem 6.9 show that the $U(\kappa, n)$'s are the only generalized multi-dimensional universal functions properties. Clause (3) of Proposition 6.6 says that $U(\kappa, n)$ implies $U(\kappa, n+1)$ and we will show in Corollary 6.13 that none of these implications can be reversed.

Proposition 6.8. *Let X be an infinite X, $n \in \omega \setminus 2$, and $\Sigma, \Sigma_0, \Sigma_1$ subsets of $\mathcal{P}(n)$.*

(1) *If $\Sigma_0 \subseteq \Sigma_1$, then $U(X, n, \Sigma_0)$ implies $U(X, n, \Sigma_1)$.*

(2) *If $Q_0 \subseteq Q_1 \in \Sigma$, then $U(X, n, \Sigma)$ is equivalent to*

$$U(X, n, \Sigma \cup \{Q_0\}).$$

(3) *Suppose that Σ is closed under taking subsets, every element of n is in some element of Σ, and $n = \{0, 1, 2, \ldots, n-1\} \notin \Sigma$. Let $m+1$ be the size of the smallest subset of n not in Σ. Then $U(X, n, \Sigma)$ is equivalent to $U(X, m)$.*

Proof

(1) This follows from the fact that the F which works for Σ_0 also works for Σ_1 by ignoring the values of h_Q for $Q \in \Sigma_1 \setminus \Sigma_0$.

(2) One direction follows from part (1). For the other suppose that $F\colon X^{\Sigma \cup \{Q_0\}} \to X$ witnesses $U(X, n, \Sigma \cup \{Q_0\})$. Let $\pi\colon X \to X \times X$ be a bijection, and let $\pi_0\colon X \to X$ and $\pi_1\colon X \to X$ be such that $\pi(x) = (\pi_0(x), \pi_1(x))$ for all $x \in X$. Define $F'\colon X^\Sigma \to X$ by setting $F'(\langle x_Q : Q \in \Sigma\rangle)$ to be $F(\langle y_R : R \in \Sigma \cup \{Q_0\}\rangle)$, where $y_{Q_0} = \pi_0(x_{Q_1})$, $y_{Q_1} = \pi_1(x_{Q_1})$ and $y_R = x_R$ if $R \notin \{Q_0, Q_1\}$. Fix $G\colon X^n \to X$, and let h_R $(R \in \Sigma \cup \{Q_0\})$ be as in the definition of $U(X, n, \Sigma \cup \{Q_0\})$. For each $Q \in \Sigma \setminus \{Q_1\}$, let $h'_Q = h_Q$. Let h'_{Q_1} be such that

$$h'_{Q_1}(\langle x_j : j \in Q_1\rangle) = \pi(h_{Q_0}(\langle x_j : j \in Q_0\rangle), h_{Q_1}(\langle x_j : j \in Q_1\rangle))$$

for all $x_0, \ldots, x_{n-1}$ from X. Then h'_Q $(Q \in \Sigma)$ are as desired.

(3) First suppose that $U(X, n, \Sigma)$ holds. Choose $R \subseteq \{0, 1, \ldots n-1\}$ not in Σ with $|R| = m+1$. By the choice of m all subsets of R of size m are in Σ. By restricting to the case where $x_i = 0$ for $i \notin R$, we see that $U(X, m, [m]^{m+1})$ holds.

Now assume that $U(\kappa, m)$ holds. Then $U(\kappa, n, [n]^m)$ holds. Since $[n]^m \subseteq \Sigma$, part (1) implies that $U(\kappa, n, \Sigma)$ holds.

QED

Proposition 6.8 gives the following. The first two statements in the theorem are trivial.

Theorem 6.9. *For any $n \in \omega$, any infinite cardinal κ and any $\Sigma \subseteq \mathcal{P}(n)$, if*

$$\bigcup \Sigma \neq n = \{0, 1, 2, \ldots, n-1\}$$

then $U(\kappa, n, \Sigma)$ fails. If $n \in \Sigma$, then $U(\kappa, n, \Sigma)$ holds. If neither of these is true, then by the Proposition 6.8 there exists m with $U(\kappa, n, \Sigma)$ equivalent to $U(\kappa, m)$.

The following fact will be used in the proof of Proposition 6.11. Recall that a linear preorder on a set X is a binary relation $\preceq$ on X which is reflexive, transitive and total (i.e., for all $x, y \in X$, at least one of $x \preceq y$ and $y \preceq x$ holds). Every linear preorder is a superset of a linear order.

Proposition 6.10. *Let κ be an infinite cardinal and let $m < n$ be integers, with $m \geq 2$. Suppose that $F_0 \colon \kappa^{([n]^m)} \to \kappa$ has the property that for each $G \colon \kappa^n \to \kappa$ there exist $h_Q \colon \kappa^Q \to \kappa$ ($Q \in [n]^m$) such that*

$$G(y_0, \ldots, y_{n-1}) = F(\langle h_Q(\langle x_i : i \in Q\rangle) : Q \in [n]^m\rangle)$$

for all nondecreasing $\langle y_0, \ldots, y_{n-1}\rangle \in \kappa^n$. Then $U(\kappa, n, [n]^m)$ holds.

Proof
Let κ, m and n be as given. As in Remark 1.1, we may assume that F_0 has the property that for each $G \colon \kappa^n \to \kappa$ there exists a single function $h \colon \kappa^Q \to \kappa$ such that

$$G(y_0, \ldots, y_{n-1}) = F(\langle h(\langle x_i : i \in Q\rangle) : Q \in [n]^m\rangle)$$

for all nondecreasing $\langle y_0, \ldots, y_{n-1}\rangle \in \kappa^n$.

Let P be the set of all permutations of n. Let $\pi \colon \kappa \to \kappa^P$ be a bijection, and let $F = \pi \circ F_0$. Then for each $G \colon \kappa^n \to \kappa^P$ there exist an $h \colon \kappa^Q \to \kappa$ such that

$$G(y_0, \ldots, y_{n-1}) = F(\langle h(\langle x_i : i \in Q\rangle) : Q \in [n]^m\rangle)$$

for all nondecreasing $\langle y_0, \ldots, y_{n-1}\rangle \in \kappa^n$.

Let L be the set of all linear preorders on members of $[n]^m$. Let $r \colon L \times \kappa \to \kappa$ be a bijection. Let e be a function taking linear preorders on n to linear orders contained in them.

Let $F^* \colon \kappa^{([n]^m)} \to \kappa$ be the function which takes in a sequence

$$\langle r(l_Q, \alpha_Q) : Q \in [n]^m\rangle,$$

where each l_Q is a linear preorder on Q and each α_Q is in κ, and returns a value in κ defined as follows. If $\bigcup\{l_Q : Q \in [n]^m\}$ is not a linear preorder, then let F^* take any value in κ. Otherwise, let

$$l = e(\bigcup\{l_Q : Q \in [n]^m\}),$$

and let $s \colon n \to n$ be the function that takes each $i \in n$ to its l-rank. For each $Q \in [n]^m$, let β_Q be $\alpha_{s^{-1}[Q]}$. Finally, let

$$F^*(\langle r(l_Q, \alpha_Q) : Q \in [n]^m\rangle) = F(\langle \beta_Q : Q \in [n]^m\rangle)(s^{-1}).$$

Let us see that this F^* works. Fix a function $G^*\colon \kappa^n \to \kappa$. Let $G\colon \kappa^n \to \kappa^P$ be the function defined by letting

$$G(y_0,\ldots,y_{n-1}) = \langle G^*(y_{p(0)},\ldots,y_{p(n-1)}) : p \in P\rangle.$$

Let $h\colon \kappa^Q \to \kappa$ be such that

$$G(y_0,\ldots,y_{n-1}) = F(\langle h(\langle y_i : i \in Q\rangle) : Q \in [n]^m\rangle)$$

for all nondecreasing $\langle y_0,\ldots,y_{n-1}\rangle \in \kappa^n$. For each $Q \in [n]^m$, define $h^*_Q\colon \kappa^Q \to \kappa$ by setting $h^*_Q(\langle x_i : i \in Q\rangle)$ to be

$$r(l_Q(\langle x_i : i \in Q\rangle), h(\langle z_i : i \in Q\rangle)),$$

where l_Q is the linear order on Q induced by $\langle x_i : i \in Q\rangle$ and $\langle z_i : i \in Q\rangle$ lists $\{x_i : i \in Q\}$ in nondecreasing order.

Now each $\langle x_0,\ldots,x_{n-1}\rangle \in \kappa^n$ is $\langle y_{p(0)},\ldots,y_{p(n-1)}\rangle$ for some $p \in P$ and a unique nondecreasing $\langle y_0,\ldots,y_{n-1}\rangle$ in κ^n. Furthermore, p can be taken to be s^{-1}, where $l = e(\bigcup\{l_Q : Q \in [n]^m\})$, each l_Q is the linear order on Q given by $\{x_i : i \in Q\}$ and s is the function taking each element of n to its l-rank. Then $G^*(x_0,\ldots,x_{n-1}) = G(y_0,\ldots,y_{n-1})(p)$, which is

$$F(\langle h(\langle y_i : i \in Q\rangle) : Q \in [n]^m\rangle)(p).$$

Finally, each $\langle y_i : i \in Q\rangle = \langle x_i : i \in s^{-1}[Q]\rangle$, so that

$$F(\langle h(\langle y_i : i \in Q\rangle) : Q \in [n]^m\rangle)(p)$$

equals

$$F^*(\langle\langle r(l_Q(\langle x_i : i \in Q\rangle), h(z_i : i \in Q)) : Q \in [n]^m\rangle)$$

(where $\langle z_i : i \in Q\rangle$ is $\langle x_i : i \in Q\rangle$ listed in increasing order), which is equal to

$$F^*(\langle h^*_Q(\langle x_i : i \in Q\rangle) : Q \in [n]^m\rangle).$$

QED

Since $|\omega^{<\omega}| = \omega$, $U(\omega, 1)$ follows from Theorem 5.1. The following proposition allows us to propagate this fact, showing for instance that $U(\omega_n, n+1)$ holds for every $n \in \omega$.

Proposition 6.11. *For any infinite cardinal κ, and any $n \in \omega$, $U(\kappa, n)$ implies $U(\kappa^+, n+1)$*

Proof

Let $\Sigma = \{a \cup \{n+1\} : a \in [n+1]^n\}$. Applying Proposition 6.10, and the idea behind the first part of Proposition 6.8, it suffices to produce a function

$$F\colon (\kappa^+)^\Sigma \to \kappa^+$$

such that for every $G\colon (\kappa^+)^{n+2} \to \kappa^+$ there exist

$$H_Q\colon (\kappa^+)^{n+1} \to \kappa^+ (\mathrm{Q} \in \Sigma)$$

for which $G(\xi_0, \ldots, \xi_{n+1}) = F(\langle H_Q(\langle \xi_i : i \in Q\rangle) : Q \in \Sigma\rangle)$ for all nondecreasing sequences $\langle \xi_0, \ldots, \xi_{n+1}\rangle$ from κ^+.

Suppose that $f : \kappa^{([n+1]^n)} \to \kappa$ witnesses $U(\kappa, n+1, [n+1]^n)$. For each $\alpha \in [\kappa, \kappa^+)$, let $B_\alpha\colon \alpha \to \kappa$ be a bijection. Let $r\colon \kappa \times \kappa^+ \to \kappa^+$ be a bijection.

Let F be a function which takes in a sequence $\langle r(\alpha_Q, \beta_Q) : Q \in \Sigma\rangle$ and returns a value in κ^+ as follows. If β_Q is not the same value for all Q, then F returns any value. Otherwise, letting β be this constant value, and letting γ_R be $\alpha_{R\cup\{n+1\}}$ for each $R \in [n+1]^n$, F returns the value

$$B_\beta^{-1}(f(\langle \gamma_R : R \in [n+1]^n\rangle)).$$

Let us check that this definition works. Suppose we are given

$$G\colon (\kappa^+)^{n+2} \to \kappa^+.$$

For each $\delta < \kappa^+$, let $k(\delta)$ be $\sup(G[(\delta+1)^{n+2}]) + 1$. For each $\delta < \kappa^+$ there exist $h_R^\delta\colon \kappa^n \to \kappa$ $(R \in [n+1]^n)$ such that

$$f(\langle h_R^\delta(\langle B_{\delta+1}(\alpha_i) : i \in R\rangle) : R \in [n+1]^n\rangle) = B_{k(\delta)}(G(\alpha_0, \ldots, \alpha_n, \delta))$$

for all $\alpha_0, \ldots, \alpha_n \leq \delta$.

For each $Q \in \Sigma$, let $H_Q\colon (\kappa^+)^{n+1} \to \kappa^+$ be defined as follows. Given $\langle \zeta_i : i \in Q\rangle$ from κ^+, if $\zeta_{n+1} < \zeta_i$ for some $i \in Q$, then let $H_Q(\langle \zeta_i : i \in Q\rangle)$ take any value in κ^+. Otherwise, let $H_Q(\langle \zeta_i : i \in Q\rangle)$ take the value $r(\alpha_Q, k(\zeta_{n+1}))$, where

$$\alpha_Q = h_{Q\cap(n+1)}^{\zeta_{n+1}}(\langle B_{\zeta_{n+1}+1}(\zeta_i) : i \in Q \cap (n+1)\rangle).$$

Now let $\xi_0, \ldots, \xi_{n+1}$ be a nondecreasing sequence from κ^+. Then $G(\xi_0, \ldots, \xi_{n+1})$ is equal to

$$B_{k(\xi_{n+1})}^{-1}(f(\langle h_R^{\xi_{n+1}}(\langle B_{\xi_{n+1}+1}(\xi_i) : i \in R\rangle) : R \in [n+1]^n\rangle)).$$

Then we are done, since, as written above, each γ_R is $\alpha_{R\cup\{n+1\}}$, which is $h_R^{\xi_{n+1}}(\langle B_{\xi_{n+1}+1}(\xi_i) : i \in R\rangle)$.
QED

The partial order $\mathrm{Fn}(X, Y, \kappa)$ was defined before Theorem 5.5. For any $n \in \omega$ and any infinite cardinal κ, the partial order $\mathrm{Fn}(\kappa^n, 2, \aleph_0)$ is forcing-equivalent to the partial order which adds a subset of κ by finite conditions.

Proposition 6.12. *Suppose that $n \in \omega \setminus \{0\}$ and that $\gamma < \kappa$ are cardinals with $\aleph_n \leq \gamma$. Then $U(\gamma, n)$ fails after forcing with* $\mathrm{Fn}(\kappa^{n+1}, 2, \aleph_0)$.

Proof
Let $G \subseteq \mathrm{Fn}(\kappa^{n+1}, 2, \aleph_0)$ be a V-generic filter, and fix a function

$$F\colon \gamma^{n+1} \to \gamma$$

in $V[G]$. Since $\mathrm{Fn}(\kappa^{n+1}, 2, \aleph_0)$ is c.c.c., we may fix an $\eta < \kappa$ such that F is in $V[G \restriction \eta^{n+1}]$. Supposing toward a contradiction that F witnesses $U(\gamma, n+1, [n+1]^n)$ in $V[G]$, there exist functions

$$h_i\colon (\prod_{j \in (n+1)\setminus\{i\}} \omega_j) \to \gamma \quad (i < n+1)$$

such that for all $(\alpha_0, \ldots, \alpha_n) \in \prod_{i<n+1} \omega_i$, $G(\alpha_0, \ldots, \eta + \alpha_n)$ is equal to

$$F(h_0(\alpha_1, \ldots, \alpha_n), h_1(\alpha_0, \alpha_2, \ldots, \alpha_n), \ldots, h_n(\alpha_0, \ldots, \alpha_{n-1})).$$

Again applying the fact that $\mathrm{Fn}(\kappa^{n+1}, 2, \aleph_0)$ is c.c.c., there exists a $\delta_n < \omega_n$ such that $h_n \in V[G^{\{n\}}]$, where $G^{\{n\}}$ is the restriction of G to those members of κ^{n+1} whose last element is not $\eta + \delta_n$. For each $i < n$, let $h_i^{\{n\}}$ be the function on $\prod_{j \in n\setminus\{i\}} \omega_j$ defined by setting

$$h_i^{\{n\}}(\alpha_0, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_{n-1})$$

to be

$$h_i(\alpha_0, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_{n-1}, \eta + \delta_n).$$

Applying the c.c.c. of $\mathrm{Fn}(\kappa^{n+1}, 2, \aleph_0)$ again, we can find a $\delta_{n-1} < \omega_{n-1}$ such that, letting $G^{\{n-1,n\}}$ be the restriction of G to those members of κ^{n+1} whose last two elements are not δ_{n-1} and $\eta + \delta_n$, $h_{n-1}^{\{n\}} \in V[G^{\{n-1,n\}}]$. For each $i < n$, let $h_i^{\{n-1,n\}}$ be the function on

$$\prod_{j \in (n-1)\setminus\{i\}} \omega_j$$

defined by setting

$$h_i^{\{n-1,n\}}(\alpha_0, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_{n-2})$$

to be

$$h_i(\alpha_0, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_{n-2}, \delta_{n-1}, \eta + \delta_n).$$

Continuing in this fashion, we can find $(\delta_1, \ldots, \delta_{n-1}) \in \omega_1 \times \ldots \times \omega_{n-1}$ such that,

- letting $G^{\{1,\ldots,n\}}$ be the restriction of G to those elements of κ^{n+1} whose last n elements are not $\delta_1, \ldots, \delta_{n-1}, \eta + \delta_n$, and
- letting, for each positive $i < n$, $h_i^{\{1,\ldots,n\}}$ be the function on ω whose value at n is $h_i(n, \delta_1, \ldots, \delta_{i-1}, \delta_{i+1}, \ldots, \delta_{n-1}, \eta + \delta_n)$,

each $h_i^{\{1,\dots,n-1\}}$ is in $V[G^{\{1,\dots,n\}}]$.

Finally, we see that the function $g\colon \omega \to 2$ defined by setting $g(n) = G(n, \delta_1, \dots, \delta_{n-1}, \eta + \delta_n)$ is Cohen-generic over $V[G^{\{1,\dots,n\}}]$. However, as $h_0(\delta_1, \dots, \delta_{n-1}, \eta + \delta_n)$ is a fixed member of ω_2, our assumptions on F and $h_0, \dots, h_n$ give that g is an element of $V[G^{\{1,\dots,n\}}]$.
QED

Putting together Propositions 6.11 and 6.12, we have the following.

Corollary 6.13. *Let $\gamma < \kappa$ be cardinals, with $\aleph_\omega \leq \gamma$. After forcing to add a subset of κ by finite conditions, we have that*

$$U(\omega_n, n+1) + \neg U(\omega_n, n) + \neg U(\gamma, n),$$

for all positive $n \in \omega$.

If we start with a model M_1 of GCH and force with the set of countable partial functions from $\kappa = \aleph_{\omega+1}$ into 2, then in the resulting model M_2 we have CH so $U(\omega_1, 1)$ holds by Theorem 5.1. Proposition 6.11 then gives $U(\omega_n, n)$ for all positive $n \in \omega$. By an argument similar to Proposition 6.12 but raised up one cardinal, we have $\neg U(\omega_n, n-1)$ for $n \geq 2$. If we then add $\kappa = \omega_3$ Cohen reals to M_2 to get M_3, then we will have in M_3 that $|2^\omega| = \omega_3$ and $\neg U(\omega_3, 2)$ by the argument of Proposition 6.12 lifted by one cardinal. By Proposition 6.11, $U(\omega_3, 4)$ is true in ZFC. This leaves open the question of whether $U(\omega_3, 3)$ holds in M_3.

Definition 6.14. *For Borel universal functions of higher dimensions, we let $U(\text{Borel}, \Sigma, n)$ and $U(\text{Borel}, n)$ denote the versions of Definition 6.5 and 6.7 where X is 2^ω and F is required to be Borel.*

The following proposition follows from the proofs of part (3) of Propositions 6.6 and 6.8, using the fact that the composition of Borel functions is Borel, and the fact that there exist continuous pairing and unpairing functions. The reader interested in constructing a complete proof will also have to verify that the universal functions constructed in Propositions 6.2 and 6.4 are Borel assuming the functions in the hypothesis are, thus yielding the same conclusion for Proposition 6.6 and 6.8.

Proposition 6.15. *The following hold for any $n \in \omega$.*

1. *$U(Borel, n)$ implies $U(Borel, n+1)$*
2. *$U(Borel, \Sigma, n)$ is equivalent to $U(Borel, m)$ for $m+1$ the size of the smallest subset of n not in the downward closure of Σ.*

We can further refine $U(\text{Borel}, n)$ in the special case that our universal function F is a level α Borel function. The composition of level α-functions is not necessarily level α, i.e., $F(F(x,y),z)$ need be at the level α just because F is. Hence it is not immediately obvious that the binary case of the next proposition implies the n-ary case. The proof here is similar to that of Rao [22]. Recall from Subsection 1.1 that the hypothesis of the proposition is implied by Martin's Axiom. The hypotheses on cardinal characteristics in Proposition 6.16 (and Proposition 7.14) are used only to get a set of reals of cardinality $\mathfrak{c}$ which is wellordered by a Borel relation.

Proposition 6.16. *Suppose that $\mathfrak{t} = \mathfrak{q} = \mathfrak{c}$. Then for every $n > 1$ there is a level 2 Borel function $F : (2^\omega)^n \to 2^\omega$ which is universal, i.e., such that for every $G : (2^\omega)^n \to 2^\omega$ there exist $h_i : 2^\omega \to 2^\omega$ $(1 \le i \le n)$ such that for every x in $(2^\omega)^n$*

$$G(x_1, \ldots, x_n) = F(h_1(x_1), \ldots, h_n(x_n))$$

Proof

By Proposition 1.5 and the remarks before, it suffices to find an F_σ set $H \subseteq (2^\omega)^n$ such that for each $A \subseteq \mathfrak{c}^n$ there exists an $h\colon \mathfrak{c} \to 2^\omega$ such that for all $(\alpha_0, \ldots, \alpha_{n-1}) \in \mathfrak{c}^n$, $(\alpha_0, \ldots, \alpha_{n-1}) \in A$ if and only if $(h(\alpha_0), \ldots, h(\alpha_{n-1})) \in H$.

Let $F \subseteq (2^\omega)^{n+1}$ be an F_σ set with the property that for every F_σ set $K \subseteq (2^\omega)^n$ there exists $x \in 2^\omega$ with $K = F_x$, i.e., the set of $(y_1, \ldots, y_n)$ in $(2^\omega)^n$ with $(x, y_1, \ldots, y_{n-1}) \in F$.

Define the binary relation $\le^*$ on 2^ω by setting $x \le^* y$ if

$$x^{-1}[\{1\}] \setminus y^{-1}[\{1\}]$$

is finite. Let $g : \mathfrak{c} \to 2^\omega$ be an injection such that for each pair α, β from $\mathfrak{c}$, $\alpha \le \beta$ if and only if $g(\alpha) \le^* g(\beta)$. The existence of such a function follows from the statement $\mathfrak{t} = \mathfrak{c}$.

For each $\beta < \mathfrak{c}$, let D_β be the set of $(\alpha_0, \ldots, \alpha_{n-1}) \in \mathfrak{c}^n$ such that $\max\{\alpha_0, \ldots, \alpha_{n-1}\} \le \beta$. Since $\mathfrak{q} = \mathfrak{c}$, every set $X \subseteq 2^\omega$ with $|X| < \mathfrak{c}$ is a Q-set (see Subsection 1.1). Thus given $A \subseteq \mathfrak{c}^n$, there exists a function $k : \mathfrak{c} \to 2^\omega$ with the property that for each $\beta < \mathfrak{c}$ and every $(\alpha_0, \ldots, \alpha_{n-1})$ in D_β, $(\alpha_0, \ldots, \alpha_{n-1}) \in A$ if and only if

$$(k(\beta), g(\alpha_0), \ldots, g(\alpha_{n-1})) \in F.$$

Let $\langle \cdot, \cdot \rangle$ be the pairing function on 2^ω such that $\langle x, y\rangle(2n) = x(n)$ and $\langle x, y\rangle(2n+1) = y(n)$, for each $n \in \omega$. Now let H be the set of $(\langle x_0, y_0\rangle, \ldots, \langle x_{n-1}, y_{n-1}\rangle) \in (2^\omega)^n$ such that, for some $i < n$, $x_j \le^* x_i$ for all $j < n$, and $(y_i, x_0, \ldots, x_{n-1}) \in F$. Then H is F_σ.

Given $A \subseteq \mathfrak{c}^n$, for each $\alpha < \mathfrak{c}$, let $h(\alpha)$ be $\langle g(\alpha), k(\alpha)\rangle$, where k is as above with respect to A.
QED

7. Model-theoretic universality

In this section we consider the relationship between the existence of abstract universal functions and the existence of universal models. The key difference is that if one were to consider a universal function as the model of some theory, then embedding would require embedding the range as well as the domain of the function. This is different than the notion of universality being considered here since the values in the range remain fixed. Nevertheless, there is insight to be gained from the model theoretic perspective. It is well known that saturated models are universal in the sense of elementary substructures and that saturated models of cardinality κ exist if $\kappa^{<\kappa} = \kappa$, see Chapter 5 of Chang and Keisler [5], for instance.

Definition 7.1. *For any cardinal κ let $\mathcal{L}_\kappa$ be the first order language consisting of a single binary function symbol Φ and constant symbols $\{c_\gamma\}_{\gamma\in\kappa}$ for distinct constants. Let $\mathcal{T}_\kappa$ be the $\mathcal{L}_\kappa$-theory consisting of the sentences $c_\gamma \neq c_\beta$ for $\gamma \neq \beta$ and the sentences $\Phi(c_\gamma, c_\beta) = c_0$ for all γ, β.*

There is some overlap between the following proposition and Theorem 5.1.

Proposition 7.2. *If $\mathcal{T}_\kappa$ has a model of cardinality κ which is universal in the model theoretic sense, then there is a universal function $F : \kappa \times \kappa \to \kappa$.*

Proof
Let $(X, \Phi, c_\alpha)_{\alpha<\kappa}$ be a universal $\mathcal{T}_\kappa$ model of cardinality κ, and let $C = \{c_\alpha : \alpha < \kappa\}$. Universality implies that $Y = X \setminus C$ has cardinality κ. Let $\langle d_\alpha : \alpha < \kappa\rangle$ enumerate Y. Define $F : \kappa^2 \to \kappa$ by setting $F(\alpha, \beta)$ to be the unique γ such that $\Phi(d_\alpha, d_\beta) = c_\gamma$, if one exists, and 0 otherwise. Given an arbitrary $f : \kappa \times \kappa \to \kappa$, construct a $\mathcal{T}_\kappa$ model $(\{b_\alpha : \alpha < \kappa\} \cup C, \Phi_f, c_\alpha)_{\alpha<\kappa}$ where $\{b_\alpha : \alpha < \kappa\}$ is disjoint from C and $\Phi_f(b_\alpha, b_\beta) = c_\gamma$ if and only if $f(\alpha, \beta) = \gamma$. Since a model theoretic embedding fixes the constant symbols, we get an h showing that $f(\alpha, \beta) = F(h(\alpha), h(\beta))$ for all $\alpha, \beta \in \kappa$.
QED

In this section we will use the term *Sierpiński universal* for the notion of universal function which is the subject of this paper, to distinguish it from model theoretic universality.

Definition 7.3. *A function $U : \kappa\times\kappa \to \kappa$ is Sierpiński universal if for every $f : \kappa \times \kappa \to \kappa$ there exists $h : \kappa \to \kappa$ such that for all $\alpha, \beta \in \kappa$,*

$$f(\alpha, \beta) = U(h(\alpha), h(\beta)).$$

Definition 7.4. *A function $U : \kappa \times \kappa \to \kappa$ is model theoretically universal if for every $f : \kappa \times \kappa \to \kappa$ there exists $h : \kappa \to \kappa$ one-to-one such that for all $\alpha, \beta \in \kappa$,*

$$h(f(\alpha, \beta)) = U(h(\alpha), h(\beta)).$$

In other words, a function $U\colon \kappa \times \kappa \to \kappa$ is model theoretically universal if and only if every structure (X, f) where $f : X^2 \to X$ and $|X| = \kappa$ is isomorphic to a substructure of (κ, U).

We define a common weakening of these notions, as follows.

Definition 7.5. *A function $U : \kappa \times \kappa \to \kappa$ is weakly universal if for every $f : \kappa \times \kappa \to \kappa$ there exist $h : \kappa \to \kappa$ and $k : \kappa \to \kappa$ one-to-one such that for all $\alpha, \beta \in \kappa$,*

$$k(f(\alpha, \beta)) = U(h(\alpha), h(\beta)).$$

Remark 7.6. *The existence of a weakly universal function on $\kappa \times \kappa$ is not changed if we allow the codomains of U and k to be any set of cardinality κ.*

Remark 7.7. *Model theoretically universal functions are weakly universal with $h = k$, and Sierpiński universal functions are weakly universal with k the identity function. implies weakly universal. For maps into 2 (or binary relations) all three notions are equivalent.*

Question 7.8. *Is the existence of a model theoretically universal function from $\kappa\times\kappa$ to κ equivalent to the existence of a Sierpiński universal one? Does the existence of either one imply the existence of the other?*

Proposition 7.9. *If κ is a singular strong limit cardinal then there is a model theoretically universal function from $\kappa \times \kappa$ to κ.*

Proof
Let γ be the cofinality of κ, and let $\langle \kappa_\alpha : \alpha < \gamma\rangle$ be an increasing sequence of cardinals cofinal in κ. Let G be the set of functions g from $\kappa \times \kappa$ to κ such that $g[\kappa_\alpha \times \kappa_\alpha] \subseteq \kappa_\alpha$ for each $\alpha < \gamma$. For any function $f\colon \kappa \times \kappa \to \kappa$, there exist a bijection $h\colon \kappa \to \kappa$ and a function $g \in G$ such that for all $\beta, \delta < \kappa$, $h(f(\beta, \delta)) = g(h(\beta), h(\delta))$ (that is, (κ, f)

is isomorphic to (κ, g) via h). This follows from the fact that we can write κ as a continuous increasing union of a sequence of sets X_α, each closed under f and having size κ_α.

It suffices then to find a $U\colon \kappa \times \kappa \to \kappa$ which is model theoretically universal with respect to functions in G. Since κ is a strong limit cardinal, we can recursively build U so that for each $\alpha < \gamma$ and each function $f\colon \kappa_\alpha \times \kappa_\alpha \to \kappa_\alpha$ there exist $X_f \subseteq \kappa$ and a bijection $h_f\colon \kappa_\alpha \to X_f$ such that $h(f(\beta, \delta)) = U(h(\beta), h(\delta)$ for all β, δ in κ_α. Furthermore, we can build U so that for all $\alpha < \alpha' < \gamma$ and all $f\colon \kappa_{\alpha'} \times \kappa_{\alpha'} \to \kappa_{\alpha'}$ such that $f[\kappa_\alpha \times \kappa_\alpha] \subseteq \kappa_\alpha$, $h_{f\restriction\kappa_\alpha\times\kappa_\alpha} = h_f \restriction \kappa_\alpha$.

Then for each $g \in G$, $\bigcup\{h_{g\restriction\kappa_\alpha\times\kappa_\alpha} : \alpha, \gamma\}$ is the desired function h witnessing that U is model theoretically universal with respect to g.
QED

Question 7.10. *Suppose that $\kappa = \aleph_\omega$ is a strong limit cardinal. For each $\alpha < \kappa$ we have a Sierpiński universal $U : \kappa^2 \to \alpha$ universal for all maps of the same type, by Theorem 5.1. By Proposition 5.4 we have a map $U : \kappa^2 \to \kappa$ which is Sierpiński universal for all maps of the form $G : \omega^2 \to \kappa$. By Proposition 7.9 we have $U : \kappa^2 \to \kappa$ which is model-theoretically universal for all maps of the same type. Is there a Sierpiński universal $U : \kappa \times \kappa \to \kappa$ for maps of type $G : \omega \times \omega_1 \to \kappa$?*

Let $\mathcal{E}_4$ be the theory in the language of a single 4-ary relation A that is an equivalence relation between the first two and last two coordinates. In other words, it has the following axioms:

- $A(a, b, c, d) \to A(c, d, a, b)$
- $A(a, b, a, b)$
- $A(a, b, c, d)$ & $A(c, d, e, f) \to A(a, b, e, f)$

The transitivity condition on A implies that $\mathcal{E}_4$ does not have the 3-amalgamation property, so Mekler's argument of [16] (see Theorem 5.12) can not be applied to produce a universal model for this theory of cardinality $\aleph_1$ along with $2^{\aleph_0} > \aleph_1$. Nevertheless, the following observation highlights the connection between Sierpiński universality and model theoretic universality.

Proposition 7.11. *There is a universal model for $\mathcal{E}_4$ of cardinality κ if and only if there is a function $U : \kappa \times \kappa \to \kappa$ which is weakly universal.*

Proof

Let (κ, A) be a universal model of $\mathcal{E}_4$. Let E be the equivalence relation on $\kappa \times \kappa$ induced by A and let $\{E_\xi\}_{\xi\in\kappa}$ enumerate the equivalence classes of E. Define $U : \kappa \times \kappa \to \kappa \times 2$ by setting $U(\alpha, \beta) = (\xi, 0)$ if and only if $(\alpha,\beta)\in E_\xi$. We will show that U is weakly universal (using Remark 7.6).

Given $g\colon \kappa\times\kappa\to\kappa$ let G be the 4-ary relation defined by letting $G(\alpha,\beta,\delta,\gamma)$ hold if and only if $g(\alpha,\beta)=g(\delta,\gamma)$. It is clear that G satisfies the axioms of $\mathcal{E}_4$ hence there exists an injective $h:\kappa\to\kappa$ such that $G(\alpha,\beta,\delta,\gamma)$ holds if and only if $A(h(\alpha),h(\beta),h(\delta),h(\gamma))$ does. It follows that $g(\alpha,\beta)=g(\delta,\gamma)$ if and only if $U(h(\alpha),h(\beta))=U(h(\delta),h(\gamma))$. Then any injection $k\colon \kappa\to\kappa\times 2$ such that $k(g(\alpha,\beta))=U(h(\alpha),h(\beta))$ for all $\alpha,\beta<\kappa$ is as desired.

The converse is proved by running the preceding argument backwards — given a Sierpiński universal function $U:\kappa\times\kappa\to\kappa$ satisfying the hypothesis, define $A(\alpha,\beta,\gamma,\delta)$ to hold precisely when $U(\alpha,\beta)=U(\gamma,\delta)$.
QED

The next three propositions concern Borel universal functions.

Proposition 7.12. *There exists a Borel $U:2^\omega\times 2^\omega\to 2^\omega$ which is model theoretically universal with respect to all $F:\omega_1\times\omega_1\to\omega_1$.*

Proof
We first describe U. We use some Borel encoding of the structures below for which the exact details are not important.

Suppose that we are given $a,b\in 2^\omega$. There are four cases:

Case 1. a codes a pair (n,x), and b codes a pair (m,x), where $n,m\in\omega$ and $x\in 2^\omega$ codes a tuple $(f,B,<,\langle c_k:k\in B\rangle)$, where f is a function from $\omega\times\omega$ to ω, B is subset of ω, $<$ is a linear order on ω and each c_k is in 2^ω;

Case 2. Case 1 fails and a codes a pair (n,x), where $n\in\omega$, $x\in 2^\omega$ and x codes a tuple $(f,B,<,\langle c_k:k\in B\rangle)$ as in Case 1, with $b=c_m$ for some $m\in B$;

Case 3. Cases 1 and 2 fail, and b codes a pair (m,x), where $m\in\omega$, $x\in 2^\omega$ and x codes a tuple $(f,B,<,\langle c_k:k\in B\rangle)$ as in Case 1, with $a=c_n$ for some $n\in B$;

Case 4. none of the previous cases hold.

In the first three cases, if $f(n,m)=k$ and $k\in B$, then we let $U(a,b)=c_k$, otherwise we let it be (a code for) (k,x). In the fourth case we let $U(a,b)=0$.

Now we verify that this works. Given $F:\omega_1\times\omega_1\to\omega_1$, let C be a closed unbounded subset of the countable limit ordinals, such that $F[\alpha\times\alpha]\subseteq\alpha$, for each $\alpha\in C$. We recursively define $h\colon\omega_1\to\omega_1$ by assuming that we have $h \restriction \alpha$ as desired, for some $\alpha \in C \cup \{0\}$, and defining $h \restriction \min(C \setminus (\alpha+1))$. Fix such an α, and let α^+ denote the least member of C above α. Let $j : \omega \to \alpha^+$ be a bijection. Define $f : \omega \times \omega \to \omega$ by setting $f(n,m) = j^{-1}(F(j(n), j(m)))$, and the binary relation $<$ on ω by setting $n < m$ if and only if $j(n) < j(m)$. Let $B = j^{-1}[\alpha]$, and, for each $k \in B$, let $c_k = a_{j(k)}$. Let $x \in 2^\omega$ be a code for the tuple $(f, B, <, \langle c_k : k \in B\rangle)$. For each $\beta \in [\alpha, \alpha^+)$, let $h(\beta)$ be (a code for) the pair $(j^{-1}(\beta), x)$.

Now suppose that we are given $\beta, \gamma < \omega_1$. Fix $\alpha \in C \cup \{0\}$ such that $\max(\beta,\gamma)$ is in $[\alpha, \min(C \setminus (\alpha+1)))$, and let $\alpha^+ = \min(C \setminus (\alpha+1))$ as above. If $\{\beta,\gamma\} \subseteq [\alpha,\alpha^+)$, then the pair $(h(\beta), h(\gamma))$ is in Case 1 above. If $\beta = \max(\beta,\gamma)$ and $\gamma < \alpha$, then $(h(\beta), h(\gamma))$ is in Case 2, and if $\gamma = \max(\beta,\gamma)$ and $\beta < \alpha$, then $(h(\beta), h(\gamma))$ is in Case 3. In any case, $U(h(\beta), h(\gamma))$ will be $h(F(\beta,\gamma))$.
QED

Remark 7.13. *The second author has recently shown that the version of Proposition 7.12 with ω_2 in place of ω_1 can consistently hold. Moreover, he has shown the following: if there is a Borel Sierpinski universal function and $2^{<\mathfrak{c}} = \mathfrak{c}$, then there is a Borel map $H : 2^\omega \times 2^\omega \to 2^\omega$ such that for every cardinal $\kappa < \mathfrak{c}$, and for every function $G\colon \kappa \times \kappa \to \kappa$, there exist x_α ($\alpha < \kappa$) in the Cantor space such that for all $\alpha, \beta, \gamma < \omega_2$ $G(\alpha,\beta) = \gamma$ if and only if $H(x_\alpha, x_\beta) = x_\gamma$. Whether this can hold for $\kappa = \mathfrak{c}$ is still open, as far as we know.*

Given an ordinal γ, say that a function $f\colon \gamma \times \gamma$ *weakly pushes down* if $f(\alpha,\beta) < \max(\alpha,\beta)$ for all $\alpha,\beta < \gamma$.

Proposition 7.14. *If $\mathfrak{t} = \mathfrak{ap} = \mathfrak{c}$, then there is a Borel function $U : 2^\omega \times 2^\omega \to 2^\omega$ such that for every $f : \mathfrak{c} \times \mathfrak{c} \to \mathfrak{c}$ which weakly pushes down, there exists a one-to-one $h : \mathfrak{c} \to 2^\omega$ such that $h(f(\alpha,\beta)) = U(h(\alpha), h(\beta))$ for all $\alpha, \beta < \mathfrak{c}$.*

Proof
Assuming $\mathfrak{ap} = \mathfrak{c}$, by the standard almost-disjoint forcing technique there exists a Borel function $F : 2^\omega \times 2^\omega \to 2^\omega$ such that for every function $g : X \to 2^\omega$ with $X \subseteq 2^\omega$ and $|X| < \mathfrak{c}$, there exists $y \in 2^\omega$ with $g(x) = F(x,y)$ for all $x \in X$ (see Lemma 3.7, of [7], for instance).

As in the proof of Proposition 6.16, since $\mathfrak{t} = \mathfrak{c}$ we may fix an injection $h : \mathfrak{c} \to 2^\omega$ such that for each pair α, β from $\mathfrak{c}$, $\alpha \leq \beta$ if and only if $h(\alpha) \leq^* h(\beta)$.

Now given any $f : \mathfrak{c} \times \mathfrak{c} \to \mathfrak{c}$ which pushes down, recursively choose $x_\alpha \in 2^\omega$ so that $x_\alpha = \langle y_\alpha, z_\alpha, h(\alpha), t_\alpha)$ where

(1) t_α is $\langle 0, x_{f(\alpha,\alpha)}\rangle$ if $f(\alpha,\alpha) < \alpha$ and $\langle 1,1\rangle$ otherwise;
(2) for all $\beta < \alpha$,
- if $f(\alpha,\beta) < \alpha$, then $F(x_\beta, z_\alpha) = \langle 0, x_{f(\alpha,\beta)}\rangle$;
- if $f(\beta,\alpha) < \alpha$, then $F(x_\beta, y_\alpha) = \langle 0, x_{f(\beta,\alpha)}\rangle$;
- if $f(\alpha,\beta) = \alpha$, then $F(x_\beta, z_\alpha) = \langle 1,1\rangle$;
- if $f(\beta,\alpha) = \alpha$, then $F(x_\beta, y_\alpha) = \langle 1,1\rangle$.

Let π_0 and π_1 be such that $x = \langle \pi_0(x), \pi_1(x)\rangle$ for all $x \in 2^\omega$. Using this we may define the Borel function U by setting $U(x, x')$, where $x = \langle y, z, s, t\rangle$ and $y = (y', z', s', t')$ to be

- $\pi_1(F(x, y'))$ if $s <^* s'$ and $\pi_0(F(x, y')) = 0$;
- $\pi_1(F(x', z))$ if $s' <^* s$ and $\pi_0(F(x', z)) = 0$;
- x' if $s <^* s'$ and $\pi_0(F(x, y')) = 1$;
- x if $s <^* s'$ and $\pi_0(F(x', z)) = 1$;
- x if $x = x'$ and $\pi_0(F(x, x)) = 1$;
- t otherwise.

One can now verify that $f(\alpha,\beta) = \gamma$ if and only if $U(x_\alpha, x_\beta) = x_\gamma$ for all $\alpha, \beta, \gamma < \mathfrak{c}$ by considering the cases $\alpha < \beta$, $\beta < \alpha$, and $\alpha = \beta$. QED

The identity function satisfies the analogous notion of Sierpiński universality for unary maps. The corresponding result for model theoretic universality appear to be more difficult.

Proposition 7.15. *Define $\pi : 2^\omega \to 2^\omega$ by setting $\pi(x) = y$ if and only if $\forall n\ y(n) = x(2n)$. Then π is a model theoretically universal for all maps $f : \mathfrak{c} \to \mathfrak{c}$.*

Proof
Any function $g\colon X \to X$ from a set X to itself induces a partition $\{Q(x) : x \in X\}$ of X, where each $Q(x)$ is the smallest subset of X closed under g-images and g-preimages with x as a member. We will refer to the sets $Q(x)$ as *g-components*. For each $x \in X$, the *pre-image tree of x* (according to g) is the tree of height at most ω whose root is x, and for which the immediate successors of each node y are the members of $g^{-1}[\{y\}]$. Let $T_g(x)$ denote the set of nodes of this tree. A g-component Q either contains a unique cycle of length n, for some positive $n \in \omega$, or it contains none. In the former case, Q consists of the union of the sets $T_g(x)$, for each member x of the cycle, and we say that the component has *type* n. In the latter case, for each x in Q, $Q = \bigcup\{T_g(g^i(x)) : x \in \omega\}$, and we say that the component has type ω.

Fix a function $f\colon \mathfrak{c} \to \mathfrak{c}$. We seek a function $h\colon \mathfrak{c} \to 2^\omega$ such that $h(f(\alpha)) = \pi(h(\alpha)$ for all $\alpha \in \mathfrak{c}$. Since the π-preimage of each singleton from 2^ω has size continuum, the analysis of the previous paragraph shows that it suffices to prove that there are continuum many π-components of type n, for each positive $n \in \omega$, and continuum many π-components of type ω, as then the components of f can be embedded into distinct π-components.

For each $x \in 2^\omega$, and all $i, n \in \omega$, $\pi^n(x)(i) = x(2^n(i))$. Given a positive $n \in \omega$, x is then part of a cycle of length n if $x(2^n i) = x(i)$ for all $i \in \omega$. There are continuum many such x, as the values $x(i)$ can be chosen freely for each odd $i \in \omega$. Since each component of type n contains exactly n such x's, there are continuum many π-components of type n.

On the other hand, one can build by recursion an increasing sequence of natural numbers $\langle p_i : i < \omega\rangle$ and a collection of sequences $t_\sigma \in 2^{p_{|\sigma|}}$ for each $\sigma \in 2^{<\omega}$ such that for each pair $n, m \in \omega$ there exists an $i \in \omega$ such that for each pair $\sigma, \sigma' \in 2^i$, if either $n \neq m$ or $\sigma \neq \sigma'$ then there exists $j \in \omega$ such that $2^n j, 2^m j < p_i$ and $t_\sigma(2^m j) \neq t_{\sigma'}(2^n j)$. Then the sets $\bigcup\{t_{y\restriction p_i} : i < \omega\}$ $(y \in 2^\omega)$ are members of distinct π-components of type ω.
QED

Finally, we indicate another possible distinction between Sierpiński universal functions and model theoretically universal ones. Let us say that a function $f : \kappa \times \kappa \to \kappa$ is *Sierpiński universal for regressive functions* if for every function $g : \kappa \times \kappa \to \kappa$ such that $g(\alpha, \beta) < \max(\alpha, \beta)$ for all α, β (other than $\alpha = \beta = 0$) there exists $h : \kappa \to \kappa$ such that $f(h(\alpha), h(\beta)) = g(\alpha, \beta)$ for all α, β in κ.

Proposition 7.16. *If κ is regular, then every $f\colon \kappa \times \kappa \to \kappa$ which is Sierpiński universal for regressive functions is Sierpiński universal.*

Proof
Let $f : \kappa \times \kappa \to \kappa$ be Sierpiński universal for regressive functions and fix $g : \kappa \times \kappa \to \kappa$. Let $j : \kappa \to \kappa$ be an increasing function such that $g(\xi, \eta) < j(\alpha)$ for all $(\xi, \eta) \in (\alpha + 1)^2$, and let

$$g^*(\xi, \eta) = \begin{cases} g(j^{-1}(\xi), j^{-1}(\eta)) & \text{if } \xi \text{ and } \eta \text{ are in the range of j} \\ 0 & \text{otherwise.} \end{cases}$$

Since $g^*(\alpha, \beta)$ is either 0 or equal to $g(j^{-1}(\alpha), j^{-1}(\beta)) < \max(\alpha, \beta)$ and f is weakly Sierpiński universal there exists an $h : \kappa \to \kappa$ such that $f(h(\alpha), h(\beta)) = g^*(\alpha, \beta)$ for all α, β in κ. Then $f(h(j(\alpha)), h(j(\beta))) =$

$g^*(j(\alpha), j(\beta)) = g(\alpha, \beta)$ for all α, β in κ, so $h \circ j$ is the required embedding.
QED

Question 7.17. *Is Proposition 7.16 is true for the analogous notion of model theoretically universal for regressive functions?*

In the proof of Proposition 7.16, the given function g was embedded into the regressive function g^*. Note however that if $g : \kappa \times \kappa \to \kappa$ is such that $g(\alpha, \beta) < \max(\alpha, \beta)$ for all α, β (aside from $\alpha = \beta = 0$), then no substructure of (κ, g) is isomorphic to the positive integers under addition. To see this, suppose toward a contradiction that $\pi : \omega \to \kappa$ is one-to-one and $\forall n, m, k > 0$

$$n + m = k \text{ iff } g(\pi(n), \pi(m)) = \pi(k)$$

then

$$\pi(2n) = \pi(n + n) = g(\pi(n), \pi(n)) < \pi(n)$$

and therefore $(\pi(2^n) : n < \omega)$ is an infinite descending sequence of ordinals.

8. Appendix

We conclude this section with an argument, due to Justin Moore, which shows that under the Proper Forcing Axiom there are no functions with property R.[5] We begin by introducing some notation.

Given a function $\Phi\colon [\omega_1]^2 \to \omega$, a finite set $F \subseteq \omega_1$ and $k \in \omega$, we let $B_k(\Phi, F)$ denote the set

$$\{\beta \in \omega_1 \mid (\forall \alpha \in F)\ \Phi(\{\alpha, \beta\}) > k\}.$$

Lemma 8.1. *Suppose that $\Phi\colon [\omega_1]^2 \to \omega$ is a function with Property R. Then for each $k \in \omega$ there exists an $\alpha < \omega_1$ such that for each finite $F \subseteq \omega_1$ either $F \cap \alpha \neq \emptyset$ or $B_k(\Phi, F)$ is uncountable.*

Proof
Otherwise, there exist infinitely many pairwise disjoint F for which $B_k(\Phi, F)$ is countable. Then there exist $\beta \in \omega_1$ and an infinite pairwise disjoint family of finite sets F for which $F \subseteq \beta$ and $\beta \notin B_k(\Phi, F)$. This yields infinitely many $\xi \in \beta$ such that $\Phi(\{\beta, \xi\}) \leq k$ contradicting that Φ satisfies Property R.

[5]We thank Alan Dow for discussions clarifying this argument.

QED

Applying Lemma 8.1, we can find for any function Φ with Property R a minimal ordinal $\alpha(\Phi)$ with the property that for any $k \in \omega$ and any finite $F \subseteq \omega_1$ either $F \cap \alpha(\Phi) \neq \emptyset$ or $B_k(\Phi, F)$ is uncountable.

Given a function $\Phi\colon [\omega_1]^2 \to \omega$ with Property R let $\mathbb{P}(\Phi)$ be the partial order consisting of pairs (A, M) such that:

- A is a finite set of pairs from $\omega_1 \setminus \alpha(\Phi)$, and for all distinct $a, b \in A$, $a \subseteq \min(b)$ or $b \subseteq \min(a)$;
- M is a finite $\in$-chain of elementary submodels of $H(\aleph_2)$, each having Φ as a member.
- For all $a \in A$, there is $\mathfrak{M} \in M$ such that $|\mathfrak{M} \cap a| = 1$.
- For all $a \in A$ and $b \in A$ such that $a \subseteq \min(b)$, there is $\mathfrak{M} \in M$ such that $a \subseteq \mathfrak{M}$ and $b \cap \mathfrak{M} = \emptyset$.
- For all distinct a, b from A,

$$\Phi(\{\min(a), \min(b)\}) < \Phi(\{\max(a), \max(b)\}).$$

The ordering on $\mathbb{P}(\Phi)$ is : $(A, M) \leq (B, N)$ if $B \subseteq A$, $N \subseteq M$ and, for all $\mathfrak{M} \in N$ and all $a \in A$, if $|\mathfrak{M} \cap a| = 1$, then $a \in B$.

The partial order $\mathbb{P}(\Phi)$ adds an uncountable set of pairs from ω_1 witnessing the failure of Property R for Φ.

Claim 8.2. *Given any $(A, M) \in \mathbb{P}(\Phi)$ and $\xi \in \omega_1$ there exists a condition $(A', M') \leq (A, M)$ such that $(\bigcup A') \setminus \xi \neq \emptyset$.*

Proof

By adding a model to the top of M if necessary, we may assume that there is $\mathfrak{M} \in M$ such that $A \in \mathfrak{M}$ and $\xi < \omega_1 \cap \mathfrak{M}$. Let γ be any element of ω_1 greater than $\omega_1 \cap \mathfrak{M}$, and extend M to M' by adding an elementary submodel $\mathfrak{M}'$ on top with $\gamma < \omega_1 \cap \mathfrak{M}'$. Let $k > \Phi(\{\alpha, \beta\})$ for all distinct α, β from $(\bigcup A) \cup \{\gamma\}$. Then $B_k(\Phi, (\bigcup A) \cup \{\gamma\})$ is uncountable since $((\bigcup A) \cup \{\gamma\}) \cap \alpha(\Phi) = \emptyset$. Let $\delta \in B_k(\Phi, \bigcup A) \setminus \mathfrak{M}'$. Then $(A \cup \{\{\gamma, \delta\}\}, M') \in \mathbb{P}(\Phi)$.

Claim 8.3. $\mathbb{P}(\Phi)$ *is proper.*

Proof

Let $(A, M) \in \mathbb{P}(\Phi)$ and $(A, M) \in \mathfrak{M} \prec H(\kappa)$ for some uncountable κ. Since $\mathfrak{M} \cap H(\aleph_2) \prec H(\aleph_2)$ it suffices to show that $(A, M \cup \{\mathfrak{M} \cap H(\aleph_2)\})$ is $\mathbb{P}(\Phi)$-generic for $\mathfrak{M}$. To see this, let $D \in \mathfrak{M}$ be a dense subset of $\mathbb{P}(\Phi)$ and suppose that $(B, N) \in D$ is such that $(B, N) \leq (A, M \cup \{\mathfrak{M}\})$. Since there is no $a \in A$ with $|a \cap \mathfrak{M}| = 1$, by the definition of the order on $\mathbb{P}(\Phi)$, there is also no $b \in B$ with $|b \cap \mathfrak{M}| = 1$. Let $\{b_1, \ldots, b_j\}$

enumerate $(\bigcup B) \setminus \mathfrak{M}$ in such a way that $\min(b_i)$ increases with i, and for each $i \in \{1, \ldots, j\}$, let $\beta_{2i+1} = \min(b_i)$ and $\beta_{2i+2} = \max(b_i)$.

Let S be the set of increasing sequences of ordinals $\langle \gamma_1, \ldots, \gamma_{2j} \rangle$ such that

(1) γ_1 is greater than every member of $\bigcup(B \cap \mathfrak{M})$;
(2) letting $\pi \colon \bigcup B \to (\bigcup(B \cap \mathfrak{M})) \cup \{\gamma_1, \ldots, \gamma_{2j}\}$ be an order-preserving bijection, $\Phi(\{\alpha_1, \alpha_2\}) = \Phi(\{\pi(\alpha_1), \pi(\alpha_2)\})$ for all $\alpha_1 < \alpha_2$ from $\bigcup B$;
(3) for some N' containing $\mathfrak{M} \cap N$,

$$((B \cap \mathfrak{M}) \cup \{\{\gamma_{2i+1}, \gamma_{2i+2}\} : i \in j\}, N')$$

is an element of D.

Notice that while the Condition 2 mentions an object outside $\mathfrak{M}$, this object is finite and so the condition can be described by a first order formula in $\mathfrak{M} \cap H(\aleph_2)$. Since the theory $H(\aleph_2)$ can be coded by a real in transitive models, the existence of N' posited in Condition 3 can also be described in $\mathfrak{M} \cap H(\aleph_2)$.

As a consequence, if T_0 is defined to be the tree consisting of all initial segments of members of S, then, since $T_0 \in H(\aleph_2)$, $T_0 \in \mathfrak{M}$ and T_0 is an element of every model of N containing $\mathfrak{M} \cap H(\aleph_2)$. Since $\langle \beta_1, \ldots, \beta_j \rangle \in S$, and since each β_{2i+1} is separated by elementary submodels in N from β_{2i+2}, T_0 can be thinned (in $\mathfrak{M}$) to a tree T_1, still containing $\langle \beta_1, \ldots, \beta_j \rangle$, such that every node of T_1 on an odd level (where the least level is the 1st level) has uncountably many immediate successors. Finally, still in $\mathfrak{M}$, thin T_1 to a tree T, still of height j, such that each node of T on an even level has infinitely many immediate successors.

We wish to pick a sequence $\langle \gamma_1, \ldots, \gamma_{2j} \rangle$ from T such that, for some N' containing N, $(B \cup \{\{\gamma_{2i+1}, \gamma_{2i+2}\} : i \in j\}, N')$ is a condition. We pick the γ_i's recursively, picking any available ordinal when i is odd. When i is even, we need to pick γ_i so that, for each $b \in B \setminus \mathfrak{M}$ $\Phi(\{\gamma_{i-1}, \min(b)\}) < \Phi(\{\gamma_i, \max(b)\})$. Since B is finite, and we have infinitely many possibilities for γ_i, we can meet this condition, using the finite-to-one property of Φ.

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