POINT-COFINITE COVERS IN THE LAVER MODEL

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Abstract. Let $\mathsf{S}_1(\Gamma,\Gamma)$ be the statement: For each sequence of point-cofinite open covers, one can pick one element from each cover and obtain a point-cofinite cover. $\mathfrak{b}$ is the minimal cardinality of a set of reals not satisfying $\mathsf{S}_1(\Gamma,\Gamma)$. We prove the following assertions:

(1) If there is an unbounded tower, then there are sets of reals of cardinality $\mathfrak{b}$, satisfying $\mathsf{S}_1(\Gamma,\Gamma)$.
(2) It is consistent that all sets of reals satisfying $\mathsf{S}_1(\Gamma,\Gamma)$ have cardinality smaller than $\mathfrak{b}$.

These results can also be formulated as dealing with Arhangel'skiĭ's property α_2 for spaces of continuous real-valued functions.

The main technical result is that in Laver's model, each set of reals of cardinality $\mathfrak{b}$ has an unbounded Borel image in the Baire space ω^ω.

1. Background

Let P be a nontrivial property of sets of reals. The *critical cardinality* of P, denoted $\mathrm{non}(P)$, is the minimal cardinality of a set of reals not satisfying P. A natural question is whether there is a set of reals of cardinality at least $\mathrm{non}(P)$, which satisfies P, i.e., a *nontrivial* example.

We consider the following property. Let X be a set of reals. $\mathcal{U}$ is a *point-cofinite* cover of X if $\mathcal{U}$ is infinite, and for each $x \in X$, $\{U \in \mathcal{U} : x \in U\}$ is a cofinite subset of $\mathcal{U}$.[1] Having X fixed in the background, let Γ be the family of all point-cofinite *open* covers of X. The following properties were introduced by Hurewicz [8], Tsaban [19], and Scheepers [15], respectively.

$\mathsf{U}_{\mathrm{fin}}(\Gamma,\Gamma)$: For all $\mathcal{U}_0,\mathcal{U}_1,\dots \in \Gamma$, none containing a finite subcover, there are finite $\mathcal{F}_0 \subseteq \mathcal{U}_0, \mathcal{F}_1 \subseteq \mathcal{U}_1,\dots$ such that $\{\bigcup \mathcal{F}_n : n \in \omega\} \in \Gamma$.

$\mathsf{U}_2(\Gamma,\Gamma)$: For all $\mathcal{U}_0,\mathcal{U}_1,\dots \in \Gamma$, there are $\mathcal{F}_0 \subseteq \mathcal{U}_0, \mathcal{F}_1 \subseteq \mathcal{U}_1,\dots$ such that $|\mathcal{F}_n| = 2$ for all n, and $\{\bigcup \mathcal{F}_n : n \in \omega\} \in \Gamma$.

[1] Historically, point-cofinite covers were named *γ-covers*, since they are related to a property numbered γ in a list from α to ϵ in the seminal paper [7] of Gerlits and Nagy.

$\mathsf{S}_1(\Gamma,\Gamma)$**:** For all $\mathcal{U}_0, \mathcal{U}_1, \dots \in \Gamma$, there are $U_0 \in \mathcal{U}_0, U_1 \in \mathcal{U}_1, \dots$ such that $\{U_n : n \in \omega\} \in \Gamma$.

Clearly, $\mathsf{S}_1(\Gamma,\Gamma)$ implies $\mathsf{U}_2(\Gamma,\Gamma)$, which in turn implies $\mathsf{U}_{\mathrm{fin}}(\Gamma,\Gamma)$. None of these implications is reversible in ZFC [19]. The critical cardinality of all three properties is $\mathfrak{b}$ [9].[2]

Bartoszyński and Shelah [1] proved that there are, provably in ZFC, totally imperfect sets of reals of cardinality $\mathfrak{b}$ satisfying the Hurewicz property $\mathsf{U}_{\mathrm{fin}}(\Gamma,\Gamma)$. Tsaban proved the same assertion for $\mathsf{U}_2(\Gamma,\Gamma)$ [19]. These sets satisfy $\mathsf{U}_{\mathrm{fin}}(\Gamma,\Gamma)$ in all finite powers [2].

We show that in order to obtain similar results for $\mathsf{S}_1(\Gamma,\Gamma)$, hypotheses beyond ZFC are necessary.

2. Constructions

We show that certain weak (but not provable in ZFC) hypotheses suffice to have nontrivial $\mathsf{S}_1(\Gamma,\Gamma)$ sets, even ones which possess this property in all finite powers.

Definition 2.1. A *tower* of cardinality κ is a set $T \subseteq [\omega]^\omega$ which can be enumerated bijectively as $\{x_\alpha : \alpha < \kappa\}$, such that for all $\alpha < \beta < \kappa$, $x_\beta \subseteq^* x_\alpha$.

A set $T \subseteq [\omega]^\omega$ is *unbounded* if the set of its enumeration functions are unbounded, i.e., for any $g \in \omega^\omega$ there is an $x \in T$ such that for infinitely many n, $g(n)$ is less than the n-th element of x.

Scheepers [16] proved that if $\mathfrak{t} = \mathfrak{b}$, then there is a set of reals of cardinality $\mathfrak{b}$, satisfying $\mathsf{S}_1(\Gamma,\Gamma)$. If $\mathfrak{t} = \mathfrak{b}$, then there is an unbounded tower of cardinality $\mathfrak{b}$, but the latter assumption is weaker.

Lemma 2.2 (folklore)**.** *If* $\mathfrak{b} < \mathfrak{d}$*, then there is an unbounded tower of cardinality* $\mathfrak{b}$*.*

Proof. Let $B = \{b_\alpha : \alpha < \mathfrak{b}\} \subseteq \omega^\omega$ be a $\mathfrak{b}$-scale, that is, each b_α is increasing, $b_\alpha \leq^* b_\beta$ for all $\alpha < \beta < \mathfrak{b}$, and B is unbounded.

As $|B| < \mathfrak{d}$, B is not dominating. Let $g \in \omega^\omega$ exemplify that. For each $\alpha < \mathfrak{b}$, let $x_\alpha = \{n : b_\alpha(n) \leq g(n)\}$. Then $T = \{x_\alpha : \alpha < \mathfrak{b}\}$ is an unbounded tower: Clearly, $x_\beta \subseteq^* x_\alpha$ for $\alpha < \beta$. Assume that T is bounded, and let $f \in \omega^\omega$ exemplify that. For each α, writing $x_\alpha(n)$ for the n-th element of x_α:

$$b_\alpha(n) \leq b_\alpha(x_\alpha(n)) \leq g(x_\alpha(n)) \leq g(f(n))$$

for all but finitely many n. Thus, $g \circ f$ shows that B is bounded. A contradiction. □

[2]Blass's survey [4] is a good reference for the definitions and details about the special cardinals mentioned in this paper.

Theorem 2.3. *If there is an unbounded tower (of any cardinality), then there is a set of reals X of cardinality $\mathfrak{b}$, which satisfies $\mathsf{S}_1(\Gamma,\Gamma)$.*

Theorem 2.3 follows from the following two propositions.

Proposition 2.4. *If there is an unbounded tower, then there is one of cardinality $\mathfrak{b}$.*

Proof. By Lemma 2.2, it remains to consider the case $\mathfrak{b}=\mathfrak{d}$. Let T be unbounded tower of cardinality κ. Let $\{f_\alpha : \alpha<\mathfrak{b}\}\subseteq\omega^\omega$ be dominating. For each $\alpha<\mathfrak{b}$, pick $x_\alpha\in T$ which is not bounded by f_α. $\{x_\alpha : \alpha<\mathfrak{b}\}$ is unbounded, being unbounded in a dominating family. □

Scheepers's mentioned proof actually establishes the following result, to which we give an alternative proof.

Proposition 2.5 (essentially, Scheepers [16])**.** *For each unbounded tower T of cardinality $\mathfrak{b}$, $T\cup[\omega]^{<\omega}$ satisfies $\mathsf{S}_1(\Gamma,\Gamma)$.*

Proof. Let $T=\{x_\alpha : \alpha<\mathfrak{b}\}$ be an unbounded tower of cardinality $\mathfrak{b}$. For each α, let $X_\alpha=\{x_\beta : \beta<\alpha\}\cup[\omega]^{<\omega}$. Let $\mathcal{U}_0,\mathcal{U}_1,\dots$ be point-cofinite covers of $X_\mathfrak{b}=T\cup[\omega]^{<\omega}$. We may assume that each $\mathcal{U}_n$ is countable and that $\mathcal{U}_i\cap\mathcal{U}_j=\emptyset$ whenever $i\neq j$.

By the proof of Lemma 1.2 of [6], for each k there are distinct $U^k_0,U^k_1,\dots\in\mathcal{U}_k$, and an increasing sequence $m^k_0<m^k_1<\dots$, such for each n and k,

$$\{x\subseteq\omega\ :\ x\cap(m^k_n,m^k_{n+1})=\emptyset\}\subseteq U^k_n.$$

As T is unbounded, there is $\alpha<\mathfrak{b}$ such that for each k, $I_k=\{n : x_\alpha\cap(m^k_n,m^k_{n+1})=\emptyset\}$ is infinite.

For each k, $\{U^k_n : n\in\omega\}$ is an infinite subset of $\mathcal{U}_k$, and thus a point-cofinite cover of X_α. As $|X_\alpha|<\mathfrak{b}$, there is $f\in\omega^\omega$ such that

$$\forall x\in X_\alpha\ \exists k_0\ \forall k\geq k_0\ \forall n>f(k)\ \ x\in U^k_n.$$

For each k, pick $n_k\in I_k$ such that $n_k>f(k)$ and $U^k_{n_k}\neq U^l_{n_l}$ for $l<k$.

We claim that $\{U^k_{n_k} : k\in\omega\}$ is a point-cofinite cover of $X_\mathfrak{b}$: If $x\in X_\alpha$, then $x\in U^k_{n_k}$ for all but finitely many k, because $n_k>f(k)$ for all k. If $x=x_\beta$, $\beta\geq\alpha$, then $x\subseteq^* x_\alpha$. For each large enough k, $m^k_{n_k}$ is large enough, so that $x\cap(m^k_{n_k},m^k_{n_k+1})\subseteq x_\alpha\cap(m^k_{n_k},m^k_{n_k+1})=\emptyset$, and thus $x\in U^k_{n_k}$. □

Remark 2.6. Zdomskyy points out that for the proof to go through, it suffices that $\{x_\alpha : \alpha<\mathfrak{b}\}$ is such that there is an unbounded $\{y_\alpha : \alpha<\mathfrak{b}\}\subseteq[\omega]^\omega$ such that for each α, x_α is a pseudointersection of

$\{y_\beta : \beta < \alpha\}$. We do not know whether the assertion mentioned here is weaker than the existence of an unbounded tower.

We now turn to nontrivial examples of sets satisfying $\mathsf{S}_1(\Gamma,\Gamma)$ in all finite powers. In general, $\mathsf{S}_1(\Gamma,\Gamma)$ is not preserved by taking finite powers [9], and we use a slightly stronger hypothesis in our construction.

Definition 2.7. Let $\mathfrak{b}_0$ be the additivity number of $\mathsf{S}_1(\Gamma,\Gamma)$, that is, the minimum cardinality of a family $\mathcal{F}$ of sets of reals, each satisfying $\mathsf{S}_1(\Gamma,\Gamma)$, such that the union of all members of $\mathcal{F}$ does not satisfy $\mathsf{S}_1(\Gamma,\Gamma)$.

$\mathfrak{t} \leq \mathfrak{h}$, and Scheepers proved that $\mathfrak{h} \leq \mathfrak{b}_0 \leq \mathfrak{b}$ [17]. It is open whether $\mathfrak{b}_0 = \mathfrak{b}$. If $\mathfrak{t} = \mathfrak{b}$ or $\mathfrak{h} = \mathfrak{b} < \mathfrak{d}$, then there is an unbounded tower of cardinality $\mathfrak{b}_0$.

Theorem 2.8. *For each unbounded tower T of cardinality $\mathfrak{b}_0$, all finite powers of $T \cup [\omega]^{<\omega}$ satisfy $\mathsf{S}_1(\Gamma,\Gamma)$.*

Proof. We say that $\mathcal{U}$ is an *ω-cover* of X if no member of $\mathcal{U}$ contains X as a subset, but each finite subset of X is contained in some member of $\mathcal{U}$. We need a multidimensional version of Lemma 1.2 of [6].

Lemma 2.9. *Assume that $[\omega]^{<\omega} \subseteq X \subseteq P(\omega)$, and let $e \in \omega$. For each open ω-cover $\mathcal{U}$ of X^e, there are $m_0 < m_1 < \dots$ and distinct $U_0, U_1, \dots \in \mathcal{U}$, such that for all $x_0, \dots, x_{e-1} \subseteq \omega$, $(x_0, \dots, x_{e-1}) \in U_n$ whenever $x_i \cap (m_n, m_{n+1}) = \emptyset$ for all $i < e$.*

Proof. As $\mathcal{U}$ is an open ω-cover of X^e, there is an open ω-cover $\mathcal{V}$ of X such that $\{V^e : V \in \mathcal{V}\}$ refines $\mathcal{U}$ [9].

Let $m_0 = 0$. For each $n \geq 0$: Assume that $V_0, \dots, V_{n-1} \in \mathcal{V}$ are given, and $U_0, \dots, U_{n-1} \in \mathcal{U}$ are such that $V_i^e \subseteq U_i$ for all $i < n$. Fix a finite $F \subseteq X$ such that F^e is not contained in any of the sets $U_0, \dots, U_{n-1}$. As $\mathcal{V}$ is an ω-cover of X, there is $V_n \in \mathcal{V}$ such that $F \cup P(\{0, \dots, m_n\}) \subseteq V_n$. Take $U_n \in \mathcal{U}$ such that $V_n^e \subseteq U_n$. Then $U_n \notin \{U_0, \dots, U_{n-1}\}$. As V_n is open, for each $s \subseteq \{0, \dots, m_n\}$ there is k_s such that for each $x \in P(\omega)$ with $x \cap \{0, \dots, k_s - 1\} = s$, $x \in V_n$. Let $m_{n+1} = \max\{k_s : s \subseteq \{0, \dots, m_n\}\}$.

If $x_i \cap (m_n, m_{n+1}) = \emptyset$ for all $i < e$, then $(x_0, \dots, x_{e-1}) \in V_n^e \subseteq U_n$. □

The assumption in the theorem implies that $\mathfrak{b}_0 = \mathfrak{b}$. The proof is by induction on the power e of $T \cup [\omega]^{<\omega}$. The case $e = 1$ follows from Theorem 2.5.

Let $\mathcal{U}_0, \mathcal{U}_1, \dots \in \Gamma((T \cup [\omega]^{<\omega})^e)$. We may assume that these covers are countable. As in the proof of Theorem 2.5 (this time using Lemma

2.9), there are for each k $m_0^k < m_1^k < \dots$ and distinct $U_0^k, U_1^k, \dots \in \mathcal{U}_k$ (so that $\{U_n^k : n \in \omega\} \in \Gamma((T \cup [\omega]^{<\omega})^e))$, such that all $y_0, \dots, y_{e-1} \subseteq \omega$, $(y_0, \dots, y_{e-1}) \in U_n^k$ whenever $y_i \cap (m_n^k, m_{n+1}^k) = \emptyset$ for all $i < e$.

Let α_0 be such that X_{α_0} is not contained in any member of $\bigcup_n \mathcal{U}_n$. As T is unbounded, there is α such that $\alpha_0 \le \alpha < \mathfrak{b}$, and for each k, $I_k = \{n : x_\alpha \cap (m_n^k, m_{n+1}^k) = \emptyset\}$ is infinite.

Let $Y = \{x_\beta : \beta \ge \alpha\}$. $(T \cup [\omega]^{<\omega})^e \setminus Y^e$ is a union of fewer than $\mathfrak{b}_0$ homeomorphic copies of $(T \cup [\omega]^{<\omega})^{e-1}$. By the induction hypothesis, $(T \cup [\omega]^{<\omega})^{e-1}$ satisfies $\mathsf{S}_1(\Gamma, \Gamma)$, and therefore so does $(T \cup [\omega]^{<\omega})^e \setminus Y^e$. For each k, $\{U_n^k : n \in I_k\}$ is a point-cofinite cover of $(T \cup [\omega]^{<\omega})^e \setminus Y^e$, and thus there are infinite $J_0 \subseteq I_0, J_1 \subseteq I_1, \dots$, such that $\{\bigcap_{n \in J_k} U_n^k : k \in \omega\}$ is a point-cofinite cover of $(T \cup [\omega]^{<\omega})^e \setminus Y^e$.[3] For each k, pick $n_k \in J_k$ such that: $m_{n_k}^k > m_{n_{k-1}+1}^{k-1}$, $x_\alpha \cap (m_{n_k}^k, m_{n_k+1}^k) = \emptyset$, and $U_{n_k}^k \notin \{U_{n_0}^0, \dots, U_{n_{k-1}}^{k-1}\}$.

$\{U_{n_k}^k : k \in \omega\} \in \Gamma(T \cup [\omega]^{<\omega})$: If $x \in (T \cup [\omega]^{<\omega})^e \setminus Y^e$, then $x \in U_{n_k}^k$ for all but finitely many k. If $x = (x_{\beta_0}, \dots, x_{\beta_{e-1}}) \in Y$, then $\beta_0, \dots, \beta_{e-1} \ge \alpha$, and thus $x_{\beta_0}, \dots, x_{\beta_{e-1}} \subseteq^* x_\alpha$. For each large enough k, $m_{n_k}^k$ is large enough, so that $x_{\beta_i} \cap (m_{n_k}^k, m_{n_k+1}^k) \subseteq x_\alpha \cap (m_{n_k}^k, m_{n_k+1}^k) = \emptyset$ for all $i < e$, and thus $x \in U_{n_k}^k$. □

There is an additional way to obtain nontrivial $\mathsf{S}_1(\Gamma, \Gamma)$ sets: The hypothesis $\mathfrak{b} = \mathrm{cov}(\mathcal{N}) = \mathrm{cof}(\mathcal{N})$ provides $\mathfrak{b}$-Sierpiński sets, and $\mathfrak{b}$-Sierpiński sets satisfy $\mathsf{S}_1(\Gamma, \Gamma)$, even for *Borel* point-cofinite covers. Details are available in [18].

We record the following consequence of Theorem 2.3 for later use.

Corollary 2.10. *For each unbounded tower T of cardinality $\mathfrak{b}$, $T \cup [\omega]^{<\omega}$ satisfies $\mathsf{S}_1(\Gamma, \Gamma)$ for open covers, but not for Borel covers.*

Proof. The latter property is hereditary for subsets [18]. By a theorem of Hurewicz, a set of reals satisfies $\mathsf{U}_{\mathrm{fin}}(\Gamma, \Gamma)$ if, and only if, each continuous image of X in ω^ω is bounded. It follows that the set $T \subseteq T \cup [\omega]^{<\omega}$ does not even satisfy $\mathsf{U}_{\mathrm{fin}}(\Gamma, \Gamma)$. □

3. A consistency result

By the results of the previous section, we have the following.

Lemma 3.1. *Assume that every set of reals with property $\mathsf{S}_1(\Gamma, \Gamma)$ has cardinality $< \mathfrak{b}$, and $\mathfrak{c} = \aleph_2$. Then $\aleph_1 = \mathfrak{t} = \mathrm{cov}(\mathcal{N}) < \mathfrak{b} = \aleph_2$.*

[3] Choosing infinitely many elements from each cover, instead of one, can be done by adding to the given sequence of covers all cofinite subsets of the given covers.

Proof. As there is no unbounded tower, we have that $\mathfrak{t} < \mathfrak{b} = \mathfrak{d}$. As $\mathfrak{c} = \aleph_2$, $\aleph_1 = \mathfrak{t} < \mathfrak{b} = \aleph_2$. Since there are no $\mathfrak{b}$-Sirepiński sets and $\mathfrak{b} = \mathrm{cof}(\mathcal{N}) = \mathfrak{c}$, $\mathrm{cov}(\mathcal{N}) < \mathfrak{b}$. □

In Laver's model [11], $\aleph_1 = \mathfrak{t} = \mathrm{cov}(\mathcal{N}) < \mathfrak{b} = \aleph_2$. We will show that indeed, $\mathsf{S}_1(\Gamma, \Gamma)$ is trivial there. Laver's model was constructed to realize Borel's Conjecture, asserting that "strong measure zero" is trivial. In some sense, $\mathsf{S}_1(\Gamma, \Gamma)$ is a dual of strong measure zero. For example, the canonical examples of $\mathsf{S}_1(\Gamma, \Gamma)$ sets are Sierpiński sets, a measure theoretic object, whereas the canonical examples of strong measure zero sets are Luzin sets, a Baire category theoretic object. More about that can be seen in [18].

The main technical result of this paper is the following.

Theorem 3.2. *In the Laver model, if $X \subseteq 2^\omega$ has cardinality $\mathfrak{b}$, then there is a Borel map $f : 2^\omega \to \omega^\omega$ such that $f[X]$ is unbounded.*

Proof. The notation in this proof is as in Laver [11]. We will use the following slightly simplified version of Lemma 14 of [11].

Lemma 3.3 (Laver)**.** *Let $\mathbb{P}_{\omega_2}$ be the countable support iteration of Laver forcing, $p \in \mathbb{P}_{\omega_2}$, and $\mathring{a}$ be a $\mathbb{P}_{\omega_2}$-name such that*

$$p \Vdash \mathring{a} \in 2^\omega.$$

Then there are a condition q stronger than p, and finite $U_s \subseteq 2^\omega$ for each $s \in q(0)$ extending the root of $q(0)$, such that for all such s and all n:

$$q(0)_t \,\hat{}\, q \restriction [1, \omega_2) \Vdash \text{“} \exists u \in \check{U}_s \;\; u = \mathring{a} \restriction n \text{ ”}$$

for all but finitely many immediate successors t of s in $q(0)$.

Assume that $X \subseteq 2^\omega$ has no unbounded Borel image in $\mathcal{M}[G_{\omega_2}]$, Laver's model. For every code $u \in 2^\omega$ for a Borel function $f : 2^\omega \to \omega^\omega$ there exists $g \in \omega^\omega$ such that for every $x \in X$ we have that $f(x) \leq^* g$.

By a standard Löwenheim-Skolem argument, see Theorem 4.5 on page 281 of [3], or section 4 on page 580 of [12], we may find $\alpha < \omega_2$ such that for every code $u \in \mathcal{M}[G_\alpha]$ there is an upper bound $g \in \mathcal{M}[G_\alpha]$. By the arguments employed by Laver [11, Lemmata 10 and 11], we may assume that $\mathcal{M}[G_\alpha]$ is the ground model $\mathcal{M}$.

Since the continuum hypothesis holds in $\mathcal{M}$ and $|X| = \mathfrak{b} = \aleph_2$, there are $p \in G_{\omega_2}$ and $\mathring{a}$ such that

$$p \Vdash \mathring{a} \in \mathring{X} \text{ and } \mathring{a} \notin \mathcal{M}.$$

Work in the ground model $\mathcal{M}$.

Let $q \leq p$ be as in Lemma 3.3. Define

$$Q = \{s \in q(0) \;:\; \mathrm{root}(q(0)) \subseteq s\}$$

and let U_s, $s \in Q$, be the finite sets from the Lemma. Let $U = \bigcup_{s\in Q} U_s$. Define a Borel map $f : 2^\omega \to \omega^Q$ so that for every $x \in 2^\omega \setminus U$ and for each $s \in Q$: If $f(x)(s) = n$, then $x \restriction n \neq u \restriction n$ for each $u \in U_s$. For $x \in U$, $f(x)$ may be arbitrary. There must be a $g \in \omega^Q \cap \mathcal{M}$ and $r \leq q$ such that

$$r \Vdash f(\mathring{a}) \leq^* \check{g}.$$

Since p forced that a is not in the ground model, it cannot be that a is in U. We may extend $r(0)$ if necessary so that if $s = \mathrm{root}(r(0))$, then

$$r \Vdash f(\mathring{a})(s) \leq \check{g}(s).$$

But this is a contradiction to Lemma 3.3, since for all but finitely many $t \in r(0)$ which are immediate extensions of s:

$$r(0)_t \,\hat{}\, q \restriction [1, \omega_2) \Vdash f(\mathring{a})(s) > \check{g}(s).$$

□

In [20], Tsaban and Zdomskyy prove that $\mathsf{S}_1(\Gamma,\Gamma)$ for Borel covers is equivalent to the Kočinac property $\mathsf{S}_{\mathrm{cof}}(\Gamma,\Gamma)$ [10], asserting that for all $\mathcal{U}_0, \mathcal{U}_1, \dots \in \Gamma$, there are cofinite subsets $\mathcal{V}_0 \subseteq \mathcal{U}_0, \mathcal{V}_1 \subseteq \mathcal{U}_1, \dots$ such that $\bigcup_n \mathcal{V}_n \in \Gamma$. The main result of [5] can be reformulated as follows.

Theorem 3.4 (Dow [5])**.** *In Laver's model,* $\mathsf{S}_1(\Gamma,\Gamma)$ *implies* $\mathsf{S}_{\mathrm{cof}}(\Gamma,\Gamma)$.

For the reader's convenience, we give Dow's proof, adopted to the present notation.

Proof. A family $\mathcal{H} \subseteq [\omega]^\omega$ is *ω-splitting* if for each countable $\mathcal{A} \subseteq [\omega]^\omega$, there is $H \in \mathcal{H}$ which splits each element of $\mathcal{A}$, i.e.,

$$|A \cap H| = |A \setminus H| = \omega \text{ for all } A \in \mathcal{A}.$$

The main technical result in [5] is the following.

Lemma 3.5 (Dow)**.** *In Laver's model, each ω-splitting family contains an ω-splitting family of cardinality* $< \mathfrak{b}$.

Assume that X satisfies $\mathsf{S}_1(\Gamma,\Gamma)$. Let $\mathcal{U}_0, \mathcal{U}_1, \dots$ be open point-cofinite countable covers of X. We may assume[4] that $\mathcal{U}_i \cap \mathcal{U}_j = \emptyset$ whenever $i \neq j$. Put $\mathcal{U} = \bigcup_{n<\omega} \mathcal{U}_n$. We identify $\mathcal{U}$ with ω, its cardinality.

Define $\mathcal{H} \subseteq [\mathcal{U}]^\omega$ as follows. For $H \in [\mathcal{U}]^\omega$, put $H \in \mathcal{H}$ if and only if there exists $\mathcal{V} \in [\mathcal{U}]^\omega$, a point-cofinite cover of X, such that $H \cap \mathcal{U}_n \subseteq^* \mathcal{V}$ for all n. We claim that $\mathcal{H}$ is an ω-splitting family. As $\mathcal{H}$ is closed under taking infinite subsets, it suffices to show that it is *ω-hitting*, i.e., for any countable $\mathcal{A} \subseteq [\mathcal{U}]^\omega$ there exists $H \in \mathcal{H}$ which

[4] To see why, replace each $\mathcal{U}_n$ by $\mathcal{U}_n \setminus \bigcup_{i<n} \mathcal{U}_i$, and discard the finite ones. It suffices to show that $\mathsf{S}_{\mathrm{cof}}(\Gamma,\Gamma)$ applies to those that are left.

intersects each $A \in \mathcal{A}$. (It is enough to intersect each $A \in \mathcal{A}$, since we may assume that $\mathcal{A}$ is closed under taking cofinite subsets.)

Let $\mathcal{A} \subseteq [\mathcal{U}]^\omega$ be countable. For each n, choose sets $\mathcal{U}_{n,m} \in [\mathcal{U}_n]^\omega$, $m \in \omega$, such that for each $A \in \mathcal{A}$, if $A \cap \mathcal{U}_n$ is infinite, then $\mathcal{U}_{n,m} \subseteq A$ for some m. Apply the $\mathsf{S}_1(\Gamma,\Gamma)$ to the family $\{\mathcal{U}_{n,m} : n, m \in \omega\}$, to obtain a point-cofinite $\mathcal{V} \subseteq \mathcal{U}$ such that $\mathcal{V} \cap \mathcal{U}_{n,m}$ is nonempty for all n, m.

Next, choose finite subsets $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n \in \omega$, such that for each $A \in \mathcal{A}$ with $A \cap \mathcal{U}_n$ finite for all n, then $A \subseteq^* \bigcup_n \mathcal{F}_n$. Take $H = \mathcal{V} \cup \bigcup_n \mathcal{F}_n$. Then H is in $\mathcal{H}$ and meets each $A \in \mathcal{A}$. This shows that $\mathcal{H}$ is an ω-splitting family.

By Lemma 3.5, there is an ω-splitting $\mathcal{H}' \subseteq \mathcal{H}$ of cardinality $< \mathfrak{b}$. For each $H \in \mathcal{H}'$, let $\mathcal{V}_H$ witness that H is in $\mathcal{H}$, i.e., $\mathcal{V}_H \subseteq \mathcal{U}$ is a point-cofinite cover of X and $H \cap \mathcal{U}_n \subseteq^* \mathcal{V}_H$ for all n.

By the definition of $\mathfrak{b}$, we may find finite $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n \in \omega$, such that for each $H \in \mathcal{H}'$,

$$H \cap \mathcal{U}_n \subseteq \mathcal{V}_H \cup \mathcal{F}_n$$

for all but finitely many n. We claim that $\mathcal{W} = \bigcup_n \mathcal{U}_n \setminus \mathcal{F}_n$ is point-cofinite. Suppose it is not. Then there is $x \in X$ such that for infinitely many n, there is $U_n \in \mathcal{U}_n \setminus \mathcal{F}_n$ with $x \notin U_n$. Let $H \in \mathcal{H}'$ contain infinitely many of these U_n. By the above inclusion, all but finitely many of these U_n are in $\mathcal{V}_H$. This contradicts the fact that $\mathcal{V}_H$ is point-cofinite. □

We therefore have the following.

Theorem 3.6. *In Laver's model, each set of reals X satisfying $\mathsf{S}_1(\Gamma,\Gamma)$ has cardinality less than $\mathfrak{b}$.*

Proof. By Dow's Theorem, $\mathsf{S}_1(\Gamma,\Gamma)$ implies $\mathsf{S}_{\mathrm{cof}}(\Gamma,\Gamma)$, which in turn implies $\mathsf{S}_1(\Gamma,\Gamma)$ for Borel covers [20]. The latter property is equivalent to having all Borel images in ω^ω bounded [18]. Apply Theorem 3.2. □

Thus, it is consistent that strong measure zero and $\mathsf{S}_1(\Gamma,\Gamma)$ are both trivial.

The proof of Dow's Theorem 3.4 becomes more natural after replacing, in Lemma 3.5 "ω-splitting" by "ω-hitting". This is possible, due to the following fact (cf. Remark 4 of [5]).

Proposition 3.7. *For each infinite cardinal κ, the following are equivalent:*

(1) *Each ω-splitting family contains an ω-splitting family of cardinality $< \kappa$.*

(2) *Each ω-hitting family contains an ω-hitting family of cardinality $< \kappa$.*

Proof. ($1 \Rightarrow 2$) Suppose $\mathcal{A}$ is an ω-hitting family. Let $\mathcal{B} = \bigcup_{A \in \mathcal{A}} [A]^\omega$. Then $\mathcal{B}$ is ω-splitting. By (1) there exists $\mathcal{C} \subseteq \mathcal{B}$ of size $< \kappa$ which is ω-splitting. Choose $\mathcal{D} \subseteq \mathcal{A}$ of size $< \kappa$ such that for every $C \in \mathcal{C}$ there exists $D \in \mathcal{D}$ with $C \subseteq D$. Then $\mathcal{D}$ is ω-hitting.

($2 \Rightarrow 1$) Suppose $\mathcal{A}$ is an ω-splitting family. For each $A \subseteq \omega$ define

$$A^* = \{2n : n \in A\} \cup \{2n+1 : n \in \overline{A}\}.$$

Then the family $\mathcal{A}^* = \{A^* : A \in \mathcal{A}\}$ is ω-hitting. To see this, suppose that $\mathcal{B}$ is countable. Without loss we may assume that $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1$ where each element of $\mathcal{B}_0$ is a subset of the evens and each element of $\mathcal{B}_1$ is a subset of the odds. For $B \in \mathcal{B}_0$ let $C_B = \{n : 2n \in B\}$ and for $B \in \mathcal{B}_1$ let $C_B = \{n : 2n+1 \in B\}$. Now put

$$\mathcal{C} = \{C_B : B \in \mathcal{B}\}.$$

Since $\mathcal{A}$ is ω-splitting there is $A \in \mathcal{A}$ which splits $\mathcal{C}$. If $B \in B_0$, then $A \cap C_B$ infinite implies $B \cap A^*$ infinite. If $B \in B_1$ then $\overline{A} \cap C_B$ infinite implies $B \cap A^*$ infinite.

By (2) there exists $\mathcal{A}_0 \subseteq \mathcal{A}$ of cardinality $< \kappa$ such that $\mathcal{A}_0^*$ is ω-hitting. We claim that $\mathcal{A}_0$ is ω-splitting. Given any $B \subseteq \omega$ let $B' = \{2n : n \in B\}$ and let $B'' = \{2n+1 : n \in B\}$. Given $\mathcal{B} \subseteq [\omega]^\omega$ countable, there exists $A \in \mathcal{A}_0$ such that A^* hits each B' and B'' for $B \in \mathcal{B}$. But this implies that A splits B. □

4. Applications to Arhangel'skiĭ's α_i spaces

Let Y be a general (not necessarily metrizable) topological space. We say that a countably infinite set $A \subseteq Y$ converges to a point $y \in Y$ if each (equivalently, some) bijective enumeration of A converges to y. The following concepts are due to Arhangel'skiĭ. Y is an α_1 *space* if for each $y \in Y$ and each sequence $A_0, A_1, \dots$ of countably infinite sets, each converging to y, there are cofinite $B_0 \subseteq A_0, B_1 \subseteq A_1, \dots$, such that $\bigcup_n B_n$ converges to y. Replacing "cofinite" by "singletons" (or equivalently, by "infinite"), we obtain the definition of an α_2 *space*.

We first consider countable spaces.

Definition 4.1. Let X be a set of reals, and let $\mathcal{U}_0, \mathcal{U}_1, \dots$ be countable point-cofinite covers of X. For each n, enumerate bijectively $\mathcal{U}_n = \{U^n_m : m \in \omega\}$. We associate to X a (new) topology τ on the fan $S_\omega = \omega \times \omega \cup \{\infty\}$ as follows: ∞ is the only nonisolated point of S_ω, and a neighborhood base at ∞ is given by the sets

$$[\infty]_F = \{(n,m) : F \subseteq U^n_m\}$$

for each finite $F \subseteq X$.

Lemma 4.2. *In the notation of Definition 4.1: A converges to ∞ in τ if, and only if, $\mathcal{U}(A) = \{U^n_m : (n,m) \in A\}$ is a point-cofinite cover of X.* □

Assume that there is an unbounded tower. By Corollary 2.10, there is a set of reals X satisfying $\mathsf{S}_1(\Gamma,\Gamma)$ but not $\mathsf{S}_{\mathrm{cof}}(\Gamma,\Gamma)$. Let $\mathcal{U}_0, \mathcal{U}_1, \dots$ be countable open point-cofinite covers of X witnessing the failure of $\mathsf{S}_{\mathrm{cof}}(\Gamma,\Gamma)$. Then, by Lemma 4.2, (S_ω, τ) is α_2 but not α_1. In particular, we reproduce the following.

Corollary 4.3 (Nyikos [13])**.** *If there is an unbounded tower of cardinality $\mathfrak{b}$, then there is a countable α_2 space, which is not an α_1 space.* □

Recall that by Proposition 2.4, it suffices to assume in Corollary 4.3 the existence of any unbounded tower.

Next, we consider spaces of continuous functions. Consider $C(X)$, the family of continuous real-valued functions, as a subspace of the Tychonoff product $\mathbb{R}^X$, i.e., with the topology of pointwise convergence. Sakai [14] proved that X satisfies $\mathsf{S}_1(\Gamma,\Gamma)$ for *clopen* covers if, and only if, $C(X)$ is an α_2 space. The main result of [20] is that $C(X)$ is α_1 if, and only if, X satisfies $\mathsf{S}_1(\Gamma,\Gamma)$ for Borel covers (equivalently, each Borel image of X in ω^ω is bounded).

The *Scheepers Conjecture* is that for subsets of $\mathbb{R} \setminus \mathbb{Q}$, $\mathsf{S}_1(\Gamma,\Gamma)$ for clopen covers implies $\mathsf{S}_1(\Gamma,\Gamma)$ for open covers. Dow [5] proved that in Laver's model, every α_2 space is α_1. By Theorem 3.2, we can add the last item in the following list.

Corollary 4.4. *In Laver's model, the following are equivalent for sets of reals X:*

1. *$C(X)$ is an α_2 space;*
2. *$C(X)$ is an α_1 space;*
3. *X satisfies $\mathsf{S}_1(\Gamma,\Gamma)$ for clopen covers;*
4. *X satisfies $\mathsf{S}_1(\Gamma,\Gamma)$ for open covers;*
5. *X satisfies $\mathsf{S}_1(\Gamma,\Gamma)$ for Borel covers;*
6. $|X| < \mathfrak{b}$. □

On the other hand, Corollary 2.10 implies the following.

Corollary 4.5. *If there is an unbounded tower, then there is a set of reals X such that $C(X)$ is α_2 but not α_1.* □

Essentially, Corollary 4.3 is a special case of Corollary 2.10, whereas Corollary 4.5 is equivalent to Corollary 2.10.

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