

PARTITIONS OF 2^ω AND COMPLETELY ULTRAMETRIZABLE SPACES

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ABSTRACT. We prove that, for every n , the topological space ω_n^ω (where ω_n has the discrete topology) can be partitioned into \aleph_n copies of the Baire space. Using this fact, the authors then prove two new theorems about completely ultrametrizable spaces. We say that Y is a condensation of X if there is a continuous bijection $f : X \rightarrow Y$. First, it is proved that ω^ω is a condensation of ω_n^ω if and only if ω^ω can be partitioned into \aleph_n Borel sets, and some consistency results are given regarding such partitions. It is also proved that it is consistent with *ZFC* that, for any $n < \omega$, $\mathfrak{c} = \omega_n$ and there are exactly $n + 3$ similarity types of perfect completely ultrametrizable spaces of size \mathfrak{c} . These results answer two questions of the first author from [1].

1. INTRODUCTION

Every zero-dimensional Polish space can be represented as the end space of a countable tree. This fact is exploited in many classical proofs: that every perfect Polish space is a condensation of ω^ω , the Alexandrov-Urysohn characterization of ω^ω , Brouwer's characterization of 2^ω . All these can be proved by combining a bit of topology with a bit of clever reasoning with trees.

In [1], the first author explores to what extent these classical proofs involving countable trees can be translated to the context of uncountable trees. The end spaces of arbitrary trees are precisely the completely ultrametrizable spaces, and several classical results for Polish spaces extend to this class. However, many proofs break down in the uncountable case. In general, it seems that, while many strong results for countable trees are provable in *ZFC*, the corresponding results for uncountable trees often require extra set-theoretic hypotheses.

One recurring theme in the analysis of uncountable trees is that of partitioning the spaces 2^ω and ω^ω . For example, it is proved in [1] (Theorem 6.2) that ω^ω is a condensation of κ^ω whenever ω^ω can be

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partitioned into κ Borel sets. It is also proved (Theorem 5.1) that 2^ω is a condensation (that is, a continuous bijective image) of every perfect completely ultrametrizable space of size $|2^\omega|$ if and only if 2^ω can be partitioned into κ copies of 2^ω for every $\kappa \leq \mathfrak{c}$.

Theorem 3.5 below, which is our main lemma for what follows, can be viewed as another partition theorem for completely ultrametrizable spaces. It states that $\omega_{\alpha+1}^\omega$ can be partitioned into $\aleph_{\alpha+1}$ copies of ω_α^ω . For κ below the first singular cardinal, a simple induction then allows us to write κ^ω as a union of κ copies of the Baire space ω^ω .

The second author, in [7], develops methods for finding models of *ZFC* in which ω^ω can be partitioned in nice ways. Here we refine these techniques (see Theorem 3.11) and, using the partition theorem of the previous paragraph, obtain nice partitions of the spaces κ^ω for $\kappa < \aleph_\omega$. This allows us to obtain several new consistency results for completely ultrametrizable spaces. These consistency results will be the topic of Section 3.

In Section 4, we present a more complex application of Theorem 3.11 to answer an open question from [1]. Specifically, we show that, for any value of $\mathfrak{c} < \omega_\omega$, it is consistent to have the minimum possible number of similarity classes of perfect completely ultrametrizable spaces of size \mathfrak{c} (X and Y are similar if each is a condensation of the other).

2. COMPLETELY ULTRAMETRIZABLE SPACES AND TREES

In what follows, if an ordinal or cardinal is treated as a topological space then it is assumed to have the discrete topology. As usual, every ordinal is equal to the set of its predecessors. Most of our terminology is standard, and the rest follows [1]. In the interest of making this paper self-contained, we will review in this section some definitions and preliminaries from [1] concerning trees and completely ultrametrizable spaces.

A **tree** is a connected, nonempty, infinite graph in which every two nodes are connected by exactly one path, together with a distinguished node called the **root**. If T is a tree and $s, t \in T$, we say that t **extends** s if the unique path from the root to t goes through s . We denote the extension relation by \leq .

The extension relation allows us to think of trees as partial orders, and in what follows we will freely confuse trees with “tree-like” partial orders. Especially ubiquitous is $\kappa^{<\omega}$, the set of all finite sequences in κ , ordered by inclusion.

Two nodes of a tree T are **incomparable** if neither one extends the other. A tree is **pruned** if each of its elements has a proper extension, and is **perfect** if each of its elements has two incomparable proper extensions. In what follows, a “tree” will always mean a pruned, nonempty tree.

If T is a tree and s is a node of T , then

$$T_s = \{t \in T : t \leq s \text{ or } s \leq t\}$$

is the set of all nodes of T that compare with s under the extension relation. If $s \leq t$ and there is no r such that $s \leq r \leq t$, then t is a **child** of s .

In every tree T , there is a unique path from the root to a given node. This naturally divides T into levels. We say that a node s is at **level** n , denoted $\text{lev}(s) = n$, if the unique path from the root to s has $n - 1$ elements. Thus the root is the unique node at level 0, the children of the root are all at level 1, etc. We write $\text{Lev}_n(T)$ for $\{s \in T : \text{lev}(s) = n\}$.

A **branch** of a tree T is an infinite sequence x of nodes in T such that $x(0)$ is the root and $x(n+1)$ is a child of $x(n)$ for every n . $[[T]]$ is the set of all branches of T . $[[T]]$ has a natural topology defined by taking $\{[[T_s]] : s \in T\}$ to be a basis. This space is sometimes called the **end space** of T .

Proposition 2.1. *If T is a (perfect) tree, then $[[T]]$ is a (perfect) completely ultrametrizable space. If X is a (perfect) completely ultrametrizable space, then there is a (perfect) tree T such that $[[T]] \cong X$.*

Proof. See [3] for a thorough treatment. □

We will say that a space X is **represented** by a tree T whenever $X \cong [[T]]$. Thus Proposition 2.1 can be rephrased by saying that the (perfect) completely ultrametrizable spaces are precisely those representable by (perfect) trees. Note that, for all cardinals κ , $\kappa^{<\omega}$ represents κ^ω . In particular, $2^{<\omega}$ represents the Cantor space and $\omega^{<\omega}$ represents the Baire space.

Let T be a tree and let X be a topological space. A **T -scheme** on X is a family $(B_s)_{s \in T}$ of subsets of X such that

- $B_t \subseteq B_s$ whenever t is an extension of s .
- $B_s \cap B_t = \emptyset$ whenever s and t are incompatible.

If d is a metric on X then $(B_s)_{s \in T}$ has **vanishing diameter** (with respect to d) if $\lim_{n \rightarrow \infty} \text{diam}(B_{x(n)}) = 0$ whenever $x \in [[T]]$. If X is a metric space and $(B_s)_{s \in T}$ is a T -scheme with vanishing diameter,

then let $D = \{x \in \llbracket T \rrbracket : \bigcap_{n \in \omega} A_{x(n)} \neq \emptyset\}$ and define $f : D \rightarrow X$ by $\{f(x)\} = \bigcap_{n < \omega} B_{x(n)}$. We call f the **associated map**.

Lemma 2.2. *Let $(B_s)_{s \in T}$ be a T -scheme with vanishing diameter on a metric space (X, d) . If $f : D \rightarrow X$ is the associated map, then*

- (1) f is injective and continuous.
- (2) if $B_s = \bigcup \{B_t : t \text{ is a child of } s\}$ for all $s \in T$, then f is surjective.
- (3) if B_s is open for every $s \in T$, then f is open.

Proof. See [1], Lemma 2.3. □

3. PARTITIONS OF LARGE SPACES

As mentioned in the introduction, the existence of well-behaved partitions of completely ultrametrizable spaces determines a good deal about the maps between them. In this section we prove our main lemma concerning partitions of the spaces κ^ω , and use this lemma to derive some consistency results.

Lemma 3.1. *Let T be any tree. Then every open subset of $\llbracket T \rrbracket$ can be written as a disjoint union of sets of the form $\llbracket T_s \rrbracket$.*

Proof. Let U be any open subset of $\llbracket T \rrbracket$. For each $s \in \llbracket T \rrbracket$, say $s \in A$ if and only if $\llbracket T_s \rrbracket \subseteq U$ and $\llbracket T_t \rrbracket \not\subseteq U$ for any t such that $t \preceq s$. No two elements of A are comparable, and if $x \in U$ then $x(n) \in A$ for some n , namely the smallest n such that $\llbracket T_{x(n)} \rrbracket \subseteq U$. Therefore $U = \bigcup_{s \in A} \llbracket T_s \rrbracket$, and this is a disjoint union. □

Lemma 3.2. *Let κ be an infinite cardinal. Every nonempty open subset of κ^ω is homeomorphic to κ^ω .*

Proof. Let U be an open subset of κ^ω . Since $\kappa^\omega \cong \llbracket \kappa^{<\omega} \rrbracket$, we may apply Lemma 3.1 to obtain U as a disjoint union of sets of the form $\llbracket \kappa_s^{<\omega} \rrbracket$. Each such set is clearly a copy of κ^ω , and we have at most κ of them since $|\kappa^{<\omega}| = \kappa$. Thus, for some $\lambda \leq \kappa$, $U \cong \lambda \times \kappa^\omega \cong \kappa^\omega$. □

Lemma 3.3. *Let κ be an infinite cardinal and d any metric on κ^ω compatible with its topology. If $\varepsilon > 0$, then there is a partition $\{B_\alpha : \alpha \in \kappa\}$ of κ^ω into κ clopen sets such that $\text{diam}(B_\alpha) < \varepsilon$ for every α .*

Proof. See [1], Lemma 6.7. □

Lemma 3.4. *Let κ be an infinite cardinal. If X is a dense G_δ subset of κ^ω , then $X \cong \kappa^\omega$.*

Proof. Recall (e.g., from [6], Corollary 5) that a metric space is completely ultrametrizable if and only if it is zero-dimensional and Čech complete. Since metrizability, zero-dimensionality, and Čech completeness are all inherited by G_δ subsets, X is completely ultrametrizable.

Suppose that U is any clopen subset of X . There is some $s \in \kappa^{<\omega}$ such that $[[\kappa_s^{<\omega}]] \cap X \subseteq U$. Because X is dense in κ^ω , $[[\kappa_s^{<\omega}]] \cap X$ is a nonempty clopen subset of U for every $\alpha < \kappa$. Consequently, $\{X \setminus [[\kappa_s^{<\omega}]]\} \cup \{[[\kappa_s^{<\omega}]] \cap X : \alpha \in \kappa\}$ is a partition of U into clopen sets. Thus every open neighborhood of U can be partitioned into κ disjoint clopen subsets.

Fix a compatible complete metric on X . We will show that $X \cong \kappa^\omega$ by constructing an appropriate $\kappa^{<\omega}$ -scheme in X .

Let $B_\emptyset = X$ and fix $s \in \kappa^{<\omega}$. Suppose B_s has been defined, is a clopen subset of X , and has diameter at most $\frac{1}{\text{lev}(s)+1}$. As we have already noted, it is possible to partition B_s into κ disjoint clopen subsets. If necessary, we may use Lemma 3.3 to partition each of these further into sets smaller than $\frac{1}{\text{lev}(s)+2}$; thus we may assume that B_s is partitioned into κ clopen sets, each smaller than $\frac{1}{\text{lev}(s)+2}$. Enumerate these as $\{B_{s \frown \alpha} : \alpha \in \kappa\}$ to define B_t for every child t of s . By recursion, this defines a $\kappa^{<\omega}$ -scheme $(B_s)_{s \in \kappa^{<\omega}}$ in X .

By construction, this scheme has vanishing diameter, $B_\emptyset = X$, each B_s is clopen, and $B_s = \bigcup \{B_t : t \text{ is a child of } s\}$ for every s . By Lemma 2.2, the associated map of $(B_s)_{s \in \kappa^{<\omega}}$ is a homeomorphism. It remains to show that the domain of this map is X : that is, we must show that $\bigcap_{n \in \omega} B_{x(n)} \neq \emptyset$ for every $x \in [[\kappa^{<\omega}]]$. Since each B_s is clopen, we have $\bigcap_{n \in \omega} B_{x(n)} = \bigcap_{n \in \omega} \overline{B_{x(n)}}$, and this is nonempty because the $B_{x(n)}$ have decreasing diameter with respect to a complete metric. \square

Note that, if we set $\kappa = \omega$, then Lemma 3.4 reduces to a variant of a classical theorem of Alexandrov and Urysohn (see [4], Theorem 7.7).

Theorem 3.5. *Let α be any ordinal. Then $\omega_{\alpha+1}^\omega$ can be partitioned into $\aleph_{\alpha+1}$ homeomorphic copies of ω_α^ω .*

Proof. Fix an ordinal α . For every β with $\omega_\alpha \leq \beta < \omega_{\alpha+1}$, let

$$X_\beta = \beta^\omega \setminus \bigcup \{\gamma^\omega : \omega_\alpha \leq \gamma < \beta\}.$$

Because $\omega_{\alpha+1}$ is a regular uncountable cardinal, the range of every $x \in \omega_{\alpha+1}^\omega$ is bounded by some $\beta < \omega_{\alpha+1}$, so that $x \in \beta^\omega$. Therefore $\{X_\beta : \omega_\alpha \leq \beta < \omega_{\alpha+1}\}$ is a partition of $\omega_{\alpha+1}^\omega$. Of course $X_{\omega_\alpha} = \omega_\alpha^\omega$. For $\beta \neq \omega_\alpha$, we will show that X_β is empty if and only if β has uncountable cofinality, and otherwise is homeomorphic to ω_α^ω .

First suppose that β has uncountable cofinality. Then every element of β^ω is bounded inside some γ^ω , $\gamma < \beta$, so that $\beta^\omega = \bigcup_{\gamma < \beta} \gamma^\omega$. If $\beta \neq \omega_\alpha$, then this implies $X_\beta = \emptyset$.

Next suppose that $\beta = \gamma + 1$ is a successor ordinal. Because γ is closed in β , γ^ω is closed in β^ω . Also, $\beta^\omega \setminus \gamma^\omega \neq \emptyset$ (explicitly, $\beta^\omega \setminus \gamma^\omega$ is the set of all sequences in β with the point γ in their range). By Lemma 3.2, $X_\beta \cong \omega_\alpha^\omega$.

Finally, suppose that β is a limit ordinal with countable cofinality. Let $\langle \gamma_n : n < \omega \rangle$ be a sequence of ordinals with limit β . As in the previous paragraph, each γ_n^ω is closed in β^ω . Also, each γ_n^ω is nowhere dense in β^ω . To see this, let $s \in \beta^{<\omega}$ so that $U = \llbracket \beta_s^{<\omega} \rrbracket$ is a basic open set of β^ω , and consider that $\llbracket \beta_s^{<\omega} \rrbracket_{\gamma_n}$ is an open subset of U that is disjoint from γ_n^ω . Since $X_\beta = \beta^\omega \setminus \bigcup_{n < \omega} \gamma_n^\omega$, X is G_δ in β^ω , and X is dense in β^ω by the Baire Category Theorem. Since $\beta^\omega \cong \omega_\alpha^\omega$, it follows from Lemma 3.4 that $X_\beta \cong \omega_\alpha^\omega$. \square

Theorem 3.5 can be seen as a topological version of the basic fact of cardinal arithmetic that $\aleph_{\alpha+1}^{\aleph_0} = \aleph_{\alpha+1} \cdot \aleph_\alpha^{\aleph_0}$.

Corollary 3.6. *Let $n < \omega$. Then ω_n^ω can be partitioned into \aleph_n copies of ω^ω .*

Proof. By induction, using Theorem 3.5 \square

Corollary 3.6 cannot be extended to $\kappa \geq \aleph_\omega$ using *ZFC* alone, because the induction breaks down at the first singular cardinal. In fact, it always fails when κ has countable cofinality:

Theorem 3.7. *If κ is an uncountable cardinal with cofinality ω , then κ^ω is not the union of κ many subspaces homeomorphic to ω^ω .*

Proof. If $X \subseteq \kappa^\omega$ is homeomorphic to ω^ω , then $X \subseteq \Gamma^\omega$ for some countable $\Gamma \subseteq \kappa$. To see this, note that for any $n < \omega$ if $\pi_n : \kappa^\omega \rightarrow \kappa$ is the projection map $\pi(x) = x(n)$, then $\pi_n(X)$ must be countable since otherwise X would contain an uncountable family of pairwise disjoint open sets.

Suppose $X_\alpha \subseteq \kappa^\omega$ for $\alpha < \kappa$ are homeomorphic copies of ω^ω . Let $\Gamma_\alpha \subseteq \kappa$ be countable with $X_\alpha \subseteq \Gamma_\alpha^\omega$. Let κ_n be a cofinal sequence in κ and choose

$$x(n) \in \kappa \setminus \bigcup_{\alpha < \kappa_n} \Gamma_\alpha$$

for each n . Then $x \in \kappa^\omega \setminus \bigcup_{\alpha < \kappa} X_\alpha$. \square

Note that for any κ the space κ^ω can be partitioned into $|\kappa^\omega|$ copies of ω^ω . This is because $\omega^\omega \times \kappa^\omega$ is homeomorphic to κ^ω . Hence, assuming

GCH, for every κ with uncountable cofinality the space κ^ω can be partitioned into κ many copies of ω^ω . This would also follow from the weaker assumption that $|\kappa^\omega| = \kappa$ for every $\kappa > \aleph_\omega$ with uncountable cofinality.

Question 3.8. *Suppose $\aleph_\omega < \kappa < \mathfrak{c}$ and κ has uncountable cofinality, then can κ^ω be partitioned into κ many copies of ω^ω ?*

In [1] it is proved that ω^ω is a condensation of κ^ω whenever ω^ω can be partitioned into κ Borel sets. Theorem 3.5 allows us to prove a partial converse.

Theorem 3.9. *Let $n < \omega$. Then ω^ω is a condensation of ω_n^ω if and only if ω^ω can be partitioned into \aleph_n Borel sets.*

Proof. Theorem 6.2 of [1] states that ω^ω is a condensation of κ^ω whenever ω^ω can be partitioned into κ Borel sets. For the converse, suppose that $f : \omega_n^\omega \rightarrow \omega^\omega$ is a condensation. By Corollary 3.6, there is a partition $\{N_\alpha : \alpha \in \omega_n\}$ of ω_n^ω into \aleph_n copies of ω^ω . By an old theorem of Lusin and Souslin, every bijective continuous image of ω^ω is Borel (see [4], Theorem 15.1). Thus $\{f(N_\alpha) : \alpha \in \omega_n\}$ is a partition of ω^ω into \aleph_n Borel sets. \square

The following corollary answers Question 6.4 from [1].

Corollary 3.10. *It is relatively consistent with ZFC that $\mathfrak{c} = \omega_3$ and there is no condensation $\omega_2^\omega \rightarrow \omega^\omega$.*

Proof. In [8], Theorem 3.7, the second author proves that if ω_3 Cohen reals are added to a model of CH then there is no partition of 2^ω into \aleph_2 Borel sets. Since ω^ω can be identified with a co-countable subset of 2^ω , this model also has no partition of ω^ω into \aleph_2 Borel sets. It follows from Theorem 3.9 that this model has no condensation $\omega_2^\omega \rightarrow \omega^\omega$. \square

Contrast this result with the result of Hausdorff in [2], where it is proved from ZFC that ω^ω can be partitioned into \aleph_1 Borel sets (and hence ω^ω is a condensation of ω_1^ω). The next theorem gives the opposite consistency result:

Theorem 3.11. *It is consistent with any possible value of \mathfrak{c} that for every $\kappa \leq \mathfrak{c}$ there is a partition of 2^ω into κ closed sets.*

Proof. It is proved in [7], Theorem 4, that for any possible value of \mathfrak{c} and any fixed $\kappa < \mathfrak{c}$, there is a model in which 2^ω can be partitioned into κ copies of 2^ω . Here we show how to modify that construction to obtain a partition into κ copies of 2^ω for all $\kappa < \mathfrak{c}$ simultaneously.

If CH holds then the conclusion is trivial. Let M be any model in which CH fails. We will show how to construct a finite support iterated forcing such that extending M by this forcing preserves cardinals, does not change the value of \mathfrak{c} , and adds for every $\omega_1 \leq \kappa < \mathfrak{c}$ a partition of 2^ω into κ closed sets.

For $X \subseteq 2^\omega$, define $\mathbb{P}(X)$ as follows. Conditions are finite mutually consistent sets of sentences of the form “ $[[2_s^{<\omega}]] \cap C_n = \emptyset$ ” or “ $x \in C_n$ ” where $n \in \omega$, $x \in X$, and $s \in 2^{<\omega}$ (this is as in [7]). In an extension of M by $\mathbb{P}(X)$, $\bigcup_{n \in \omega} C_n$ will be an F_σ set that covers X and misses every element of $(2^\omega)^M \setminus X$. For any X , $\mathbb{P}(X)$ has the c.c.c.

Our iterated forcing will have length $\alpha \cdot \omega_1$, where α is the unique ordinal such that $\mathfrak{c} = \omega_\alpha$ (alternatively, this can be thought of as a length- ω_1 iteration of length- α iterations). For each κ with $\omega_1 \leq \kappa < \mathfrak{c}$, let X_κ^0 be a subset of 2^ω with $|X_\kappa^0| = \kappa$. Let $\delta < \alpha \cdot \omega_1$, and let β and γ be the unique ordinals such that $\gamma < \alpha$ and $\delta = \alpha \cdot \beta + \gamma$ (see [5], Ch. I, Ex. 3 to see that such ordinals exist and are unique). In our iteration, $M_{\delta+1}$ is obtained by forcing with $\mathbb{P}(2^\omega \setminus (X_{\omega_\gamma} \cup \bigcup \{F_{\omega_\gamma}^\xi : \xi < \beta\}))$ in M_δ ; this creates a generic F_σ set which we call $F_{\omega_\gamma}^\beta$.

Since each $\mathbb{P}(X)$ has the c.c.c., our iteration has the c.c.c., and since $|\alpha \cdot \omega_1| \leq \omega_\alpha = \mathfrak{c}^M$, $M_{\alpha \cdot \omega_1} \models “\mathfrak{c} = \omega_\alpha”$. Fix $\gamma < \alpha$ and let $\kappa = \omega_\gamma < \mathfrak{c}$. For every $x \in 2^\omega \setminus X_\kappa$, there is smallest $\beta < \omega_1$ such that $x \in M_{\alpha \cdot \beta + \gamma}$, in which case $x \in F_\kappa^\beta$. Thus $M_{\alpha \cdot \omega_1} \models “2^\omega = X_\kappa \cup \bigcup_{\beta < \omega_1} F_\kappa^\beta”$. In other words, $\{\{x\} : x \in X_\kappa\} \cup \{F_\kappa^\beta : \beta < \omega_1\}$ is a partition of 2^ω in $M_{\alpha \cdot \omega_1}$, and (because our iteration preserves cardinals) this partition has cardinality κ when κ is uncountable.

To obtain a partition of 2^ω into κ closed sets in $M_{\alpha \cdot \omega_1}$, it is sufficient to note that every F_σ subset of 2^ω can be partitioned into countably many compact sets. This is showed in [7] (the last part of the proof of Theorem 4) or, alternatively, in [1] (Proposition 3.5). \square

Corollary 3.12. *It is consistent with any possible value of \mathfrak{c} that whenever $\omega \leq \kappa \leq \mathfrak{c}$ there is a condensation $\kappa^\omega \rightarrow \omega^\omega$.*

Proof. Because ω^ω can be identified with a co-countable subset of 2^ω , the model in Theorem 3.11 has, for every $\kappa < \mathfrak{c}$, a partition of ω^ω into κ Borel sets. The corollary now follows from Theorem 3.9. \square

One might notice that the κ -sized partition given by our forcing consists of κ singletons and at most \aleph_1 nontrivial closed sets. However, it is easy to modify these partitions to obtain κ copies of 2^ω :

Proposition 3.13. *If 2^ω can be partitioned into κ closed sets, then 2^ω can be partitioned into κ copies of 2^ω .*

Proof. Let $\{K_\alpha : \alpha \in \kappa\}$ be a partition of 2^ω into closed sets. Then $\{K_\alpha \times 2^\omega : \alpha \in \kappa\}$ is a partition of $2^\omega \times 2^\omega \cong 2^\omega$ into copies of 2^ω . \square

Corollary 3.14. *It is consistent with any possible value of \mathfrak{c} that 2^ω is a condensation of every perfect completely ultrametrizable space X with $|X| = \mathfrak{c}$.*

Proof. Combine Theorem 3.11, Proposition 3.13, and Theorem 5.1 of [1]. \square

A result of Hausdorff states that there is always a partition of ω^ω into \aleph_1 Borel sets (see [2]). Theorem 3.7 of [8], together with the comments that follow it, says that it is consistent with any value of \mathfrak{c} that every uncountable partition of ω^ω into Borel sets has size \aleph_1 or \mathfrak{c} (this is strengthened below in Corollary 3.16). Theorem 3.11 gives the opposite result, but it remains an open problem to find models in which some intermediate property holds. Is it consistent, for example, to have $\mathfrak{c} = \omega_9$ with uncountable partitions of sizes \aleph_1 , \aleph_4 , \aleph_8 , and \aleph_9 , but of no other sizes? The following propositions provide partial answers to such questions.

Let $\text{FIN}(\kappa, 2)$ be the partial order of finite partial functions from κ to 2, i.e., Cohen forcing.

Theorem 3.15. ([8] 3.7) *Suppose M is a countable transitive model of ZFC + GCH. Let κ be any cardinal of M of uncountable cofinality. Suppose that G is $\text{FIN}(\kappa, 2)$ -generic over M , then in $M[G]$ the continuum is κ and for every family \mathcal{F} of Borel subsets of ω^ω with size $\omega_1 < |\mathcal{F}| < \kappa$, if $\bigcup \mathcal{F} = \omega^\omega$ then there exists $\mathcal{F}_0 \in [\mathcal{F}]^{\omega_1}$ with $\bigcup \mathcal{F}_0 = \omega^\omega$.*

This is only stated in [8] for $\kappa = \omega_3$ but it is clear from the proof that it is true in more generality.

Corollary 3.16. *Suppose M is a countable transitive model of ZFC + GCH. Let κ be any cardinal of M of uncountable cofinality which is not the successor of a cardinal of countable cofinality. Suppose that G is $\text{FIN}(\kappa, 2)$ -generic over M , then in $M[G]$ the continuum is κ and for every uncountable $\gamma < \kappa$ if $F : \gamma^\omega \rightarrow \omega^\omega$ is continuous and onto, then there exists a $Q \in [\gamma]^{\omega_1}$ such that $F(Q^\omega) = \omega^\omega$.*

Proof. Let $\Sigma = [\gamma]^\omega \cap M$. Note that $|\Sigma| < \kappa$ since in M $|\gamma^\omega| > \gamma$ if and only if γ has cofinality ω , but in that case $|\gamma^\omega| = \gamma^+ < \kappa$. Since the forcing is c.c.c.

$$M[G] \models \gamma^\omega = \bigcup \{X^\omega : X \in \Sigma\}$$

For any $X \in \Sigma$ the continuous image $F(X^\omega)$ is a Σ_1^1 set, and hence the union of ω_1 Borel sets. Given a family \mathcal{F} of $|\Sigma|$ -many Borel sets whose union is ω^ω there is a subfamily \mathcal{F}_0 of size ω_1 whose union is ω^ω and hence a Q as required. \square

If $\kappa = \lambda^+$ where λ has cofinality ω , then the result holds for $\gamma < \lambda$ but since $\lambda^\omega = \kappa$ holds in M we do not know whether it is true for λ .

Proposition 3.17. *It is consistent that the continuum be arbitrarily large, ω^ω can be partitioned into ω_2 Borel sets, and ω^ω is not the condensation of κ^ω whenever $\omega_2 < \kappa < \mathfrak{c}$*

Proof. Let M be a model of *ZFC* satisfying Martin's Axiom and $\mathfrak{c} = \omega_2$. Using transfinite induction in M , it is possible to construct a sequence $\langle C_\alpha : \alpha < \omega_2 \rangle$ of closed nowhere dense subsets of 2^ω such that for every non-atomic Borel probability measure μ on 2^ω there are countably many of the C_α whose union has μ -measure 1.

In M , force with the measure algebra on 2^λ for any λ with uncountable cofinality not the successor of a cardinal of countable cofinality. In the generic extension we have $\mathfrak{c} = \lambda$, and every new real is random with respect to some non-atomic Borel probability measure in the ground model. Because of our choice of the C_α , this implies that every new real will be in some C_α . Therefore $\{C_\alpha : \alpha < \omega_2\} \cup \{\{x\} : x \in 2^\omega \setminus \bigcup_{\alpha < \omega_2} C_\alpha\}$ has size \aleph_2 in the generic extension. Because the C_α are disjoint, it is a partition of 2^ω into closed sets.

The proof of Theorem 3.7 in [8] uses Cohen reals, but the same idea shows that this generic extension has the property that for every family \mathcal{F} of Borel subsets of ω^ω with size $\aleph_2 < |\mathcal{F}| < \lambda$, if $\bigcup \mathcal{F} = \omega^\omega$ then there exists $\mathcal{F}_0 \in [\mathcal{F}]^{\omega_2}$ with $\bigcup \mathcal{F}_0 = \omega^\omega$. As in the proof of Corollary 3.16 we get that ω^ω is not the condensation of any κ^ω whenever $\aleph_2 < \kappa < \mathfrak{c}$. \square

Note the similarity of this argument to the argument of Stern in [9] (later rediscovered by Kunen), where he proves that 2^ω can be partitioned into \aleph_1 closed sets in any random real extension of a model of *CH*.

Note also that trivial modifications to the proof of Proposition 3.17 allow us to replace ω_2 with any cardinal μ of uncountable cofinality. However, doing so will not guarantee partitions of all sizes smaller than μ . This is because it is not currently known whether Martin's Axiom implies (or even permits) Borel set partitions of ω^ω of all sizes less than \mathfrak{c} . Thus this proof does not give “small partitions without large,” but only “partitions of a given size without larger ones.”

Question 3.18. *If $\omega_1 < \kappa < \mathfrak{c}$, does Martin's Axiom imply the existence of a size- κ partition of ω^ω into Borel sets? Is this consistent with Martin's Axiom?*

4. SIMILARITY TYPES

This section deals mostly with perfect completely ultrametrizable spaces of size \mathfrak{c} . For the sake of brevity, we will henceforth refer to such spaces as **PCU spaces**.

Say that two spaces are **similar** if each condenses onto the other. In [1], it is proved that there are exactly three similarity types of separable PCU spaces, that is, of perfect zero-dimensional Polish spaces. Furthermore, it is proved that these three types are totally ordered by the relation “ X condenses onto Y ” (this is called the **condensation relation**). It is also proved in [1] that if CH is assumed then there are exactly four similarity types of PCU spaces, and these four types are totally ordered by the condensation relation. That is, the inclusion of non-separable spaces (equivalently, of uncountable trees) introduces only one new similarity type, namely those spaces representable with trees of size \mathfrak{c} .

In general, neither

- (1) the total orderability of similarity types by condensation
- (2) the similarity of spaces represented by uncountable trees of the same size

necessarily holds when CH fails. If MA holds, for example, then ω_1^ω and $\omega_1 \times 2^\omega$ are not similar, and neither of $\omega_1 \times 2^\omega$ and ω^ω condenses onto the other (see [1], Proposition 5.10 and Corollary 6.3). It is left an open question in [1] whether it is consistent for (1) and (2) to hold when CH fails.

We will show in this section that it is consistent for (1) and (2) to hold with \mathfrak{c} equal to any $\omega_n < \omega_\omega$. In fact the appropriate model has already been constructed in Theorem 3.11, and here we merely need to show that (1) and (2) hold in this model.

Lemma 4.1. *The following are equivalent:*

- (1) 2^ω can be partitioned into κ closed sets.
- (2) 2^ω can be partitioned into κ copies of 2^ω
- (3) ω^ω can be partitioned into κ copies of 2^ω .

Proof. If κ is countable then the result is trivial, so suppose $\kappa \leq \mathfrak{c}$ is uncountable.

(3) \Rightarrow (1): Let D be a countable dense subset of 2^ω ; then $2^\omega \setminus D \cong \omega^\omega$. If $\{K_\alpha: \alpha \in \kappa\}$ is a partition of $2^\omega \setminus D$ into copies of 2^ω , then $\{K_\alpha: \alpha \in \kappa\} \cup \{\{x\}: x \in D\}$ is a partition of 2^ω into κ closed sets.

(1) \Rightarrow (2): This is given by Proposition 3.13.

(2) \Rightarrow (3): Let $\{K_\alpha: \alpha \in \kappa\}$ be a partition of 2^ω into κ copies of 2^ω . We say that a partition is “nice” if each partition element is nowhere dense. Assuming our partition is nice, we can pick a sequence $\langle x_n: n < \omega \rangle$ of points in 2^ω such that $\{x_n: n \in \omega\}$ is dense in 2^ω and no two points of our sequence are in the same K_α . Then $X = 2^\omega \setminus \{x_n: n \in \omega\} \cong \omega^\omega$, and $\{K_\alpha \cap X: \alpha \in \kappa\}$ is a partition of X . For each α , either K_α contains no x_n , in which case $K_\alpha \cap X = K_\alpha \cong 2^\omega$, or $K_\alpha \cap X = K_\alpha \setminus \{x_n\}$ for some $x_n \in K_\alpha$, in which case $K_\alpha \cap X \cong \omega \times 2^\omega$. This lets us obtain a partition of X into κ copies of 2^ω .

Given any partition $\{K_\alpha: \alpha \in \kappa\}$ of 2^ω into closed sets, we will now show how to find a nice partition. The basic idea is to do something like a Cantor-Bendixson derivative to eliminate partition elements with non-empty interior. Set $C_0 = 2^\omega$. Given C_α , let

$$C_{\alpha+1} = C_\alpha \setminus \bigcup \{U \subseteq C_\alpha: U \text{ is clopen and, for some } \alpha, U \subseteq K_\alpha\},$$

and if α is a limit ordinal take $C_\alpha = \bigcap_{\beta < \alpha} C_\beta$. By transfinite recursion, this defines a decreasing sequence $\langle C_\alpha: \alpha \in Ord \rangle$ of closed subspaces of C_0 . Because each C_α is closed and 2^ω is second countable, there is some countable ordinal α such that $C_\beta = C_\alpha$ for all $\beta \geq \alpha$. If x were an isolated point of C_α then we would have $x \notin C_{\alpha+1}$, so C_α is perfect. By induction, again using the fact that 2^ω is second countable, there are for any $\beta \leq \alpha$ at most countably many γ such that $K_\gamma \cap C_\beta = \emptyset$. Since κ is uncountable, $\{K_\gamma: \gamma \in \kappa\}$ restricts to a partition of size κ on C_α . In particular, $C_\alpha \neq \emptyset$; since we have already seen that C_α is closed in 2^ω and has no isolated points, $C_\alpha \cong 2^\omega$. If $K_\gamma \cap C_\alpha$ had nonempty interior, there would be some clopen $U \subseteq C_\alpha$ with $U \subseteq K_\gamma$, contradicting the fact that $C_\alpha = C_{\alpha+1}$. Thus each $K_\gamma \cap C_\alpha$ is closed and nowhere dense in C_α . As in the proof of Proposition 3.13, $\{(K_\gamma \cap C_\alpha) \times 2^\omega: K_\gamma \cap C_\alpha \neq \emptyset\}$ is a partition of $C_\alpha \times 2^\omega \cong 2^\omega$ into nowhere dense copies of 2^ω . \square

Lemma 4.2. *If $|T| < |S|$, then $\llbracket S \rrbracket$ is not a condensation of $\llbracket T \rrbracket$.*

Proof. See [1], Proposition 5.5. \square

Theorem 4.3. *Assume $\mathfrak{c} = \omega_n < \omega_\omega$. The following are equivalent:*

- (1) *There are $n + 3$ similarity types of PCU spaces, and these are totally ordered by condensation.*
- (2) *There are $n + 3$ similarity types of PCU spaces.*

- (3) *The similarity types of PCU spaces are totally ordered by condensation.*
- (4) *2^ω is a condensation of every PCU space.*
- (5) *For every $\kappa \leq \mathfrak{c}$, there is a partition of 2^ω into κ closed sets.*

Proof. (1) implies (2) and (3) trivially. No Hausdorff space is a non-trivial condensation of 2^ω , so (3) implies (4). For every $\kappa \leq \mathfrak{c}$, $\kappa \times 2^\omega$ is a completely ultrametrizable space. If $f : \kappa \times 2^\omega \rightarrow 2^\omega$ is a condensation, then $\{f(\{a\} \times 2^\omega) : a \in \kappa\}$ is a partition of 2^ω into κ copies of 2^ω . Thus (4) implies (5).

Next we show that (2) implies (5). Suppose there are exactly $n + 3$ similarity types of PCU spaces. By Lemma 4.2, there are at least n similarity types corresponding to PCU spaces arising from uncountable trees: one for each possible tree size. Because there are exactly three similarity types for PCU spaces arising from countable trees (see [1], Theorem 3.9), we have exactly n types for uncountable trees, and $\llbracket T \rrbracket$ and $\llbracket S \rrbracket$ are similar if and only if $|S| = |T|$.

In particular, if $0 < m < n$ then $\omega_m \times 2^\omega$ and ω_m^ω are similar. Let $f : \omega_m^\omega \rightarrow \omega_m \times 2^\omega$ be a condensation. By Lemma 3.2, $f^{-1}(\{0\} \times 2^\omega) \cong \omega_m^\omega$. Thus $f \upharpoonright f^{-1}(\{0\} \times 2^\omega)$ is a condensation from (a homeomorphic copy of) ω_m^ω to $\{0\} \times 2^\omega \cong 2^\omega$. Composing with a condensation $\omega_m \times 2^\omega \rightarrow \omega_m^\omega$, we see that there is a condensation $g : \omega_m \times 2^\omega \rightarrow 2^\omega$. Since any Hausdorff condensation of 2^ω is simply 2^ω , $\{g(\{a\} \times 2^\omega) : a \in \omega_m\}$ is a partition of 2^ω into \aleph_m copies of 2^ω . Thus (2) implies (5).

It remains to show that (5) implies (1). We will show that if T is a tree with κ nodes, with $\omega < \kappa < \mathfrak{c}$, then $\llbracket T \rrbracket$ is similar to κ^ω . This shows that each uncountable $\kappa < \mathfrak{c}$ corresponds to a single similarity type, consisting precisely of the spaces arising from trees of size κ (and these types are distinct by Lemma 4.2). Given our assumptions on \mathfrak{c} and the fact that there are exactly three types corresponding to countable trees (Theorem 3.9 from [1]), we then have exactly $n + 3$ types. We will also show that κ^ω condenses onto λ^ω whenever $\omega \leq \lambda < \kappa \leq \mathfrak{c}$, which will show that these types are totally ordered by condensation. It will be convenient to prove the latter of these propositions first.

Claim. If $\omega \leq \lambda \leq \kappa \leq \mathfrak{c}$, then λ^ω is a condensation of κ^ω .

Proof of claim. Combining Corollary 3.6 and Lemma 4.1 with (5), we see that there is a partition of λ^ω into κ copies of 2^ω . In particular, λ^ω is a condensation of $\kappa \times \omega^\omega$. By Theorem 3.9 and (5), $\kappa \times \omega^\omega$ is a condensation of $\kappa \times \kappa^\omega \cong \kappa^\omega$. Composing condensations, we have a condensation from κ^ω to λ^ω . \square

Let T be a tree and $|T| = \kappa$, with $\omega < \kappa \leq \mathfrak{c}$. We will show that $\llbracket T \rrbracket$ is similar to κ^ω by showing (in the following three claims) that there are condensations $\llbracket T \rrbracket \rightarrow \kappa \times 2^\omega \rightarrow \kappa^\omega \rightarrow \llbracket T \rrbracket$.

Claim. $\kappa \times 2^\omega$ is a condensation of $\llbracket T \rrbracket$.

Proof of claim. Because κ is uncountable and regular, and because $T = \bigcup_{n \in \omega} \text{Lev}_n(T)$, there is some n such that $|\text{Lev}_n(T)| = \kappa$. $\llbracket T \rrbracket$ is the disjoint union of $\{\llbracket T_s \rrbracket : \text{lev}(s) = n\}$, and each $\llbracket T_s \rrbracket$ is clopen in $\llbracket T \rrbracket$. Thus it suffices to show that 2^ω is a condensation of each $\llbracket T_s \rrbracket$. This follows immediately from (4), and (4) follows from (5) by Theorem 5.1 of [1]. \square

Claim. κ^ω is a condensation of $\kappa \times 2^\omega$.

Proof of claim. It follows from Corollary 3.6, Lemma 4.1, and (5) that κ^ω can be partitioned into κ copies of 2^ω . This is equivalent to being a condensation of $\kappa \times 2^\omega$. \square

Claim. $\llbracket T \rrbracket$ is a condensation of κ^ω .

Proof of claim. We will prove this claim by induction. Our inductive hypothesis is that whenever S is a tree of size λ then there is a condensation $\lambda^\omega \rightarrow \llbracket S \rrbracket$. If $\lambda = \omega$ then the inductive hypothesis becomes: every perfect zero-dimensional Polish space is a condensation of ω^ω . This is a well-known classical result (see, e.g., Exercise 7.15 in [4]). Assume now that the inductive hypothesis holds for every $\lambda < \kappa$.

By Theorem 4.3 in [1], we may assume that every node s of T has exactly $|T_s|$ children. We will build a $\kappa^{<\omega}$ -scheme $(B_s)_{s \in \kappa^{<\omega}}$ in $\llbracket T \rrbracket$ by recursion.

Set $B_\emptyset = \llbracket T \rrbracket$. Assume now that B_s has been defined and is equal to $\llbracket T_t \rrbracket$ for some node $t \in T$. If $|T_t| < \kappa$, then by hypothesis there is a condensation $|T_t|^\omega \rightarrow \llbracket T_t \rrbracket$. Since we have already proved that there is a condensation $\kappa^\omega \rightarrow |T_t|^\omega$ (our first claim above), there is a condensation $g : \llbracket \kappa_s^{<\omega} \rrbracket \rightarrow \llbracket T_t \rrbracket$. Define $B_r = g(\llbracket \kappa_r^{<\omega} \rrbracket)$ for every extension r of s . If $|T_t| = \kappa$, then t has κ children in T by our choice of T . Enumerating these as $\{t_\alpha : \alpha < \kappa\}$, we let $B_{s \frown \alpha} = \llbracket T_{t_\alpha} \rrbracket$. This recursion defines a $\kappa^{<\omega}$ -scheme $(B_s)_{s \in \kappa^{<\omega}}$.

Let $x \in \llbracket \kappa^{<\omega} \rrbracket$. If there is some n such that $B_{x(n)} = \llbracket T_t \rrbracket$ for some t with fewer than κ children, then $B_{x(m)}$ is defined by some embedding $g : B_{x(n)} \rightarrow X$ for all $m \geq n$. Because g is an embedding, $\lim_{m \rightarrow \infty} \text{diam}(B_{x(m)}) = 0$ and $\bigcap_{n \in \omega} B_{x(n)} = g(x)$. Otherwise, $B_{x(n)}$ is always equal to some $\llbracket T_t \rrbracket$, where t has κ children in T . Then (by an easy induction) $B_{x(n)} = \llbracket T_{y(n)} \rrbracket$ for some $y \in \llbracket T \rrbracket$ and every n . Since $\{\llbracket T_{y(n)} \rrbracket : n < \omega\}$ is a local basis for y , $\lim_{n \rightarrow \infty} \text{diam}(B_{x(n)}) = 0$ in this

case too; also, clearly, $\bigcap_{n \in \omega} B_{x(n)} = y$. Thus $(B_s)_{s \in \kappa^{<\omega}}$ has vanishing diameter, and $\bigcap_{n \in \omega} B_{x(n)} \neq \emptyset$ for every $x \in \llbracket \kappa^{<\omega} \rrbracket$. Furthermore, it is clear from our construction that $B_s = \bigcup \{B_t : t \text{ is a child of } s\}$ for every $s \in \kappa^{<\omega}$. It follows from Lemma 2.2 that the associated map of $(B_s)_{s \in \kappa^{<\omega}}$ is a condensation.

This shows that the inductive hypothesis holds at κ and completes the induction. \square

This shows that uncountable trees of the same size represent similar spaces and completes the proof that (5) implies (1). \square

Corollary 4.4. *Let $n < \omega$. It is consistent with ZFC that $\mathfrak{c} = \omega_n$ and the five propositions listed in the statement of Theorem 4.3 all hold.*

Proof. Combine Theorem 3.11 with Theorem 4.3. \square

This corollary provides partial answers to Questions 5.11 and 5.12 from [1]. It leaves open the question of whether the similarity types can be totally ordered when $\mathfrak{c} > \omega_\omega$.

It is worth pointing out that the proof of Theorem 4.3 does not depend on \mathfrak{c} being small. If we consider a model guaranteed by Theorem 3.11 in which $\mathfrak{c} > \omega_\omega$, then the proof of Theorem 4.3 shows that two PCU spaces arising from trees of size κ with $\aleph_0 < \kappa < \aleph_\omega$ will be similar. This gives countably many similarity types of spaces with weight less than ω_ω , with the types totally ordered (with order type ω) by condensation.

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