Two remarks about analytic sets [1]

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Abstract

In this paper we give two results about analytic sets. The first is a counterexample to a problem of Fremlin. We show that there exists ω_1 compact subsets of a Borel set with the property that no σ-compact subset of the Borel set covers them. In the second section we prove that for any analytic subset A of the plane either A can be covered by countably many lines or A contains a perfect subset P which does not have three collinear points.

In his book about Martin's Axiom [2] p. 61 D. H. Fremlin shows that, assuming MA, for any analytic set X and set $A \subset X$ of cardinality less than the continuum there exists a σ-compact set L such that $A \subset L \subset X$. (σ-compact means countable union of compact sets.) Here we show that the

[1]in **Lecture Notes in Mathematics**, Springer-Verlag, 1401(1989), 68-72.
[2]Research partially supported by the Netherlands organization for the advancement of pure research
[3]Research partially supported by NSF grant MCS-8401711

set A cannot be replaced by a family of compact sets of cardinality ω_1. This answers a question of Fremlin [2] p.67.

Let $\mathbb{Q}$ be the rationals and let $E = \mathbb{Q}^\omega$. E is a Π^0_3 set, or equivalently an $F_{\sigma\delta}$ set. For each $\alpha < \omega_1$ let C_α be a compact subset of $\mathbb{Q}$ which is homeomorphic to $\alpha + 1$ and let $H_\alpha = C_\alpha^\omega$.

Theorem 1 *There does not exists a σ-compact set L such that for every $\alpha < \omega_1$, $H_\alpha \subset L \subset E$.*

Lemma 2 *For every compact set $K \subset \mathbb{Q}$ there exists $\alpha < \omega_1$ such that for all $\beta > \alpha$, $C_\beta \setminus K$ is nonempty.*

proof

This follows easily from a well-known theorem of Sierpiński that every countable scattered space is isomorphic to an ordinal. For simplicity we sketch a proof here.

Let $D(X)$ be the derivative operator, i.e. $D(X)$ is the set of nonisolated points of X. Then let $D^\alpha(X)$ be the usual α^{th} iterate of D, defined by induction as follows.

$$D^{\alpha+1}(X) = D(D^\alpha(X))$$

$$D^\lambda(X) = \cap_{\alpha<\lambda} D^\alpha(X) \text{ if } \lambda \text{ a limit ordinal}$$

Define the rank of any X (rank(X)) as the least α such that $D^\alpha(X)$ is empty.

Then the lemma follows easily from the following facts:

1. Every compact subset of $\mathbb{Q}$ has a countable rank.
2. If $X \subset Y$ then $D(X) \subset D(Y)$.
3. If $X \subset Y$ then rank(X)$\leq$rank(Y).
4. rank$(C_{\omega^\alpha+1}) = \alpha + 1$.

□

To prove the Theorem let $L = \bigcup_{n\in\omega} L_n$ where each L_n is compact. Let $K_n \subset \mathbb{Q}$ be the projection of L_n onto the n^{th} coordinate. By the lemma there exists C_β which is not covered by any K_n. It follows that H_β is not covered by L.

□

We don't know whether the theorem is true for Σ^0_3 sets ($G_{\delta\sigma}$) or even for a set which is the union of a countable set and a Π^0_2 (G_δ).

Next we prove the following theorem:

Theorem 3 *Suppose that A is an analytic subset of the plane, $\mathbb{R}^2$, which cannot be covered by countably many lines. Then there exists a perfect subset P of A such that no three points of P are collinear.*

proof

A set is perfect iff it is homeomorphic to the Cantor space 2^ω. The proof we give is similar to the classical proof that uncountable analytic sets must contain a perfect subset. A subset A of a complete separable space X is analytic iff there exists a closed set $C \subset \omega^\omega \times X$ such that A is the projection of C, i.e.

$$A = \mathrm{p}(C) = \{y \in X \mid \exists x \in \omega^\omega \ (x, y) \in C\}$$

Every Borel subset of X is analytic.

Let A be analytic subset of the plane $\mathbb{R}^2$ which cannot be covered by countably many lines. Let S be the unit square ($[0,1] \times [0,1]$) minus all lines of the form $x = r$ or $y = r$ for r a rational number. Without loss of generality we may assume that A is a subset of S. Since S is a complete separable space there exists $C \subset \omega^\omega \times S$ a closed set such that $A = p(C)$.

Give S the basis B_s for $s \in 4^{<\omega}$ described in figure 1. ($4^{<\omega}$ is the set of finite sequences of elements from $4 = \{0, 1, 2, 3\}$)

For $y \in S$ define $y \restriction n = s$ iff $y \in B_s$ for s of length n and for $x \in \omega^\omega$ let $x \restriction n$ be the restriction of x to the set $n = \{0, 1, 2, \ldots, n-1\}$. Let

$$T = \{(x \restriction n, y \restriction n) \mid (x, y) \in C\}$$

Then

$$C = [T] = \{(x, y) \mid \forall n \ (x \restriction n, y \restriction n) \in T\}$$

and

$$A = \mathrm{p}[T]$$

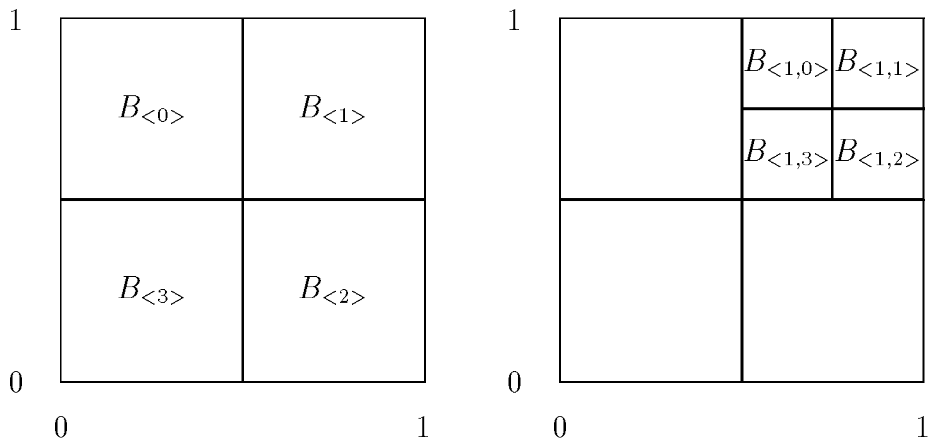

Figure 1: B_s for $s \in 4^{<\omega}$

For any $(s,t) \in T$ define

$$T_{(s,t)} = \{(\hat{s},\hat{t}) \in T \mid (\hat{s} \subset s \wedge \hat{t} \subset t) \text{ or } (s \subset \hat{s} \wedge t \subset \hat{t})\}$$

Let

$$T' = \{(s,t) \in T \mid \text{p}[T_{(s,t)}] \text{ cannot be covered by countably many lines } \}$$

Lemma 4 *For any $(s,t) \in T'$ $p[T'_{(s,t)}]$ cannot be covered by countably many lines, where $T'_{(s,t)} = T_{(s,t)} \cap T'$.*

proof

By definition p$[T_{(s,t)}]$ cannot be covered by countably many lines. For any x in p$[T_{(s,t)}]$ but not in p$[T'_{(s,t)}]$ there must be some $(\hat{s},\hat{t})$ in $T_{(s,t)}$ but not in $T'_{(s,t)}$ such that x is in p$[T_{(\hat{s},\hat{t})}]$. Since each such p$[T_{(\hat{s},\hat{t})}]$ can be covered by countably many lines, p$[T'_{(s,t)}]$ cannot be covered by countably many lines.
□

Lemma 5 *(Split and shrink) Suppose $(s_i,t_i) \in T'$ for $i = 0,1,2,\ldots,n$ are given with the properties that for $i \neq j$ B_{t_i} is disjoint from B_{t_j} and no line meets three or more of the B_{t_j}'s. Then there exists $(\hat{s}_i,\hat{t}_i) \in T'$ for*

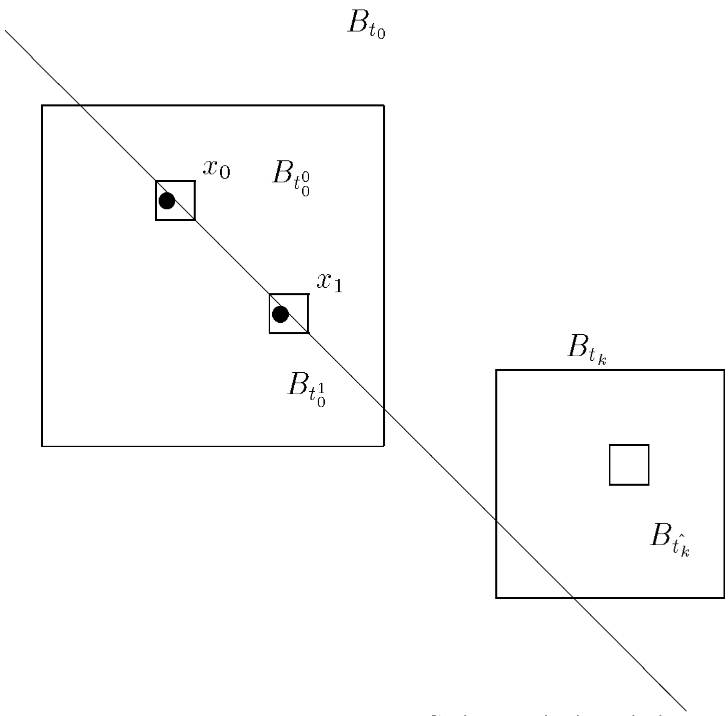

Figure 2: Split and shrink lemma

$i = 1, 2, \ldots, n$ with $(\hat{s}_i, \hat{t}_i) \supset (s_i, t_i)$ and $(s_0^j, t_0^j) \in T'$ for $j = 0, 1$ with $(s_0^j, t_0^j) \supset (s_0, t_0)$ and $B_{t_0^0}$ disjoint from $B_{t_0^1}$ and no line meets three or more of the

$$B_{t_0^0}, B_{t_0^1}, B_{\hat{t_1}}, B_{\hat{t_2}}, B_{\hat{t_3}}, \ldots, B_{\hat{t_n}},$$

proof

Since p$[T'_{(s_0,t_0)}]$ cannot be covered by countably many lines we can find distinct elements of it x_0, x_1. Let l be the line containing x_0 and x_1. Since this line cannot cover p$[T'_{(s_i,t_i)}]$ for $i = 1, 2, \ldots, n$ we can find $(\hat{s}_i, \hat{t}_i) \supset (s_i, t_i)$ with each $B_{\hat{t_i}}$ a positive distance from the line l. Now choose $(s_0^j, t_0^j) \supset (s_0, t_0)$ in T' with $B_{t_0^j}$ a small enough neighborhood of x_j so as ensure that no line meets three or more of these squares. See figure 2.

□

Lemma 6 *Suppose $(s_i, t_i) \in T'$ for $i = 0, 1, 2, \ldots, n$ are given with the properties that for $i \neq j$ B_{t_i} is disjoint from B_{t_j} and no line meets three or more* *of the B_{t_j}'s. Then there exists $(s_i^j, t_i^j) \in T'$ for $i = 0, 1, 2, \ldots, n$ and $j = 0, 1$ with $(s_i^j, t_i^j) \supset (s_i, t_i)$ and $B_{t_i^0}$ disjoint from $B_{t_i^1}$ and no line meets three or more of the $B_{t_i^j}$ for $i = 0, 1, 2, \ldots, n$ and $j = 0, 1$.*

proof

Apply the split and shrink lemma iteratively $n + 1$ times.

□

To prove the theorem construct a subtree $T^* \subset T'$ with the property that $p[T^*] = P$ is perfect and for every $n \in \omega$ no line meets three or more of the B_t with $(s, t) \in T^*$ for some s and t of length n. Then P is a perfect subset of A which does not contain three collinear points.

□

One of our original interests in this problem was the following corollary:

Corollary 7 *Suppose that A is an analytic subset of the plane and $\mathcal{L}$ is a family of fewer than continuum many lines such that $\mathcal{L}$ covers A. Then A is covered by a countable subfamily of lines from $\mathcal{L}$.*

proof

If A contains a perfect set with no three points collinear then A could not be covered by $\mathcal{L}$, since perfect sets have the cardinality of the continuum. Hence we may assume that A is covered by countably many lines. Suppose:

$$A \subset \bigcup \{l_n \mid n \in \omega\}$$

If l is any line such that $l \cap A$ is uncountable, then l is in $\mathcal{L}$. This is because $l \cap A$ is an analytic set, hence has cardinality the continuum, but every line in $\mathcal{L}$ meets l in at most one point, so $l \cap A$ could not be covered by $\mathcal{L}$.

So A is covered by the l_n which are in $\mathcal{L}$ plus at most countably many more lines in $\mathcal{L}$ which cover the points in A such that $l_n \cap A$ is countable.

□

Note that if V=L then there exists an uncountable coanalytic subset of the line which contains no perfect subsets. If this set is arranged around a circle then we see that the theorem cannot be generalized to include coanalytic sets.

However Dougherty, Jackson, and Kechris have proved the following result:

Theorem 8 *Suppose the axiom of determinacy and V=L[R] is true. Then every subset of the plane either can be covered by countably many lines or contains a perfect subset P with no three points collinear.*

Their proof uses a technique of Harrington (see Kechris and Martin [1]) to prove Silver's theorem that every coanalytic equivalence relation with uncountably many equivalence classes contains a perfect set of inequivalent points. They generalize this result and our result.

Is it true in Solovay's model [3] that every subset of the plane either can be covered by countably many lines or contains a perfect subset P with no three points collinear?

References

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