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Some Problems in Set Theory and Model Theory

By

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INTRODUCTION AND ACKNOWLEDGEMENTS

It is often the case that it is easier to prove something is consistent with ZFC than it is to prove it. This is especially true when it is independent of ZFC.

The focus of this thesis is on subsets of the set of real numbers and some of their properties. By a real number we could mean an element of 2^{ω} the Cantor space or ω^{ω} the Baire space (topologized as usual by basic open sets of the form $[s] = \{f \in \omega^{\omega} : s \subseteq f\}$ where $s \in \omega^{<\omega} = \bigcup_{n<\omega} \omega^{n}$). Also a countable structure is a real number. For example, identify a structure $\langle \omega, R \rangle$ where $R \subseteq \omega^2$ a binary relation with an element of $2^{\omega \times \omega}$. Even an ultrafilter on ω is just a set of real numbers with some peculiar property.

Some important notation:

$|A|$ usually denotes the cardinality of the set A except sometimes it's used to denote the universe of a model, the rank of an element in a well-founded tree, or any two of the above.

$\underset{\sim}{\Sigma}^0_0 = \underset{\sim}{\Pi}^0_0 = \underset{\sim}{\Delta}^0_1$ = clopen sets of reals.

$\underset{\sim}{\Sigma}^0_\alpha$ = countable unions of $\bigcup_{\beta<\alpha} \underset{\sim}{\Pi}^0_\beta$.

$\underset{\sim}{\Pi}^0_\alpha$ = countable intersections of $\bigcup_{\beta<\alpha} \underset{\sim}{\Sigma}^0_\beta$.

= complements of $\underset{\sim}{\Sigma}^0_\alpha$'s.

$\underset{\sim}{\Sigma}^0_5 = G_{\delta\sigma\delta\sigma}$.

$P(X)$ denotes the set of all subsets of X.

Each of the four Parts begins with an introduction and ends with a bibliography.

I would like to thank:

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Foremost I would like to thank John Addison for the encouragement, stimulus, and necessary prodding that I needed.

I would like to dedicate this thesis to my wife, Man-Li.

PART I. ON THE LENGTH OF BOREL HIERARCHIES

Introduction.

For any separable metric space X and α with $1 \leq \alpha \leq \omega_1$ define the Borel classes $\underset{\sim}{\Sigma}^0_\alpha$ and $\underset{\sim}{\Pi}^0_\alpha$. Let $\underset{\sim}{\Sigma}^0_1$ be the class of open sets and for $\alpha > 1$ $\underset{\sim}{\Sigma}^0_\alpha$ is the class of countable unions of elements of $\bigcup\{\underset{\sim}{\Pi}^0_\beta : \beta < \alpha\}$ where $\underset{\sim}{\Pi}^0_\beta = \{X - A: A \in \underset{\sim}{\Sigma}^0_\beta\}$. Hence $\underset{\sim}{\Sigma}^0_1$ = open = G, $\underset{\sim}{\Pi}^0_1$ = closed = F, $\underset{\sim}{\Sigma}^0_2 = F_\sigma$, $\underset{\sim}{\Pi}^0_2 = G_\delta$, etc. Note that $\underset{\sim}{\Sigma}^0_{\omega_1} = \underset{\sim}{\Pi}^0_{\omega_1}$ = set of all Borel in X subsets of X. The Baire order of X (ord(X)) is the least $\alpha \leq \omega_1$ such that every Borel in X subset of X is $\underset{\sim}{\Sigma}^0_\alpha$ in X. Since the Borel subsets of X are closed under complementation we could equally well have defined ord(X) in terms of $\underset{\sim}{\Pi}^0_\alpha$ in X or $\underset{\sim}{\Delta}^0_\alpha = \underset{\sim}{\Pi}^0_\alpha \cap \underset{\sim}{\Sigma}^0_\alpha$ in X. Note also that for $X \subseteq \mathbb{R}$ (the real numbers) ord(X) is the least α such that for every Borel set A in $\mathbb{R}$ there is a $\underset{\sim}{\Sigma}^0_\alpha$ in $\mathbb{R}$ set B such that $A \cap X = B \cap X$. Also note that ord(X) = 1 iff X is discrete, ord(Q) = 2 where Q is the space of rationals, and in general for X a countable metric space ord(X) ≤ 2 since every subset of X is $\underset{\sim}{\Sigma}^0_2(F_\sigma)$ in X.

It is a classical theorem of Lebesgue (see [11]) that for any uncountable Polish (separable and completely metrizable) space $\mathrm{ord}(X) = \omega_1$. The same is true for any uncountable analytic ($\underset{\sim}{\Sigma}_1^1$) space X since X has a perfect subspace (see [11]) and Borel hierarchies relativize.

The Baire order problem of Mazurkiewicz (see [19]) is: for what ordinals α does there exist $X \subseteq \mathbb{R}$ such that $\mathrm{ord}(X) = \alpha$. Banach conjectured (see [29]) that for any uncountable $X \subseteq \mathbb{R}$ the Baire order of X is ω_1. In §3 we review the classically known results of Sierpinski, Szpilrajn, and Poprougenko. We show that it is consistent with ZFC that for each $\alpha \leq \omega_1$ there is an $X \subseteq \mathbb{R}$ with $\mathrm{ord}(X) = \alpha$. In fact, we prove a theorem of Kunen's that CH implies this. We also show that Banach's conjecture is consistent with ZFC.

Given a set X and R a family of subsets of X ($R \subseteq P(X)$) define for every $\alpha \leq \omega_1$ $R_\alpha \subseteq P(X)$ as follows. Let $R_0 = R$ and for each $\alpha > 0$ if α is even (odd) let R_α be the family of countable intersections (unions) of elements of $\cup\{R_\beta : \beta < \alpha\}$. Generalizing Mazurkiewicz's question Kolmogorov (see [8]) asked: for what ordinals α does there exist X and $R \subseteq P(X)$ such that α is the least such that $R_\alpha = R_{\omega_1}$. Kolmogorov's question can be generalized by replacing $P(X)$ by an arbitrary σ-algebra (a countably complete boolean algebra). In §2 we prove that for any $\alpha \leq \omega_1$ there is a complete boolean algebra with the countable chain condition which is countably generated in exactly α steps. This answers a question of Tarski who had noticed that the boolean algebras Borel(2^ω) modulo the ideal of meager sets and Borel(2^ω) modulo the ideal of measure zero sets are countably generated in exactly one and two steps respectively (see [4]). Theorem 12 which is due to Kunen shows that the same answer to Kolmogorov's problem (every $\alpha \leq \omega_1$) follows from the solution of Tarski's problem.

Let $R = \{A \times B: A,B \subseteq 2^\omega\}$. In §4 we show that for any α, $2 \leq \alpha < \omega_1$, it is consistent with ZFC that α is the least ordinal such that R_α is the set of all subsets of $2^\omega \times 2^\omega$. This answers a question of Mauldin [1].

For $\alpha \leq \omega_1$ a set $X \subseteq 2^\omega$ is a Q_α set iff every subset of X is Borel in X and $\mathrm{ord}(X) = \alpha$. It is shown that it is consistent with ZFC that for every $\alpha < \omega_1$ there is a Q_α set. In §4 we also show that there are no Q_{ω_1} sets. However, we do show that it is consistent with ZFC that there is an $X \subseteq 2^\omega$ with $\mathrm{ord}(X) = \omega_1$ and every X-projective set is Borel in X. This answers a question of Ulam [31], p. 10.

Also in §4 we show that it is relatively consistent with ZFC that the universal Σ^1_1 set is not in R_{ω_1} confirming a conjecture of Mansfield [13] who had shown that the universal $\underset{\sim}{\Sigma}^1_1$ set is never in the σ-algebra generated by the rectangles with $\underset{\sim}{\Sigma}^1_1$ sides.

Given $R \subseteq P(X)$ let $K(R)$ (the Kolmogorov number of R) be the least α such that $R_\alpha = R_{\omega_1}$. It is an exercise to show that for $\alpha = 0,1$, or 2 there is an $R \subseteq P(\{0,1\})$ with $K(R) = \alpha$.

Proposition 1. Given $R \subseteq P(X)$ then (a) if R is finite or X is countable then $K(R) \leq 2$, and (b) there exists $S \subseteq P(Y)$ such that cardinality of S and Y is $\leq 2^{\aleph_0}$ and $K(R) = K(S)$.

Proof.

(a) Note $\bigcup_{\alpha<\alpha_0} \bigcap_{\beta<\beta_0} \bigcup_{\gamma<\gamma_0} A_{\alpha,\beta,\gamma} = \bigcap_{f:\alpha_0\to\beta_0} \bigcup_{\alpha<\alpha_0} \bigcup_{\gamma<\gamma_0} A_{\alpha,f(\alpha),\gamma}$

If R is finite or X countable then $\bigcap_{f:\alpha_0\to\beta_0}$ can always be taken to be a countable intersection.

(b) Let V_α be the sets of rank less than α. Choose α a limit ordinal of uncountable cofinality so that $R,X \in V_\alpha$. Let (M,ε) be an elementary substructure of (V_α,ε) containing R and X such that $M^\omega \subseteq M$ and $|M| \leq 2^{\aleph_0}$. Now let $Y = X \cap M$ and $S = \{A \cap Y: A \in R \cap M\}$. ∎

Mazurkiewicz's problem is equivalent to Kolmogorov's problem for R a countable field of sets (that is closed under finite intersection and complementation).

Proposition 2. (Sierpinski [23] also in [30]) Given $R \subseteq P(X)$ a countable field of sets there exists $Y \subseteq 2^{\omega}$ such that $K(R) = \text{ord}(Y)$. (That is we may reduce to considering subsets Y of 2^{ω} and relativizing the usual Borel hierarchy on 2^{ω} to Y.)

Proof.

Let $R = \{A_n : n \in \omega\}$ and define $F: X \to 2^{\omega}$ by $F(x)(n) = 1$ iff $x \in A_n$. Put $Y = F''X$. ∎

Define $K = \{\beta : 2 \leq \beta < \omega_1$ and there is $X \subseteq \omega^{\omega}$ uncountable with $\text{ord}(X) = \beta\}$. What can K be?

Proposition 3. K is a closed subset of ω_1.

Proof.

Given $A \subseteq \omega^{\omega}$ and $n \in \omega$ define $nA = \{x \in \omega^{\omega} : x(0) = n$ and $\exists y \in A \forall n(x(n+1) = y(n))\}$. If $X = \bigcup_{n \in \omega} nX_n$, then it is readily seen that $\text{ord}(X) = \sup\{\text{ord}(X_n) : n \in \omega\}$. ∎

Note that K is the same set of ordinals if we replace ω^{ω} by $\mathbb{R}$ the real numbers or 2^{ω}. This is true for $\mathbb{R}$ because if $X \subseteq \mathbb{R}$ and $\mathbb{R} - X$ is not dense then X contains a nonempty interval, hence $\text{ord}(X) = \omega_1$; but $\mathbb{R} - X$ dense means we may as well assume $X \subseteq$ irrationals $\cong \omega^{\omega}$.

In the definition of $K(R) = \omega$ for $R \subseteq P(X)$ we ignored the possibility that the hierarchy on R might have exactly ω levels, i.e. $R_{\omega_1} = \cup\{R_n: n < \omega\}$ but for all $n < \omega$ $R_n \neq R_{\omega_1}$. In fact a Borel hierarchy of length less than ω_1 must have a top level.

<u>Proposition</u> 4. If $R \subseteq P(X)$ is a field of sets, λ is a countable limit ordinal, and $R_{\omega_1} = \cup\{R_\alpha: \alpha < \lambda\}$ then there is $\alpha < \lambda$ such that $R_\alpha = R_{\omega_1}$.

<u>Proof</u>.

Using the proof of Proposition 2 we can assume $X \subseteq 2^\kappa$ for some κ and $R = \{[s] \cap X: S: D \to 2$ where $D \subseteq \kappa$ is finite$\}$ where $[s] = \{f \in 2^\kappa: f$ extends $s\}$. For each A in R_{ω_1} there is $T \subseteq \kappa$ countable such that for any f and g in X if $f \restriction T = g \restriction T$, then $f \in A$ iff $g \in A$. In this case we say T supports A. Choose $T \subseteq \kappa$ countable so that for any $D \subseteq T$ finite and $s: D \to 2$ if $\text{ord}(X \cap [s]) = \lambda$ then for any $\alpha < \lambda$ there is an $A \subseteq [s]$ in $R_{\alpha+1} - R_\alpha$ such that T supports A. By taking an autohomeomorphism of 2^κ we may assume $T = \omega$. Define L to be $\{s \in 2^{<\omega}: \text{ord}([s] \cap X) = \lambda\}$.

<u>Claim</u>. For any s in L there are t and $\hat{t}$ in L incompatible extensions of s.

<u>Proof</u>.

Without loss of generality assume $s = \phi$ and there is $f \in 2^\omega$ such that for every $s \in L$ $s \subseteq f$. For each $n < \omega$

define t_n in 2^{n+1} by $t_n(m) = f(m)$ for $m < n$ and $t_n(n) = 1 - f(n)$. Then $[f] \cup \bigcup\{[t_n]: n < \omega\}$ is a disjoint union covering 2^κ. If there is a $\beta_0 < \lambda$ such that for all $n < \omega$ $\mathrm{ord}([t_n] \cap X) < \beta_0$, then for all A in R_{ω_1} supported by ω A is in R_{β_0+1}. This is because $A \cap [f] = \phi$ or $X \cap [f] \subseteq A$. But this contradicts the choice of ω.

On the other hand, if there is no such bound β_0, choose $Z_n \subseteq [t_n]$ with $Z_n \in R_{\omega_1}$ so that for every $\beta < \lambda$ there is $n < \omega$ with $Z_n \notin R_\beta$. But then $\bigcup\{Z_n: n < \omega\}$ is not in $\bigcup\{R_\beta: \beta < \lambda\}$. This proves the claim and this last argument also proves the proposition from the claim. ∎

<u>Remark</u>. If $R \subseteq P(X)$ and $R_{\omega_1} = \bigcup\{R_n: n < \omega\}$ and there is $n_0 < \omega$ such that $\{X - A: A \in R\} \subseteq R_{n_0}$ then there is $n_1 < \omega$ such that $R_{n_1} = R_{\omega_1}$. Willard [32] shows that for any $\alpha < \omega_1$ there are R and X with $R \subseteq P(X)$ such that α is the least ordinal such that $\{X - A: A \in R\} \subseteq R_\alpha$.

§1. Some basic definitions and lemmas

For $T \subseteq \omega^{<\omega}$ T is a well founded tree iff T is a tree (if $t \subseteq s \in T$ then $t \in T$) and is well founded (for any $f \in \omega^{\omega}$ there is an $n < \omega$ such that $f \restriction n \notin T$). For $s \in T$ define $|s|_T$ (the rank of s in T) by $|s|_T = \sup\{|t|_T + 1 : s \subsetneq t \in T\}$. Often we drop T and let $|s| = |s|_T$. T is normal of rank α means that:

(a) T is a well founded tree;

(b) $|\phi| = \alpha$ (ϕ is the empty sequence);

(c) ($s \in T$ and $|s| > 0$) $\rightarrow$ $(\forall i < \omega(s^\frown i \in T))$;

(d) ($s \in T$ and $|s| = \beta + 1$) $\rightarrow$ $(\forall i < \omega(|s^\frown i| = \beta))$;

(e) ($s \in T$ and $|s| = \lambda$ where λ is a limit ordinal) $\rightarrow$
$(\forall \beta < \lambda \{i : |s^\frown i| < \beta\}$ is finite and
$\forall i < \omega |s^\frown i| \geq 2)$.

Note that for any $n < \omega$ the tree $\omega^{\leq n}$ is normal of rank n. If α_n for $n < \omega$ are strictly increasing to α (or $\alpha_n = \beta$ where $\alpha = \beta + 1$) and for each $n < \omega$ T_n is normal of rank α_n, then $T = \cup\{n^\frown s : n < \omega$ and $s \in T_n\}$ is normal of rank α. We often use T_α to denote some fixed normal tree of rank α.

For any $\alpha < \omega_1$ and $Y \subseteq X \subseteq \omega^{\omega}$ define the partial order $\mathbb{P}_\alpha(Y,X)$ (the order is given by inclusion). Fix some T normal of rank α. $p \in \mathbf{P}_\alpha(Y,X)$ iff $p \subseteq (T - \{\phi\}) \times (X \cup \omega^{<\omega})$ and (1) through (5) hold.

(1) p is finite.

(2) $|s| = 0$ implies that if $(s,x) \in p$ then $x \in \omega^{<\omega}$ and if $(s,y) \in p$ then $x = y$. (So if $T^* = \{s \in T: |s| = 0\}$ then $p \upharpoonright (T^* \times (X \cup \omega^{<\omega}))$ is a function from a finite subset of T^* into $\omega^{<\omega}$.)

(3) If $|s| > 0$ and $(s,x) \in p$ then $x \in X$

(4) If s and $s^\frown i \in T$ and $x \in X$ then not both (s,x) and $(s^\frown i,x)$ are in p, or if $|s^\frown i| = 0$, not there exists $k \in \omega$ such that both (s,x) and $(s^\frown i, x\upharpoonright k)$ are in p.

(5) If s of length one and $(s,x) \in p$ then x is not in Y.

Let G be $P_\alpha(Y,X)$ generic. Working in the extension define for each $s \in T$, $G_s \subseteq \omega^\omega$. For $|s| = 0$, let $G_s = \{x \in \omega^\omega: \exists t \in \omega^{<\omega}\ t \subseteq x$ and $\{(s,t)\} \in G\}$.

For $|s| > 0$, let $G_s = \{\omega^\omega - G_{s^\frown i}: i < \omega\}$.

Note that for each $s \in T$

$G_s \in \underset{\sim}{\Pi}^0_{|s|}$.

<u>Lemma</u> 5. For each x in X and s in $T - \{\phi\}$ with $|s| > 0 [x \in G_s$ iff $\{(s,x)\} \in G]$.

Proof.

Case 1. $|s| = 1$. (This is the argument from almost-disjoint-sets forcing.)

If $x \in G_s$ then $x \notin G_{s^\frown i}$ for all $i \in \omega$. Hence for all k and i in ω $(s^\frown i, x \restriction k) \notin G$. Let $D = \{p: (s,x) \in p$ or there exist k and i such that $(s^\frown i, x \restriction k) \in p\}$. D is dense since if $(s,x) \notin p$ if we let $\{x_1, x_2, \ldots, x_n\} \subseteq X$ be all the elements of ω^ω mentioned in p other than x, we can choose k sufficiently large so that $x \restriction k \neq x_i \restriction k$ for all $i \leq n$. Also we can choose j sufficiently large so that $(s^\frown j)$ is not mentioned in p and then $p \cup \{(s^\frown j, x \restriction k)\} \in (\mathbb{P}_\alpha(Y,X) \cap D)$. Since $G \cap D$ is non-empty and $x \notin G_{s^\frown i}$ all i; we conclude that $(s,x) \in G$.

If $x \notin G_s$ then $x \in G_{s^\frown i}$ for some i. Hence there exist k such that $(s^\frown i, x \restriction k) \in G$ so $(s,x) \notin G$ by clause (4).

Case 2. $|s| > 1$.

If $x \in G_s$ then $x \notin G_{s^\frown i}$ for all i, and hence by induction $(s^\frown i, x) \notin G$ for all i. Let $D = \{p: (s,x) \in p$ or there exist i such that $(s^\frown i, x) \in p\}$. D is dense hence $(s,x) \in G$.

If $x \notin G_s$ then $(s^\frown i, x) \in G$ for some i (by induction). Hence $(s,x) \notin G$ by clause (4). ∎

Corollary 6. $G_\phi \cap X = Y$ $(\alpha \geq 2)$

Proof.

If $x \in Y$ then for every $n, ((n), x) \notin G$ (by clause 5). Hence by Lemma 5 for every $n, x \notin G_{(n)}$ and so $x \in G_\phi$. If $x \notin Y$ then $\{p$: there exists n such that $((n),x) \in p\}$ is dense hence there exists n such that $x \in G_{(n)}$ (by Lemma 5) so $x \notin G_\phi$. ∎

Remarks:

(1) $\mathbb{P}_0(X,Y)$ is trivial (the empty set).

(2) $\mathbb{P}_1(X,Y)$ has nothing to do with X and Y and is isomorphic as a partial order to the usual Cohen partial order for adding a map from ω to ω.

(3) $\mathbb{P}_2(X,Y)$ is another way of viewing Solovay's "almost-disjoint-sets forcing" (see [6]).

Lemma 7. $\mathbb{P}_\alpha(X,Y)$ has the countable chain condition.

Proof.

Suppose by way of contradiction that there exist F included in $\mathbb{P}_\alpha(X,Y)$ of cardinality $\aleph_1$ of pairwise incompatible conditions. Since there are only countably many finite subsets of T, we may assume there exist $H \subseteq T - \{\phi\}$ finite so that every $p \in F$ is included in $H \times (X \cup \omega^{<\omega})$. We may also assume that for every $p \in F$ and $q \in F$ and $s \in H$ with $|s| = 0$ and $t \in \omega^{<\omega}$ that

$[(s,t) \varepsilon p$ iff $(s,t) \varepsilon q]$. Now let $(x_\beta : \beta < \aleph_1)$ be all the elements of X occurring in members of F. For each p in F let $p^*: G_p \to P(H)$ be defined by $G_p = \{\beta$: there exists s $(s,x_\beta) \varepsilon p\}$ and for $\beta \varepsilon G_p$ $p^*(\beta) = \{s: (s,x_\beta) \varepsilon p\}$. $\{p^*: p \varepsilon F\}$ is a family of $\aleph_1$ incompatible conditions in the partial order Q, where $Q = \{p$: domain of p is a finite subset of $\aleph_1$ and range of p is $P(H)\}$, ordered by inclusion. Since it is well known that Q has the countable chain condition we have a contradiction. ■

Remarks:

1) If $\mathbb{P} = \mathbb{P}_\alpha(Y,X)$ for any α,X, and Y then $\mathbb{P}$ is absolutely c.c.c. That is to say if $\mathbb{P} \varepsilon M \models$ "ZFC" then $M \models$ "P has c.c.c.". It follows that the direct sum of any combination of the $\mathbb{P}_\alpha$'s has the c.c.c. (direct sum of $Q_\alpha : \alpha < \kappa)$ is $\sum_{\alpha<\kappa} Q_\alpha = \{p: p: \kappa \to \bigcup_{\alpha<\kappa} Q_\alpha, \forall \alpha < \kappa\ p(\alpha) \varepsilon Q_\alpha$, and $p(\alpha) = 0$ for all but finitely many $\alpha\}$. $p \geq q$ iff $\forall\alpha(p(\alpha) \geq q(\alpha))$.

2) We assume the fact that iterated c.c.c. forcing is c.c.c. (Solovay-Tennenbaum [26]) and occasionally use notation and facts from [26].

I would like to prove next an heuristic proposition. Define $\mathbb{P}$ a partial order: $p \varepsilon \mathbb{P}$ iff p is a finite consistent set of sentences of the form "$[s] \subseteq G_n$", "$x \notin G_n$", or "$x \in \bigcap_{n\in\omega} G_n$" (where $s \in \omega^{<\omega}$ and $x \in \omega^\omega$). Order $\mathbb{P}$ by inclusion. Any G $\mathbb{P}$-generic determines a $\underset{\sim}{\Pi}^0_2$ set $\bigcap_{n\in\omega} G_n$.

<u>Proposition</u>. If G is $\mathbb{P}$-generic over M (transitive countable model of ZFC) then $M[G] \models$ "$\forall F \in F_\sigma (F \cap M \neq \bigcap_{n\in\omega} G_n \cap M)$".

<u>Proof</u>.

Suppose not and let C_n be closed for $n \in \omega$ and $p \in G$ be such that $p \Vdash$ "$\bigcup_{n\in\omega} C_n \cap M = \bigcap_{n\in\omega} G_n \cap M$". It is easily seen that $\mathbb{P}$ has c.c.c. (see the proof of Lemma 7). Thus working in M we can find $Q \subseteq \mathbb{P}$ countable such that for any $\hat{G}$ $\mathbb{P}$-generic, $n \in \omega$, and $s \in \omega^{<\omega}$, if $M[\hat{G}] \models$ "$[s] \cap C_n = \emptyset$" then $\exists q \in Q \cap \hat{G}$ such that $q \Vdash$ "$[s] \cap C_n = \emptyset$". Since Q is countable, we can find $z \in \omega^\omega \cap M$ not mentioned in p or any condition in Q. Since $p \cup \{z \in \bigcap_{n\in\omega} G_n\} \Vdash$ "$z \in \bigcup_{n\in\omega} C_n$" we can find $\bar{n} \in \omega$ and $\hat{p} \geq p$ and not mentioning z so that $\hat{p} \cup \{z \in \bigcap_{n\in\omega} G_n\} \Vdash$ "$z \in C_{\bar{n}}$", because the only other way to mention z is "$z \notin G_n$". By taking $\bar{m}$ large enough $\hat{p} \cup \{z \notin G_{\bar{m}}\}$ will be consistent, and since it extends p it forces "$z \notin C_{\bar{n}}$." Let G be $\mathbb{P}$-generic with $\hat{p} \cup \{z \notin G_{\bar{m}}\}$ in G. Let $k \in \omega$ and $q \in G \cap Q$ be so that $q \Vdash$ "$[z \upharpoonright k] \cap C_{\bar{n}} = \emptyset$". But $\hat{p} \cup q \cup \{z \in \bigcap_{n\in\omega} G_n\}$ is consistent because $q \in Q$ and so doesn't mention z. This is a contradiction since $q \Vdash$ "$z \notin C_n$" and $\hat{p} \cup \{$"$z \in \bigcup_{n\in\omega} G_n$"$\} \Vdash$ "$z \in C_{\bar{n}}$". ■

Define for $F \subseteq \omega^\omega$ and $p \in \mathbb{P} = \mathbb{P}_\alpha(Y,X)$,
$|p|(F) = \max(\{|s| : \text{there is } x \notin F \text{ with } (s,x) \in p\})$.
This is called the rank of p over F.

Lemma 8. For all $\beta \geq 1$ and $p \in \mathbb{P}$ there is $\hat{p} \in \mathbb{P}$ compatible with p and $|\hat{p}|(F) < \beta + 1$ so that for any $q \in \mathbb{P}$ with $|q|(F) < \beta$, if $\hat{p}$ and q are compatible then p and q are compatible.

Proof. First find an extension $p_0 \geq p$ so that for all $(s,x) \in p$ and $i < \omega$ if $|s| = \lambda$ is a limit ordinal and $|s^\frown i| \leq \beta + 1 < \lambda$ (there are only finitely many such $s^\frown i$), then there is a $j < \omega$ such that $(s^\frown i^\frown j, x) \in p_0$. Now let $\hat{p} = \{(s,x) \in p_0 : |s| < \beta + 1 \text{ or } x \in F\}$. We check that $\hat{p}$ has the requisite property. Suppose p and q are incompatible, $\hat{p}$ and q are compatible, and $|q|(F) < \beta$. Since $\beta \geq 1$ for all $(s,x) \in p$ if $|s| \leq 1$ then $(s,x) \in \hat{p}$, hence since $\hat{p}$ and q are compatible there are $s,t \in \omega^{<\omega}$, $i < \omega$, and $x \in \omega^\omega$ such that $(s,x) \in p$, $(t,x) \in q$, and $s = t^\frown i$ or $t = s^\frown i$.

Case 1. If $x \in F$ or $|s| < \beta + 1$ then $(s,x) \in \hat{p}$ and so $\hat{p}$ and q are incompatible.

Case 2. If $x \notin F$ and $|s| \geq \beta + 1$ then by definition of

$|q|(F) < \beta$, $|t| < \beta$. So $t = s^\frown i$. If $|s| = \gamma + 1$ for some γ then $|t| = \gamma \geq \beta$, contradiction. If $|s| = \lambda$ is an infinite limit ordinal then by the construction of p_0 there is $j < \omega$ with $(t^\frown j, x) \in p_0$ and hence $(t^\frown j, x) \in \hat{p}$ and so q and $\hat{p}$ are incompatible.

§2 Boolean Algebras

For $\mathbb{B}$ a complete boolean algebra, C included in $\mathbb{B}$, and $\alpha \geq 1$ define $\Sigma_\alpha(C)$, $\Pi_\alpha(C)$:

$$\Sigma_1(C) = \{\Sigma S\colon S \subseteq C\},$$
$$\Sigma_\alpha(C) = \{\Sigma S\colon S \subseteq \bigcup_{\beta<\alpha}\Pi_\beta(C)\} \text{ for } \alpha > 1, \text{ and}$$
$$\Pi_\alpha(C) = \{-a\colon a \in \Sigma_\alpha(C)\}$$

Define $K(\mathbb{B})$ to be the least ordinal α such that there exists a countable C included in $\mathbb{B}$ with $\Sigma_\alpha(C) = \mathbb{B}$.

Theorem 9. For each $\alpha \leq \omega_1$ there exists a complete boolean algebra $\mathbb{B}$ with countable chain condition and $K(\mathbb{B}) = \alpha$.

Proof.

For $\alpha = 0$ take $\mathbb{B}$ to be any finite boolean algebra.
For $\alpha = 1$ take $\mathbb{B}$ to be $(P(\omega), \wedge, \vee)$ (or more appropriately the regular open subsets of ω^ω since this corresponds to Cohen real forcing).

For α, $2 \leq \alpha < \omega_1$, $\mathbb{B}$ will be the complete boolean algebra associated with $\underset{\sim}{\Pi}^0_\alpha$-forcing. Let $\mathbb{P} = \mathbb{P}_\alpha(\emptyset, X)$. Given a partial order $\mathbb{P}$ there is a canonical way of constructing a complete boolean algebra $\mathbb{B}$ in which $\mathbb{P}$ is densely embedded (see [5]). Let $[p]$ denote the image of $p \in \mathbb{P}$ under this embedding j then if $p \geq q$ then $[p] \leq [q]$, and for every $a \in \mathbb{B}$ if $a \neq 0$ then there is a $p \in \mathbb{P}$ such that $[p] \leq a$.

<u>Lemma</u> 10. Suppose $F \subseteq X$ and $C = \{[p]: p \in \mathbb{P}$ and $|p|(F) = 0\}$. For any $\beta \geq 1$, $p \in \mathbb{P}$, and a in $\Sigma_\beta(C)$, if $[p] \leq a$ then there is $q \in \mathbb{P}$ such that $|q|(F) < \beta$, q and p are compatible, and $[q] \leq a$.

<u>Proof</u>.

The proof is by induction on β.

<u>Case</u> 1. $\beta = 1$. Suppose $a = \Sigma\{[q]: q \in \Gamma\}$ for some $\Gamma \subseteq C$. If $[p] \leq a$ then for $q \in \Gamma$, p and q are compatible.

<u>Case</u> 2. β a limit ordinal. Suppose $a = \Sigma\{b: b \in \Gamma\}$ for some $\Gamma \subseteq \cup\{\Sigma_\alpha(C): \alpha < \beta\}$. Then there is $\hat{p} \geq p$ and $b \in \Gamma \cap \Sigma_\alpha(C)$ for some $\alpha < \beta$ so that $[\hat{p}] \leq b$. Now apply the inductive hypothesis to $\hat{p}$.

<u>Case</u> 3. $\beta + 1$. Suppose $[p] \leq \Sigma\{b: b \in \Gamma\}$ for some $\Gamma \subseteq \Pi_\beta(C)$. Choose $\hat{p} \leq p$ so that for some $b \in \Gamma$,

$[\hat{p}] \leq b$. By Lemma 8 of §1, there exists q compatible with $\hat{p}$ with $|q|(F) < \beta + 1$ and for any r with $|r|(F) < \beta$, if r and q are compatible then r and $\hat{p}$ are compatible. This q works since if $[q] \not\leq b$ then there exists $q_0 \geq q$ with $[q_0] \leq -b$. Since $-b \in \Sigma_\beta(C)$ by induction there is q_1 compatible with q_0 with $|q_1|(F) < \beta$ and $[q_1] \leq -b$. But then q_1 would be compatible with $\hat{p}$, contradicting $[\hat{p}] \leq b$.

■

Now if $X = \omega^\omega$, for example, the lemma shows that $\mathbb{B}$ cannot be generated by a set of size less than the continuum in fewer than α steps. For suppose $D \subseteq \mathbb{B}$ has cardinality less than $|\omega^\omega|$, then there exists $F \subseteq \omega^\omega$ with $X - F \neq \phi$ and $D \subseteq \Sigma_1\{[p]: |p|(F) = 0\}$. Let $\beta < \alpha$, $z \in X - F$, and $s \in t - \{\phi\}$ with $|s|_T = \beta$ (where T is the normal α-tree used in the definition of $\mathbb{P}_\alpha(\phi, X)$).
$[\{(s,z)\}]$ is not in $\Sigma_\beta(D)$. Because if it were it would be in $\Sigma_\beta(C)$ and so by the lemma there exists q with $|q|(F) < \beta$ and $[q] \subseteq [\{(s,z)\}]$. But since $|s|_T = \beta$ and $z \notin F$ we know $(s,z) \notin q$. Thus there are n (and m) such that $q \cup \{(s^\frown n, z)\}$ ($q \cup \{(s^\frown n, z{\restriction}m)$ in case $|s|_T = 1$) is in $\mathbb{P}$, but this is a contradiction.

Next we show $\mathbb{B}$ is countably generated in α steps. Let $\hat{C} = \{[p]: |p|(\phi) = 0\}$.

Claim. For all $x \in X$ and $s \in T - \{\phi\}$ if $|s|_T = \beta \geq 1$ then $[\{(s,x)\}]$ is in $\Pi_\beta(\hat{C})$.

Proof.

If $|s|_T = 1$ then $[\{(s,x)\}] = \Pi\{-[\{(s^\frown n, x \restriction m)\}]: n,m \in \omega\}$. If $|s| > 1$ then $[\{(s,x)\}] = \Pi\{-[\{(s^\frown n,x)\}]: n \in \omega\}$. For $A \in B$, $-A = \{p \in P: [p] \wedge A = \emptyset\}$. If $(s,x) \in p$ then $[p] \wedge [\{(s^\frown n,x)\}] = \emptyset$ all n. On the other hand if $[p] \wedge [\{(s\ n,x)\}] = \emptyset$ for all n then easily $(s,x) \in p$. ■

Now for any $p \in \mathbb{P}$ $[p] = \Pi\{[\{(s,x)\}]: (s,x) \in p\}$, so $[p] \in \Sigma_\alpha(\hat{C})$. For any $A \in B$ $A = \Sigma\{[p]: p \in A\}$ so $A \in \Sigma_\alpha(\hat{C})$. Thus $K(\mathbb{B}) \leq \alpha$.

We are now ready to consider the case of $\alpha = \omega_1$. Let $\mathbb{P} = \sum_{\alpha<\omega_1} \mathbb{P}_\alpha(\emptyset,\omega^\omega)$. Now the complete boolean algebra associated with $\mathbb{P}$ does take ω_1 steps to close (for suitable generators), however $\mathbb{P}$ is not countably generated. So we do as follows: Let $(x_\alpha: \alpha < \omega_1)$ be any set of ω_1 distinct elements of ω^ω. Let $*: \omega^{<\omega} \times \omega^{<\omega} \to \omega$ be a 1-1 map. Let T_α be the normal tree of rank α used in the construction of $\mathbb{P}_\alpha = \mathbb{P}_\alpha(\emptyset,{}^\omega\omega)$. Any G which is $\mathbb{P}_\alpha$-generic is determined by $G \wedge \{(s,t) \in \mathbb{P}_\alpha: |s|_{T_\alpha} = 0$ and $t \in \omega^{<\omega}\}$. That is a map y from $T^*_\alpha = \{s \in T_\alpha: |s|_{T_\alpha} = 0\}$ to $\omega^{<\omega}$. Now imagine G $\mathbb{P}$-generic and let $y_\alpha: T^*_\alpha \to \omega^{<\omega}$ be the maps determined by G. Let $Y = \{(*(s,t))^\frown x_\alpha : y_\alpha(s) = t$ and $\alpha < \omega_1\}$. Form in the generic extension $\mathbb{P}_2(\omega^\omega - Y, \omega^\omega) = Q$ (in both cases we mean ω^ω formed in the ground model). The partial order we are interested in is $R = \mathbb{P} * Q$. $\mathbb{P} * Q = \{(p,q) : p \in \mathbb{P}$ and $p \Vdash$ "$q \in Q$"$\}$. $(\hat{p},\hat{q}) \geq (p,q)$ iff ($\hat{p} \geq p$ and $\hat{q} \geq q$). $p \Vdash$ "$q \in Q$" just in case whenever $((n),(*(s,t)^\frown x_\alpha)$ is in q then $(s,t) \in p(\alpha)$. Now let $\mathbb{B}$ be the complete boolean algebra associated with R. Since R has the countable chain condition so does $\mathbb{B}$.

<u>Claim</u>: $\mathbb{B}$ is countably generated

<u>Proof</u>.

The idea is that once you know what the real is gotten by Q you know all the reals gotten by $\mathbb{P}$--and hence everything. Let $C = \{[\langle \phi, q \rangle] : |q|(\phi) = 0\}$. Then C is countable and generates $\mathbb{B}$.

For $C \subseteq \omega^\omega$ and $(p,q) \in R$ define $|(p,q)|(C) = \max\{|s|_{T_\alpha} :$ there exists $x \notin C, (s,x) \in p(\alpha)$ and $\alpha < \omega_1\}$

<u>Lemma</u> 11. Given $F \subseteq \omega^\omega$ $\forall p \in R \, \forall \beta \geq 1$ $\exists \hat{p} \in R$ compatible with p, $|\hat{p}|(F) < \beta + 1$ and $\forall q \, |q|(F) < \beta$ (if $\hat{p}, q$ compatible then p, q are compatible).

Proof.

This is proved similarly to Lemma 8. Given $p = \langle p_0, p_1 \rangle$ extend each $p_0(\alpha) \leq p_0^1(\alpha)$ as in Lemma 8, then take $\hat{p} = \langle \hat{p}_0, \hat{p}_1 \rangle$, $\hat{p}_1 = p_1$, $\hat{p}_0(\alpha) = \{\langle s,x \rangle \varepsilon\ p_0^1(\alpha)$: $|s| < \beta + 1$ or $x \varepsilon C\}$. Note that $\hat{p}_0 \Vdash$ "$\hat{p}_1 \varepsilon Q$" because requirements in Q are decided by rank zero condition in $\mathbb{P}$.

From this lemma it is easily shown as before that $K(\mathbb{B}) \geq \omega_1$. Since $\mathbb{B}$ is countably generated and has the countable chain condition we have $K(\mathbb{B}) \leq \omega_1$, hence $K(\mathbb{B}) = \omega_1$.

This ends the proof of the theorem. ■

For any σ-complete boolean algebra $\mathbb{B}$ the Sikorski-Loomis theorem ([25], p. 93) says that $\mathbb{B}$ is isomorphic to a σ-field of subsets of some X modulo a σ-ideal of subsets of X.

Theorem 12. (Kunen) $\forall \alpha \leq \omega_1 \exists X, R$ with $R \subseteq P(X)$ such that $K(R) = \alpha$.

Proof.

By the Sikorski-Loomis theorem and Theorem 9 we can find $\hat{R}, X$, and I with $\hat{R} \subseteq P(X)/I$ where I is a σ-ideal and α is the least ordinal such that $\hat{R}_\alpha = \hat{R}_{\omega_1}$. Define $R \subseteq P(X)$ by $(A \in R$ iff $A/I \varepsilon \hat{R})$. It is easily shown by induction on $\beta \leq \omega_1$ that $(A \varepsilon R_\beta$ iff $A/I \varepsilon \hat{R}_\beta)$. Hence we have $K(R) = \alpha$. ■

Let $\mathbb{B}_M$ be the complete boolean algebra Borel((2^ω) modulo the ideal of meager sets.

Theorem 13. For any α, $1 \le \alpha < \omega_1$, there is a countable $C \subseteq \mathbb{B}_M$ which is closed under finite conjunction and complementation so that α is the least ordinal such that $\Sigma_\alpha(C) = \mathbb{B}_M$.

Proof.

Let $x \in \omega^\omega$ be arbitrary and $\mathbb{B}$ be the complete boolean algebra associated with $\mathbb{P}_\alpha(\phi,\{x\})$. Note that if $|p|(\phi) = 0$ then $-[p] = \Sigma\{[q] : |q|(\emptyset) = 0$ and q is incompatible with $p\}$. Let C be the closure of $\{[p]: |p|(\phi) = 0\} = \hat{C}$ under finite boolean combinations. Note that since $\hat{C}$ is closed under finite intersections and $-[p]$ is in $\Sigma_1(\hat{C})$ for any p in $\hat{C}$, we have that $\Sigma_\beta(C) = \Sigma_\beta(\hat{C})$ for all $\beta \ge 1$. By Lemma 10 α is the least such that $\Sigma_\alpha(\hat{C}) = \mathbb{B}$. Since $\mathbb{P}_\alpha(\phi,\{x\})$ is countable and separative, $\mathbb{B}$ is separable and nonatomic and hence isomorphic to $\mathbb{B}_M$. ∎

Remark: The theorem above is false for $\alpha = \omega_1$ since for any countable C which generates $\mathbb{B}_M$, at some countable stage every clopen set is generated and after one more step all of $\mathbb{B}_M$.

§3 Countably generated Borel hierarchies

A set $X \subseteq 2^\omega$ is called a Luzin set iff X is uncountable and for every meager M, $M \cap X$ is countable. The analagous definition with measure zero in place of meager is of a Sierpinski set [30]. For I a σ-ideal in Borel(2^ω) say X is I-Luzin iff $[\forall A \in \text{Borel}(2^\omega)$ $(|A \cap X| < 2^{\aleph_0}$ iff $A \in I)]$. The following theorem was first proved by Luzin [12] assuming I is the ideal of meager sets and CH.

Theorem 14.

(MA) If I is an ω_1 saturated σ-ideal in Borel(2^ω) containing singletons then there exists an I-Luzin set.

Proof.

Let $\kappa = |2^\omega|$, $\{A_\alpha : \alpha < \kappa\} = I$, and $\{B_\alpha : \alpha < \kappa\} = \text{Borel}(2^\omega) - I$ each set repeated κ-many times. Choose x_α for $\alpha < \kappa$, so that for every α x_α is in $B_\alpha - (\bigcup\{A_\beta : \beta < \alpha\} \cup \{x_\beta : \beta < \alpha\})$. Clearly if this can be done then $X = \{x_\alpha : \alpha < \kappa\}$ is I-Luzin. If $\kappa = \omega_1$ then it is trivial, and if MA then this follows from Lemma 1, p. 158 of Martin-Solovay [14]. ∎

The next theorem was proved by Poprougenko [19] and Sierpinski (see [29]).

<u>Theorem</u> 15. If $X \subseteq 2^{\omega}$ is a Luzin set then $\mathrm{ord}(X) = 3$.

<u>Proof</u>.

Since every Borel set B has the property of Baire, $B = G \triangle M$ where G is open and M is meager. But $M \cap X = F$ is countable hence F_{σ}, so $B \cap X = (G \triangle F) \cap X$ showing $\mathrm{ord}(X) \leq 3$. Now choose $s \in 2^{<\omega}$ so that $[s] \cap X$ is uncountable and dense in $[s]$. If $D \subseteq [s] \cap X$ is countable and dense in $[s]$ then $D \neq G \cap X$ for all $G \in G_{\delta}$, so $\mathrm{ord}(X) \geq 3$. ∎

A modern example of a Luzin set arises when one adds an uncountable (in M) number of product generic Cohen reals X to M a countable transitive model of ZFC. $M[X] \models$ "X is a Luzin set". See also Kunen [10] for more on Luzin sets and MA.

In contrast to the boolean algebras Szpilrajn [29] showed:

<u>Theorem</u> 16. If $X \subseteq 2^{\omega}$ is a Siezpinski set then $\mathrm{ord}(X) = 2$.

<u>Proof</u>.

The proof is similar except note that any measurable set is the union of an F_{σ} set and a set of measure zero. ∎

The following theorem generalizes these classical results using a lemma of Silver (see [14], p. 162) that assuming MA every $X \subseteq 2^{\omega}$ with $|X| < |2^{\omega}|$ is a Q set, i.e. every subset of X is an F_{σ} in X.

Theorem 17. (MA). There are uncountable $X, Y \subseteq 2^{\omega}$ such that $\mathrm{ord}(X) = 3$ and $\mathrm{ord}(Y) = 2$.

Proof.

Let X be I-Luzin where I is the ideal of meager Borel sets. For any meager set M choose F a meager F_{σ} with $M \subseteq F$. By Silver's Lemma there exists F_0 an F_{σ} set such that $F_0 \cap F \cap X = M \cap F \cap X = M \cap X$. Thus every meager set intersected with X is an F_{σ} set intersected with X and this shows as before $\mathrm{ord}(X) = 3$. For I the ideal of measure zero sets analagous arguments work. ■

After I had shown that it is consistent with ZFC that $\forall \alpha \leq \omega_1 \, \exists X \subseteq \omega^{\omega} \; \mathrm{ord}(X) = \alpha$, Kunen showed that in fact CH implies $\forall \alpha \leq \omega_1 \, \exists X \subseteq \omega^{\omega} \; \mathrm{ord}(X) = \alpha$. The following theorem sharpens his result slightly.

Theorem 18. If there exists a Luzin set, then for any α such that $2 < \alpha \leq \omega_1$ there is an $X \subseteq 2^{\omega}$ such that $\mathrm{ord}(X) = \alpha$.

Proof.

Let Y be a Luzin set with the property that for every Borel set $A \subseteq 2^{\omega}$ ($A \cap Y$ is countable iff A is meager). Such a set always exists if a Luzin set does. By Theorem 13 there is a $C \subseteq \mathbb{B}_M$ countable such that C generates $\mathbb{B}_M$ in exactly α steps and C is closed under finite Boolean combinations. Let $C = \{[C_n]: n \in \omega\}$ where the C_n are Borel subsets of 2^{ω} and $[C_n]$ is the equivalence class modulo meager of C_n. For $x,y \in 2^{\omega}$ define $x \sim y$ iff for all $n < \omega$ ($x \in C_n$ iff $y \in C_n$). We claim that for each $x \in 2^{\omega}$ the $\sim$ equivalence class of x is meager. Note that any element of the σ-algebra generated by $\{C_n: n < \omega\}$ is a union of $\sim$ equivalence. If some $\sim$ equivalence class E is not meager, then there are K_0 and K_1 disjoint nonmeager Borel sets such that $E = K_0 \cup K_1$. Since $\{[C_n]: n < \omega\}$ generates $\mathbb{B}_M$ there are L_0 and L_1 in the σ-algebra generated by $\{C_n: n < \omega\}$ such that $[L_0] = [K_0]$ and $[L_1] = [K_1]$. For some i, L_i is disjoint from E, but then L_i is meager, contradiction. By shrinking Y if necessary we may assume that for all $x,y \in Y$ ($x = y$ iff $x \sim y$). Let $R = \{C_n \cap Y: n < \omega\}$, then R_2 contains every countable subset of Y. It is easily seen that $K(R) = \alpha$, so by Proposition 2, we are done. ■

Theorem 19. (MA) For any $\alpha < \omega_1$ there is an $X \subseteq \omega^\omega$ such that $\alpha \leq \mathrm{ord}(X) \leq \alpha + 2$.

Proof.

For $\alpha < \omega_1$ let $\mathbb{P}_\alpha$ be the partial order $\mathbb{P}_\alpha(\emptyset,\omega^\omega)$. Let T_α be the normal tree of rank α used in the definition of $\mathbb{P}_\alpha$. $T^*_\alpha = \{s \in T_\alpha : |s|_{T_\alpha} = 0\}$. If G is $\mathbb{P}_\alpha$-generic, then G is completely determined by the real $y_G : T^*_\alpha \to \omega^{<\omega}$ defined by $y_G(s) = t$ iff $\{(s,t)\} \in G$. Each condition $p \in \mathbb{P}_\alpha$ can be thought of as a statement about y_G. Let $C_p = \{y \in \omega^\omega : y$ codes a map $\hat{y}: T^*_\alpha \to \omega^{<\omega}$ and $p(\hat{y})$ is true$\}$. It is easily seen that for any $p \in \mathbb{P}_\alpha$ there is $\beta < \alpha$ such that C_p is $\underset{\sim}{\Pi}^0_\beta$.

Lemma 20. If $\mathbb{B}_\alpha$ is the complete boolean algebra associated with $\mathbb{P}_\alpha$ and X_α is ω^ω with the topology generated by basic open sets $\{C_p : p \in \mathbb{P}_\alpha\}$, then $\mathbb{B}_\alpha$ is isomorphic to the boolean algebra of regular open subsets of X_α.

Proof.

Given $A \subseteq X_\alpha$ a regular open set let $D_A = \{p \in \mathbb{P}_\alpha : C_p \subseteq A\}$. The map $A \to D_A$ is an isomorphism. ∎

Define I_α to the σ-ideal generated by $\underset{\sim}{\Pi}^0_\alpha$ sets of the form $\omega^\omega - \bigcup\{C_p : p \in D\}$ where D is a maximal antichain in $\mathbb{P}_\alpha$.

Lemma 21. α is the least ordinal such that for every Borel A there is a $\underset{\sim}{\Sigma}^0_\alpha$ B such that $A \,\Delta\, B \in I_\alpha$.

<u>Proof</u>.

Note first that I_α is the ideal of meager subsets of X_α. If D is a maximal antichain in $\mathbb{P}_\alpha$, then $\bigcup\{C_p : p \in D\}$ is open dense in X_α, so every element of I_α is meager in X_α. If C is closed nowhere dense in X_α, then let $Q = \{p \in \mathbb{P} : C_p \cap C = \phi\}$. Since Q is open dense in $\mathbb{P}_\alpha$, we can pick $D \subseteq Q$ a maximal antichain. Thus $C \subseteq \omega^\omega - \bigcup\{C_p : p \in D\}$ and every meager subset of X_α is in I_α.

Since A is Borel in X_α there is a regular open set B in X_α such that $(A \Delta B) \in I_\alpha$. Let $Q = \{p \in \mathbb{P}_\alpha : C_p \subseteq B\}$. Pick $D \subseteq Q$ an antichain which is maximal with respect to being contained in Q. Since B is regular open, $B = \bigcup\{C_p : p \in D\}$, so B is $\underset{\sim}{\Sigma}^0_\alpha$ in ω^ω. To see that α is minimal note that for $s \in T_\alpha$ with $|s|_{T_\alpha} = \beta$ there is no B $\underset{\sim}{\Sigma}^0_\beta$ in ω^ω with $(C_{(s,x)} \Delta B) \in I_\alpha$. ∎

Now let $X \subseteq \omega^\omega$ be I_α-Luzin. Then $\mathrm{ord}(X) \geq \alpha$ since for any A and B Borel in ω^ω $((A \Delta B) \in I_\alpha$ iff $|(A \Delta B) \cap X| < |X|)$. But $\mathrm{ord}(X) \leq \alpha + 2$ follows from the fact that for all B in I_α there exists C in $I_\alpha \cap \underset{\sim}{\Sigma}^0_{\alpha+1}$ with $B \subseteq C$, just as in the proof of Theorem 17. This concludes the proof of Theorem 19. ∎

<u>Remarks</u>:

(1) If $V = L$, then using the Δ^1_2 well ordering of $L \cap 2^\omega$ we can get $X \subseteq 2^\omega$ a $\mathbf{\Delta}^1_2$ set with $\text{ord}(X) = \alpha$ for any $\alpha \leq \omega_1$. If X is $\mathbf{\Pi}^1_1$(or $\mathbf{\Sigma}^1_1$) then $X = A \,\Delta\, M$ where A is $\mathbf{\Pi}^0_\alpha$ and $M \in I_\alpha$, so X cannot be I_α-Luzin.

(2) A finer index can be given to a set $X \subseteq \omega^\omega$ by considering the classical Hausdorff difference hierarchies. A set $C \subseteq \omega^\omega$ is a β - $\mathbf{\Pi}^0_\alpha$ set iff there exists $D_\gamma \in \mathbf{\Pi}^0_\alpha$ for $\gamma < \beta$ such that the D_γ's are decreasing and $D_\lambda = {}_{\gamma<\lambda}D_\gamma$ for λ limit and $C = \bigcup\{D_\gamma - D_{\gamma+1} : \gamma < \beta$ and γ even$\}$. It is a theorem of Hausdorff that $\mathbf{\Delta}^0_{\alpha+1} = \bigcup\{\beta - \mathbf{\Pi}^0_\alpha : \beta < \omega_1\}$ (see p. 417, 448 [11]). It is also not hard to show, using a universal set argument, that there exists a properly β - $\mathbf{\Pi}^0_\alpha$ set for all $\alpha, \beta < \omega_1$. Accordingly define $H(X)$ to be the lexicographical least pair $(\alpha,\beta) \in \omega_1^2$ such that for any Borel set A there exists B a β - $\mathbf{\Pi}^0_\alpha$ set such that $A \cap X = B \cap X$. If X is a Luzin set (Sierpinski set) then $H(X) = (2,2)$ $(H(X) = (2,1))$. It is easily shown that in Theorem 22 $N \models$ "$H(X_{\alpha+1}) = (\alpha + 1, 1)$". It is not hard to see that for C a countable closed set $H(C) = (1,\alpha)$ where α is the Cantor-Bendixson rank of C.

<u>Theorem</u> 22. It is relatively consistent with ZFC that for any uncountable $X \subseteq 2^\omega$ $\mathrm{ord}(X) = \omega_1$. This can be generalized to show that for any successor ordinal β_0 such that $2 \leq \beta_0 < \omega_1$, it is consistent that $\{\beta: \exists X \subseteq 2^\omega \text{ uncountable } \mathrm{ord}(X) = \beta\} = \{\beta: \beta_0 \leq \beta \leq \omega_1\}$.

<u>Remark</u>: It is true in the model obtained that for any uncountable separable metric space X the Borel hierarchy on X has length ω_1. This is true, since if $|X| = \omega_1$, then since $|2^\omega| \geq \omega_2$ and X can be embedded into $\mathbb{R}^\omega$, X must be zero dimensional. But any zero dimensional space can be embedded into 2^ω.

To prove Theorem 22 let M be a countable transitive model of ZFC + GCH. Choose $(\alpha_\lambda: \lambda < \omega_2)$ in M so that for all $\beta < \omega_1$ $\{\lambda: \alpha_\lambda = \beta\}$ is unbounded in ω_2. Define $\mathbb{P}^\gamma$ for $\gamma \leq \omega_2$ by induction $\mathbb{P}^0 = \mathbb{P}_{\alpha_0}(\phi, 2^\omega \cap M)$, $\mathbb{P}^{\gamma+1} = \mathbb{P}^\gamma * Q^\gamma$ where Q^γ is a term in the forcing language of $\mathbb{P}^\gamma$ denoting $\mathbb{P}_{\alpha_\gamma}(\phi, M[G_\gamma] \cap 2^\omega)$ for any G_γ $\mathbb{P}^\gamma$-generic over M, and at limits take the direct limit.

The elements of Q^γ can be thought of as terms denoting elements of $2^\omega \cap M[G_\gamma]$ via a natural coding. Choose such a coding which has the property that for any $p, q \in Q^\gamma$ (p and q are compatible iff there is $n < \omega$ such that $p \restriction n$ and $q \restriction n$ are seen to be compatible). For $Q \subseteq \mathbb{P}$ and θ a sentence we say that Q decides θ

iff $\{p \in \mathbb{P}$: there is a $q \in Q$ such that $p \geq q$ and $(q \Vdash "\theta"$ or $q \Vdash "\neg\theta")\}$ is dense in $\mathbb{P}$. For any $H \subseteq 2^\omega$ define $|p|(H)$ and $|\tau|(H,p)$ for $p \in \mathbb{P}^\gamma$ and τ a $\mathbb{P}^\gamma$ term for an element of 2^ω by induction on γ. For $p \in \mathbb{P}^0 = \mathbb{P}_{\alpha_0}(\phi, 2^\omega \cap M)$, $|p|(H) = \max\{|s|_{T\alpha_0} : \exists x \notin H\ (s,x) \in p\}$. For $p \in \mathbb{P}^{\gamma+1}$, $|p|(H) = \max\{|p \restriction \gamma|(H), |p(\gamma)|(H, p \restriction \gamma)\}$. For $p \in \mathbb{P}^\lambda$ where λ is a limit, $|p|(H) = \max\{|p \restriction \gamma| : \gamma < \lambda\}$. For any τ, $|\tau|(H,p)$ is the least β such that for any $n \in \omega$ $\{q \in \mathbb{P}^\gamma$: q incompatible with p or $|q|(H) \leq \beta\}$ decides "$n \in \tau$".

$\mathbb{P}^{\omega 2} = \mathbb{P}$ is not a lattice however it does have one similar property:

Lemma 23. Suppose G is $\mathbb{P}^\alpha$-generic over M and for $i < n < \omega$ $q_i \in G$ and $|q_i|(H) < \beta$, then there is a $q \in G$ with $|q|(H) < \beta$ and $q \leq q_i$ for all $i < n$.

Proof.

The proof is by induction on α. For $\alpha = 0$ or a α a limit it is easy. So suppose $\alpha = \beta + 1$ and $G = G_\beta \times G^\beta$ where G_β is $\mathbb{P}^\beta$-generic over M. Find $\Gamma \subseteq G_\beta$ finite so that for any $q \in \Gamma$ with $|q|(H) < \beta$ and for any i and j less than n if $(s,\tau) \in q_i(\beta)$ and $(s^\frown k, \hat{\tau}) \in q_j(\beta)$

(or $(s^\frown k,t) \in q_j(\beta)$ where $t \in 2^{<\omega}$), then there is $r \in \Gamma$ such that $r \Vdash$ "$\tau \neq \hat{\tau}(t \not\subseteq \tau)$". By induction there is q in G_β such that $|q|(H) < \beta$, for all $q \in \Gamma$ $q \geq \hat{q}$, and for all $i < n$ $q \geq q_i \restriction \beta$. Define $q(\beta)$ to be equal to $\bigcup\{q_i(\beta): i < n\}$. ■

<u>Lemma</u> 24. Given P_0 a countable subset of $\mathbb{P}^\alpha$ and Q_0 a countable set of $\mathbb{P}^\alpha$ terms for elements of 2^ω, there exists H countable such that for every $p \in P_0$ and $\tau \in Q_0$ $|p|(H) = |\tau|(H,\phi) = 0$.

<u>Proof</u>.

This is easy using c.c.c. of $\mathbb{P}^\alpha$. ■

Let $|p| = p\ (H)$ and $|\tau|(p) = |\tau|(H,p)$, for some fixed H.

<u>Lemma</u> 25. For each $p \in \mathbb{P}^\alpha$ and β there exists $\hat{p} \in \mathbb{P}^\alpha$ compatible with p, $|\hat{p}| < \beta + 1$, and for every $q \in \mathbb{P}^\alpha$ with $|q| < \beta$, if $\hat{p}$ and q are compatible, then p and q are compatible.

<u>Proof</u>.

The proof is by induction on α. For $\alpha = 0$ this is just Lemma 8 of §1. For α limit it is easy. From now on assume the Lemma is true for α.

Define for $x,y \in 2^\omega$, $x <_\ell y$ iff $\exists n \forall m < n(x(m) = y(m)$ and $x(n) < y(n))$. This is the lexicographical order. For $C \subseteq 2^\omega$ a nonempty closed set let x_C be the lexicographically least element of C.

<u>Claim</u> 1. Let $\dot{C}$ be a term in $\mathbb{P}^\alpha$ and $p_0 \in \mathbb{P}^\alpha$ with $|p_0| < \beta + 1$ such that $p_0 \Vdash$ "$\dot{C}$ is a nonempty closed subset of 2^ω". Suppose for every G $\mathbb{P}^\alpha$-generic with $p_0 \in G$, and $s \in 2^{<\omega}$ (M[G] $\models$ "$[s] \cap \dot{C} = \emptyset$" iff $\exists q \in G$, $|q| < \beta$, and $q \Vdash$ "$[s] \cap \dot{C} = \emptyset$"). Then $|x_C|(P_0) < \beta + 1$.

<u>Proof</u>.

First we show that given any $p \in \mathbb{P}^\alpha$ with $p \geq p_0$, if $s \in 2^{<\omega}$, $p \Vdash$ "$[s] \cap \dot{C} \neq \emptyset$" then there exist $\hat{p} \in \mathbb{P}^\alpha$ compatible with p, $|\hat{p}| < \beta + 1$, and $\hat{p} \Vdash$ "$[s] \cap \dot{C} \neq \emptyset$". Let p' be as from Lemma 25 for p. By using Lemma 23 obtain $\hat{p}$ compatible with $p, \hat{p} \geq p'$, $\hat{p} \geq p_0$, and $|\hat{p}| < \beta + 1$. I claim $\hat{p} \Vdash$ "$[s] \cap \dot{C} \neq \emptyset$". Suppose not then there exists G $\mathbb{P}^\alpha$-generic, $\hat{p} \in G$, and M[G] $\models$ "$[s] \cap \dot{C} = \emptyset$". So there exists $q \in G$, $|q| < \beta$, and $q \Vdash$ "$[s] \cap \dot{C} = \emptyset$". But then since q is compatible with $\hat{p}$ it is compatible with p' and hence with p, contradiction. In order to show $|x_C|(p_0) < \beta + 1$ it suffices to show for every $p \geq p_0$ and $n \in \omega$ there exist $\hat{p} \in \mathbb{P}^\alpha$ compatible with p, $|\hat{p}| < \beta + 1$, and there exists $s \in 2^n$ such that $\hat{p} \Vdash$ "$x_C \restriction n = s$". So given p and n find

$r \geq p$ and $s \in 2^n$ such that $r \Vdash$ "$x_C \upharpoonright n = s$". We have just shown there exists $\hat{r}$ compatible with r with $|\hat{r}| < \beta + 1$ and $\hat{r} \Vdash$ "$[s] \cap C \neq \emptyset$". Let G be $\mathbb{P}^\alpha$-generic containing r and $\hat{r}$. For each $t \in 2^{m+1}$ with $m + 1 \leq n$ and for all $k < m$ $(t(k) = s(k)$ and $t(m) < s(m))$ choose $q_t \in G$ with $|q_t| < \beta$ and $q_t \Vdash$ "$[t] \cap C = \emptyset$". (There are only finitely many such t). Choose $q \in G$ with $|q| < \beta + 1$, $q \geq \hat{r}$, and $q \geq q_t$ for each such t. (q exists by Lemma 23). Then $q \Vdash$ "$x_C \upharpoonright n = s$". $\blacksquare$

For p and q compatible define $p \cup q \Vdash$ "θ" to mean that for every r, if $r \geq p$ and $r \geq q$ then $r \Vdash$ "θ". For τ a $\mathbb{P}^\alpha$ term for an element of 2^ω and $p \in \mathbb{P}^\alpha$, define $C(\tau,p)$ a $\mathbb{P}^\alpha$ term so that for any G which is $\mathbb{P}^\alpha$-generic (it need not contain p) $C^G(\tau,p) = \bigcap\{D_{\hat{\tau}}$: there exist $q \in G$, $|q| < \beta$, $|\hat{\tau}|(q) < \beta$, $q \Vdash$ "$\hat{\tau} \in 2^\omega$", p and q are compatible, and $p \cup q \Vdash$ "$\tau \in D_{\hat{\tau}}$"$\}$. D is a universal Π^0_1 subset of $2^\omega \times 2^\omega$ $(\forall K \in \underset{\sim}{\Pi}^0_1 \exists x \in 2^\omega\ K = D_x = \{y: (x,y) \in D\})$.

<u>Claim</u> 2. Let $\hat{p}$ be given by Lemma 25 for $p \in \mathbb{P}^\alpha$ (i.e. for all $q \in \mathbb{P}^\alpha$ if $|q| < \beta$, then if q and $\hat{p}$ are compatible then q and p are compatible). Then $\hat{p}$ and

$C(\tau,p)$ satisfy the hypothesis of Claim 1 for p_0 and $\dot{C}$.

Proof.

Suppose $M[G] \models$ "$[s] \cap C(\tau,p) = \emptyset$". By compactness there exists $n < \omega$, $q_i \in G$, τ_i for $i < n$ with $|q_i| < \beta$ and $|\tau_i|(q_i) < \beta$ so that $p \cup q_i \Vdash$ "$\tau \in D_{\tau_i}$" and $M[G] \models$ "$\bigcap\{D_{\tau_i} : i < n\} \cap [s] = \emptyset$". Let $\hat{\tau}$ be a term for an element of 2^ω so that $D_{\hat{\tau}} = \bigcap\{D_{\tau_i} : i < n\}$ and $q \in G$ with $q \geq q_i$ for $i < n$ and $|q| < \beta$. ($\hat{\tau}$ can be chosen so that $|\hat{\tau}|(q) < \beta$ assuming some nice properties of D). Since q and $\hat{p}$ are compatible, q and p are compatible and $q \cup p \Vdash$ "$\tau \in D_{\hat{\tau}}$". Since $M[G] \models$ "$D_{\hat{\tau}} \cap [s] = \emptyset$" by compactness there exists $m \in \omega$ so that if $t = \hat{\tau}^G \restriction m$ then for every $x \supseteq t$, $x \in 2^\omega$ $D_x \cap [s] = \emptyset$. Since $|\hat{\tau}|(q) < \beta$ there exists $\hat{q} \geq q$ an element of $G, |\hat{q}| < \beta$, and $\hat{q} \Vdash$ "$\hat{\tau}\restriction m = t$"; hence $\hat{q} \Vdash$ "$[s] \cap C(\tau,p) = \emptyset$". The fact that $\hat{p} \Vdash$ "$C(\tau,p) \neq \emptyset$" follows from this since if not there exists q compatible with $\hat{p}$, $|q| < \beta$, and $q \Vdash$ "$[\emptyset] \cap C(\tau,p) = \emptyset$". But then q is compatible with p contradiction.

We now return to the proof of the $\alpha + 1$ step of Lemma 25.

Assume $p \in \mathbb{P}^{\alpha+1}$ is given with the following property:

(*) there exists $s_\tau \in 2^{<\omega}$ for each τ such that there exist s with $(s,\tau) \in P(\alpha)$ and $|s| \geq 1$. And these s_τ have the property that $\emptyset \Vdash$ "$s_\tau \subseteq \tau$" and whenever (s,τ), $(s^\frown i,\hat{\tau}) \in p(\alpha)$ (or $(s\ i,t) \in p(\alpha)$ where $t \in 2^{<\omega}$) s_τ and $s_{\hat{\tau}}$ are incompatible (or s_τ and t are incompatible).

The set of p's with this property is dense in $\mathbb{P}^{\alpha+1}$ so it is enough to prove the Lemma 25 for them. Let (s_i,τ_i) for $i < n$ be all $(s,\tau) \in p(\alpha)$ with $|s| \geq 1$ and let $\overline{\tau} = (\tau_0,\tau_1,\ldots,\tau_{n-1})$ (where $(\ ,\ ,\ldots,\ ): (2^\omega)^n \to 2^\omega$ is some recursive coding). Let $\hat{p}\restriction_\alpha$ be as given from Lemma 25 for p_α. let $\overline{\tau}^\ell$ be the lexicographical least element of $C(\overline{\tau},p\restriction_\alpha)$. By Claim 1 and 2 $|\overline{\tau}^\ell|(\hat{p}\restriction_\alpha) < \beta + 1$. Now let $\hat{p}(\alpha) = \{(s,t) \in p(\alpha): |s| = 0\} \cup \{(s_i,\tau_i^\ell): i < n\}$ $(\overline{\tau}^\ell = (\tau_0^\ell,\ldots,\tau_{n-1}^\ell))$. Since $\emptyset \Vdash$ "$C(\overline{\tau},p_\alpha)$ is included in $\prod_{i<n}[s_{\tau_i}]$", $\hat{p}$ is a condition, $\hat{p}$ and p are compatible, also $|\hat{p}| < \beta + 1$. Now suppose $q \in \mathbb{P}^{\alpha+1}$ compatible with $\hat{p}$, $|q| < \beta$, and q and p are not compatible. Let G be $\mathbb{P}^\alpha$-generic with $\hat{p}\restriction_\alpha$ and $q\restriction_\alpha$ elements of G and $M[G] \models$ "$\hat{p}(\alpha)$ and $q(\alpha)$ are compatible". If we think of $p(\alpha)$ as a statement about $\overline{\tau}$ i.e. $p(\alpha)(\overline{\tau})$ then $\hat{p}(\alpha) = p(\alpha)(\overline{\tau}^\ell)$. Since p and q are incompatible but p_α and q_α are compatible $(p\restriction_\alpha \cup q\restriction_\alpha) \Vdash$ "$p(\alpha)$ and

$q(\alpha)$ are incompatible". $D(\bar{\tau}) \equiv$ "$p(\alpha)(\bar{\tau})$ and $q(\alpha)$ are incompatible" is a $\underset{\sim}{\Pi}^0_1$ statement with parameters from $q(\alpha)$ about $\bar{\tau}$. Thus we conclude that
$M[G] \models$ "$p(\alpha)(\bar{\tau}^{\ell})$ and $q(\alpha)$ are incompatible", contradiction. This concludes the proof of Lemma 25.

From now on let $\mathbb{P} = \mathbb{P}^{\omega_2}$.

<u>Lemma</u> 26. Suppose $|\tau| = 0$, $B(v)$ is a $\underset{\sim}{\Sigma}^0_\beta$ predicate, $\beta \geq 1$, with parameters from M, and $p \in \mathbb{P}$ is such that $p \Vdash$ "$B(\tau)$"; then there exists $q \in \mathbb{P}$ compatible with p, $|q|(H) < \beta$ and $q \Vdash$ "$B(\tau)$".

<u>Proof</u>.

The proof is by induction on β.

<u>Case</u> 1. $\beta = 1$.

Suppose $p \Vdash$ "$\exists n R(x \restriction n, \tau \restriction n)$" for R recursive and $x \in M$. Let G be $\mathbb{P}$-generic with $p \in G$. Choose $n \in \omega$ and $s \in 2^n$ so that $M[G] \models$ "$R(x \restriction n, \tau \restriction n)$ and $\tau \restriction n = s$". Choose $q \in G$ with $|q| = 0$ and $q \Vdash$ "$\tau \restriction n = s$".

<u>Case</u> 2. β is a limit ordinal.

If $p \Vdash$ "$\exists n\, B(n, \tau)$" then $\exists\, \hat{p} \geq p$ $\hat{p} \Vdash$ "$B(n_0, \tau)$" and $B(n_0, v)$ $\underset{\sim}{\Sigma}^0_\gamma$ for $\gamma < \beta$, so apply induction hypothesis to $\hat{p}$.

<u>Case</u> 3. $\beta + 1$.

Suppose $p \Vdash$ "$\exists n\, B(n, \tau)$" where $B(n, v)$ is

$\underset{\sim}{\Pi}^0_\beta$ with parameters from M. Choose $r \geq p$ and $n_0 \in \omega$ so that $r \Vdash$ "$B(n_0,\tau)$". By Lemma 25 there is q compatible with r, $|q| < \beta + 1$, and for every s, $|s| < \beta$, if q and s are compatible, then r and s are compatible. $q \Vdash$ "$B(n_0,\tau)$" because if not then there is $q' \geq q$ such that $q' \Vdash$ "$\neg B(n_0,\tau)$", and so by induction there is s with $|s| < \beta$ compatible with q' and $s \Vdash$ "$\neg B(n_0,\tau)$"; but then s is compatible with r, contradiction. ■

Now let us prove the first part of Theorem 22. Let G be $\mathbb{P}$-generic over M. We claim $M[G] \models$ "for every $X \subseteq 2^\omega$ and $\alpha < \omega_1$ if $|X| = \omega_1$ then $\text{ord}(X) \geq \alpha + 1$.". Bur since any such X is in some $M[G_\beta]$ for $\beta < \omega_2$, we may as well $X \in M$, $\alpha_0 = \alpha + 1$, and we must show $M[G] \models$ "$\text{ord}(X) \geq \alpha + 1$". Let $G_{(0)}$ be the $\underset{\sim}{\Pi}^0_\alpha$ set created by $G \cap \mathbb{P}_{\alpha_0}(\phi, 2^\omega \cap M)$. Suppose that $M[G] \models$ "there is K a $\underset{\sim}{\Sigma}^0_\beta$ set such that $K \cap X = G_{(0)} \cap X$.". Let τ be a term for the parameter of K. Choose $p \in G$ such that $p \Vdash$ "$\forall z \in X$ ($x \in K$ iff $z \in G_{(0)}$.". By Lemma 24 there exists H in M countable so that $|\tau|(H,\phi) = |p|(H) = 0$. Let $z \in X - H$. Define $\hat{p} \in \mathbb{P}$ by $\hat{p}(0) = p(0) \cup \{((0),z)\}$ and $\hat{p}(\alpha) = p(\alpha)$ for $\alpha > 0$. Since $\hat{p}$ says $z \in G_{(0)}$, $p \Vdash$ "$z \in K$". By Lemma 26 there exists q compatible with $\hat{p}$, $|q|(H) < \beta$, and $q \Vdash$ "$z \in K$". By Lemma

23 there exists $\hat{q}$ with $|\hat{q}|(H) < \beta$, $\hat{q} \geq q$, and $\hat{q} \geq p$. Since $|(0)|_{T_{\alpha_0}} = \alpha$, $((0),z) \notin \hat{q}(0)$, there exists $m \in \omega$ such that r defined by $r(0) = q(0) \cup \{((0,m),z)\}$ and $r(\alpha) = \hat{q}(\alpha)$ for $\alpha > 0$ is a condition. But this is a contradiction since $r \Vdash$ "$(z \in G_{(0)}$ iff $z \in K)$ and $z \in K$ and $z \notin G_{(0)}$".

Now we prove the second sentence of Theorem 22. Let $X = \bigcup\{X_\alpha : \beta_0 \leq \alpha < \omega_1$ and α a successor$\}$ where each X_α is a set of ω_1 product generic Cohen reals. Let $M_0 = M[X]$. Define in M_0 the partial order $\mathbb{P}^\gamma$ for $\gamma \leq \omega_2$ so that $\mathbb{P}^{\gamma+1} = \mathbb{P}^\gamma * Q_\gamma$ where Q_γ is a term denoting:

<u>Case</u> 1. $\mathbb{P}_{\beta_0}(\phi, M_0[G_\gamma] \cap 2^\omega)$ or

<u>Case</u> 2. $\mathbb{P}_\beta(Y_\gamma, X_\beta \cup F)$ where Y_γ is a Borel subset of X_β in $M_0[G_\gamma]$ and $F = \{x \in 2^\omega : x$ eventially zero$\}$.

Case 1 is done cofinally in ω_2 and Case 2 is done in such a way as to insure:

$M_0[G_{\omega_2}] \models$ "For every successor ordinal β with $\beta_0 \leq \beta < \omega_1$ and Y Borel in X_β there is a γ such that $Y = Y_\gamma$". First we show that essentially the same arguments as before show that $M_0[G_{\omega_2}] \models$ "For every $X \subseteq 2^\omega$ uncountable $\text{ord}(X) \geq \beta_0$". This will not use that the X_α are made up of Cohen reals, hence any of the intermediate models would serve as the ground model. So suppose Case 1 occurs on the first step, $Y \in M_0$ is uncountable, $\beta_0 = \gamma + 1$, and $M_0[G_{\omega_2}] \models$ "$Y \cap G_{(0)} = Y \cap J$ for some $J \in \underset{\sim}{\Sigma}{}^0_\gamma$". Given $L \subseteq \omega_2$ define $\mathbb{P}^\alpha_L$ as follows.

For $\alpha \in L$:

Case 1. $\mathbb{P}^{\alpha+1}_L = \mathbb{P}^\alpha_L * \mathbb{P}_{\beta_0}(\phi, M[G^L_\alpha] \cap 2^\omega)$ where G^L_α is $\mathbb{P}^\alpha_L$-generic over M_0.

Case 2. $\mathbb{P}^{\alpha+1}_L = \mathbb{P}^\alpha_L * \mathbb{P}_\beta(Y_\alpha - F, X_\beta \cup F)$ (where we assume L has the property that when Case 2 happens for $\alpha \in L$ then Y_α is a Borel subset of X_β coded by some term τ_α in $\mathbb{P}^\alpha_L$).

For $\alpha \notin L$:

$\mathbb{P}^{\alpha+1}_L = \mathbb{P}^\alpha_L * \mathbf{1}$ (singleton partial order).

Note that by using c.c.c. of $\mathbb{P}^{\omega_2}$ we can find $L \subseteq \omega_2$ countable, so that the Borel code for the above J is a $\mathbb{P}^{\omega_2}_L$ term and L has the property mentioned under Case 2. For α a limit $\mathbb{P}^\alpha_L$ is the direct limit of $(\mathbb{P}^\beta_L : \beta < \alpha)$.

Lemma 27. If $N \supseteq M$ is a forcing extension and G is $\mathbb{P}_\beta(\phi, N \cap 2^\omega)$ generic over N then $G \cap \mathbb{P}_\beta(\phi, M \cap 2^\omega)$ is $\mathbb{P}_\beta(\phi, M \cap 2^\omega)$ generic over M.

Proof.

It is enough to show that for any $\Delta \in M$ dense in

$\mathbb{P}_\beta(\phi, M \wedge 2^\omega)$, $\{p \in \mathbb{P}_\beta(\phi, N \wedge 2^\omega)$: $q \in \Delta, q \leq p\}$ is dense in $\mathbb{P}_\beta(\phi, N \wedge 2^\omega)$.

Let N be an extension of M via a partial order Q. Given $p \in \mathbb{P}_\beta(\phi, 2^\omega \wedge N)$ (a term in the forcing language of Q) suppose $\exists q \in Q$ $q \Vdash$ "$\forall r \in \Delta$ r and p are incompatible". View p as being coded up in some natural way by a single real in $2^\omega \wedge N$. Then we can find $\hat{p} \in \mathbb{P}_\beta(\phi, 2^\omega \wedge M)$ so that $\forall n < \omega$ $\exists \hat{q} \geq q$ $\hat{q} \Vdash$ "$p \restriction n = \hat{p} \restriction n$". Since Δ is dense $\exists r \in \Delta$ r and $\hat{p}$ are compatible. But compatibility is witnessed by $\hat{p} \restriction n$ for some $n < \omega$. Let $\hat{q} \geq q$ and $\hat{q} \Vdash$ "$p \restriction n = \hat{p} \restriction n$", then $\hat{q} \Vdash$ "p and r are compatible", contradiction.

<u>Lemma</u> 28. Suppose $\mathbb{P}_0, \mathbb{P}_1 \in M$ are partial orders and $\exists \tau$ a term in language of $\mathbb{P}_1$ such that $\forall G$ $\mathbb{P}_1$-generic over M, τ^G is $\mathbb{P}_0$-generic over M. Then $\forall G$ $\mathbb{P}_1$-generic over M, $M[G]$ is a forcing extension of $M[\tau^G]$.

<u>Proof</u>.

This is easier to prove using the cBa approach to forcing. Let $\mathbb{B}_i$ for $i = 0,1$ be the associated cBa to $\mathbb{P}_i$ for $i = 0,1$ and $\hat{\tau}$ a $\mathbb{B}_1$ term so that $\forall G$ $\mathbb{B}_1$-generic $\hat{\tau}^G$ is $\mathbb{B}_0$-generic. Define a map $j: \mathbb{B}_0 \to \mathbb{B}_1$ by $j(p) = [\![p \in \hat{\tau}]\!]_{\mathbb{B}_1}$. Then j is an isomorphism of $\mathbb{B}_0$ onto an M-complete subalgebra of $\mathbb{B}_1$. Otherwise suppose $\Gamma \subseteq \mathbb{B}_0 - \{\phi\}$, $\Gamma \in M$ and

$e = [\![\sum_{p \in \Gamma} p \in \hat{\tau}]\!]_{\mathbb{B}_1} > \sum_{p \in \Gamma} [\![p \in \hat{\tau}]\!]_{\mathbb{B}_1} = f$

Choose G $\mathbb{B}_1$-generic with $e - f \in G$. Then $\sum_{p \in \Gamma} p \in \hat{\tau}^G$ and $p \in \Gamma$ $p \notin \hat{\tau}^G$. But this means $\hat{\tau}^G$ is not $\mathbb{B}_0$-generic over M (see Lemma p. 35 Solovay [27]). But now by Lemma 5.2.4 of Solovay-Tennenbaum [26] we are done. ∎

Given any G $\mathbb{P}^{\omega_2}$-generic let G_L be the subset of $\mathbb{P}_L$ generated by the rank zero conditions in G. The two preceding lemmas enable us to prove:

<u>Lemma</u> 29. For any α if G_α is $\mathbb{P}^\alpha$-generic over M_0 then G_α^L is $\mathbb{P}_L^\alpha$-generic over M_0.

<u>Proof</u>.

This is proved by induction on α. For $\alpha + 1 \notin L$ it is immediate. For $\alpha+1 \in L$ Case 1 is handled by Lemma 27, Lemma 28, and the product lemma. Case 2 is easy as $\mathbb{P}_\beta(Y_\alpha - F, X_\beta \cup F)$ is the same partial order in either case. For α limit ordinal let $\Delta \subseteq \mathbb{P}_L^\alpha$ be dense, we show $\{q \in \mathbb{P}^\alpha : \exists p \in \Delta \ \ p \leq q\}$ is dense in $\mathbb{P}^\alpha$. If $q \in \mathbb{P}^\alpha$ then $q \in \mathbb{P}^\beta$ for some $\beta < \alpha$. Let $\Delta_\beta = \{p \restriction \beta : p \in \Delta\}$, then Δ_β is dense in $\mathbb{P}_L^\beta$. Hence if G_α is $\mathbb{P}^\alpha$- generic with $q \in G_\alpha$ then since G_β^L is $\mathbb{P}_L^\beta$-generic it meets Δ_β --say at $p \restriction \beta$. But then q and p are compatible. ∎

Define for $H \subseteq 2^{\omega}$ $|p|(H)$, $|\tau|(H,p)$ for $p \in \mathbb{P}^{\alpha}_{L}$ and τ a $\mathbb{P}^{\alpha}_{L}$-term for a subset of ω by induction on α.

Case 1. $\mathbb{P}^{\alpha+1} = \mathbb{P}^{\alpha} * \mathbb{P}_{\beta_0}(\phi, M[G^{\alpha}_{L}] \cap 2^{\omega})$
$|p|(H) = \max\{|p \upharpoonright \gamma|(H), |p(\gamma)|(H, p \upharpoonright \gamma)\}$ (same as before).
Case 2. $\mathbb{P}^{\alpha+1} = \mathbb{P}^{\alpha} * \mathbb{P}_{\beta}(Y_{\alpha} - F, X_{\alpha} \cup F)$
$|p|(H) = \max\{|p \upharpoonright \alpha|(H), |s|_{T_\phi} : x \notin H \ \langle s,x \rangle \in p(\alpha)\}$
$|\tau|(H,p)$ is defined as it was just before Lemma 23. Lemma 23 is easily proven since in Case 2 we have a lattice. Lemma 24 is also easily proven if in addition H is taken with the property that $\forall x \in H \ \ \forall \alpha \in L$ $\{p: |p|(H) = 0\}$ decides "$x \in Y_{\alpha}$" whenever Case 2 happens at stage α. Lemma 25 can be proven for $\beta < \beta_0$ by the same argument in Case 1 and by the argument of Theorem 34 in Case 2. Lemma 26 follows and so does the claim that $M_0[G_{\omega_2}] \models$ "$K \subseteq \{\beta: \beta_0 \le \beta < \omega_1\}$".

Next we show $M_0[G_{\omega_2}] \models$ "$\mathrm{ord}(X_{\beta}) = \beta$ for each β successor $\beta_0 \le \beta < \omega_1$". If not then again we can reduce to some $L \subseteq \aleph_2$ countable; and since each X_{α} is present in M_0, we can relabel L so that for some $\hat{\beta} < \omega_1$ and β_1 with $\beta_0 \le \beta_1 < \omega_1$, $M_0[G_{\hat{\beta}}] \models$ "$\mathrm{ord}(X_{\beta_1}) < \beta_1$" for $G_{\hat{\beta}}$ $\mathbb{P}^{\beta}$-generic over M_0, and on some step before $\hat{\beta}$ we force with $\mathbb{P}_{\beta_1}(\phi, X_{\beta_1} \cup F)$. Suppose $X = \{x_{\alpha}: \alpha < \omega_1\}$ and $M_0 = M[\{\langle \alpha, x_{\alpha} \rangle: \alpha < \omega_1\}]$. Given $H \subseteq \omega_1$, $H \in M$ let

$\hat{H} = \{\langle\alpha, x_\alpha\rangle : \alpha \in H\}$. Define $\mathbb{P}_H^\alpha \in M[\hat{H}]$ for each $\alpha < \hat{\beta}$.

Case 1 $\mathbb{P}_H^{\alpha+1} = \mathbb{P}_H^\alpha * \mathbb{P}_{\beta_0}(\phi, M[G_\alpha^H] \cap 2^\omega)$.

Case 2 $\mathbb{P}_H^{\alpha+1} = \mathbb{P}_H^\alpha * \mathbb{P}_\beta((Y_\beta - F) \cap \hat{H}, (X_\beta \cap \hat{H}) \cup F)$

(assuming Y_α is a Borel subset of X_β given by the term τ_α in forcing language of $\mathbb{P}_H^\alpha$).

Lemma 30. For any $\alpha \leq \hat{\beta}$ if G^α is $\mathbb{P}^\alpha$-generic over M_0 then G_H^α is $\mathbb{P}_H^\alpha$-generic over $M[\hat{H}]$.

Proof.

The proof is like Lemma 29 except on $\alpha + 1$ under Case 2.

$\mathbb{P}_1 = \mathbb{P}_\beta(Y_\alpha - F, X_\beta \cup F)$ in $M[X][G^\alpha] = M_1$

$\mathbb{P}_2 = \mathbb{P}_\beta((Y_\alpha - F) \cap \hat{H}, (X_\beta \cap \hat{H}) \cup F)$ in $M[\hat{H}][G_H^\alpha] = M_2$.

Again suppose $\Delta \in M_2$ is dense in $\mathbb{P}_2$, we show $\{p \in \mathbb{P}_1 : \exists q \in \Delta \ \ q \leq p\}$ is dense in $\mathbb{P}_1$. Given $p \in \mathbb{P}_1$ let $p = r \cup \{\langle s_n, x_n\rangle : n < N\}$ where $x_n \in X_\alpha - \hat{H}$, $N<\omega$, and $r \in \mathbb{P}_2$. Let Q_N be the partial order for adding N Cohen/reals. By the product lemma $\{x_n : n < N\}$ is Q_N-generic over M_2, and also $p \in M_2[\{x_n : n < N\}]$. Hence if $\forall q \in \Delta$ p and q are incompatible in

$\mathbb{P}_3 = \mathbb{P}_\beta((Y_\alpha - F) \cap (H \cup \{x_n : n < N\}, (X_\beta \cap H \cup \{x_n : n < N\}) \cup F)$

then $\exists \hat{p} \in Q_N$ $\hat{p} \Vdash$ "$\forall q \in \Delta$ p and q are incompatible in $\mathbb{P}_3$". Choose $y_n \in F$ for $n < N$ so that $p_0 = r \cup \{\langle s_n, y_n\rangle : n < N\} \in \mathbb{P}_2$ and $\forall m < \omega \exists \hat{p}' \geq \hat{p}$ $\forall n < N$ $\hat{p}' \Vdash$ "$y_n \restriction m = x_n \restriction m$". Since $\exists q \in \Delta$ p_0 and q are compatible, then as before p and

q can be forced compatible by an extension of $\hat{p}$. So p and q are compatible in $\mathbb{P}_3$ and hence in $\mathbb{P}_1$. ∎

Lemma 31. Given $\hat{\tau}$ a term in forcing language of $\mathbb{P}_H^{\hat{\beta}}$ if $p \in \mathbb{P}^{\hat{\beta}}$ $p \Vdash_{\mathbb{P}^{\hat{\beta}}}$ "$B(\tau)$" where $B(\mathbf{v})$ is a Σ_1^1 predicate with parameters in $M[\hat{H}]$ then $\exists\, q \in \mathbb{P}_H^{\hat{\beta}}$ compatible with p such that $q \Vdash_{\mathbb{P}^{\hat{\beta}}}$ "$B(\tau)$".

Proof.

Let G be $\mathbb{P}^{\beta}$-generic over M_0 with $p \in G$. Then by Lemma 9 G_H is $\mathbb{P}_H^{\hat{\beta}}$-generic over $M[\hat{H}]$. Since Σ_1^1 sentences are absolute and $M_0[G] \models$ "$B(\tau)$" we have $M[\hat{H}][G_H] \models$ "$B(\tau)$". So $\exists\, q \in G_H$ $q \Vdash_{\mathbb{P}_H^{\hat{\beta}}}$ "$B(\tau)$". But for any G $\mathbb{P}^{\hat{\beta}}$-generic containing q, $M[H][G_H] \models$ "$B(\tau)$" whence by absoluteness $M_0[G] \models$ "$B(\tau)$". We conclude $q \Vdash_{\mathbb{P}^{\hat{\beta}}}$ "$B(\tau)$". ∎

Lemma 32. Given $H = X - \{z\}$ where $z \in X_{\alpha+1}$, $\gamma \leq \hat{\beta}$ $1 \leq \beta < \alpha$, $p \in \mathbb{P}^{\gamma}$ then $\exists\, \hat{p} \in \mathbb{P}^{\gamma}, |\hat{p}|(M[\hat{H}] \cap 2^{\omega}) < \beta + 1$ compatible with p and $\forall\, q \in \mathbb{P}^{\gamma}$ if $|q|(M[\hat{H}] \cap 2^{\omega}) < \beta$, then ($\hat{p}$,$q$ compatible $\longrightarrow$ p,q compatible).

Proof.

This is proved by induction on γ. For γ limit it is easy, also for $\gamma + 1$ in which Case 1 occurs the proof is the same as Lemma 25. So we only have to do $\gamma + 1$ in Case 2.

$p \in \mathbb{P}^{\gamma} * \mathbb{P}_{\beta_1}(Y_{\gamma} - F, X_{\beta_1} \cup F)$. Extend $p(\gamma)$ if necessary so that $\forall \langle s,x\rangle \in p(\gamma)$ $\forall i < \omega$ if $|s| = \lambda$ infinite limit $|s^\frown i| \leq \beta + 1 < \lambda$ then $\exists j < \omega$ $\langle s^\frown i^\frown j, x\rangle \in p(\gamma)$. Let $\hat{p}(\gamma) = \{\langle s,x\rangle \in p(\gamma): |s| < \beta + 1$ or $x \neq z\}$.

If $\hat{p} = \langle \hat{p}\restriction\gamma, \hat{p}(\gamma)\rangle$ were a condition then just as in Lemma 8, $\hat{p}$ would have the required properties. To be a condition we need to know that whenever $\langle\langle n\rangle, x\rangle \in \hat{p}(\gamma)$ $\hat{p}\restriction\gamma \Vdash$ "$x \notin (Y_{\gamma} - F)$".

Note that none of these x's are equal to z because $z \in X_{\alpha+1}$ so $\langle\langle n\rangle, z\rangle \in p(\gamma) \to |\langle n\rangle| = \alpha \geq \beta + 1$ so $\langle\langle n\rangle, z\rangle \notin \hat{p}(\gamma)$. Let G be $\mathbb{P}^{\gamma}$-generic containing $p\restriction\gamma$, and $\hat{p}\restriction\gamma$. By Lemma 31 $\exists q \in \mathbb{P}^{\gamma}_{H} \cap G$. (So $|q|(M[H] \cap 2^{\omega}) = 0$) $q \Vdash$ "$x \notin Y_{\gamma} - F$" $\forall x \forall n \langle\langle n\rangle, x\rangle \in \hat{p}(\gamma)$. By Lemma 23, $\exists p_0 \geq q, \hat{p}\restriction\gamma$ so that $|p_0|(M[H] \cap 2^{\omega}) < \beta + 1$. So $\langle p_0, \hat{p}(\gamma)\rangle$ works. ■

Immediate from Lemma 32 we get that: If J is any $\Sigma^0_{\alpha+1}$ predicate with parameters ($H = X - \{z\}$, $z \in X_{\alpha+1}$, and τ is in the forcing language of $\mathbb{P}_H$) then $\forall p \in \mathbb{P}$ if $p \Vdash$ "$z \in J$" then $\exists q \in \mathbb{P}$ $|q|(M[H] \cap 2^{\omega}) < \beta$, q and p are compatible, and $q \Vdash$ "$z \in J$". So we get our result $\mathrm{ord}(X_{\alpha+1}) = \alpha + 1$ in $M_0[G_{\omega_2}]$. ■

<u>Remark</u>: Assuming large amounts of the axiom of determinacy and therefore getting more absoluteness in inner models

(see [7]) it is easy to produce an inner model N such that $N \models$ "For every $\alpha < \omega_1$ there exist $X \subseteq 2^\omega$ such that $\mathrm{ord}(X) = \alpha$ and for every $n < \omega$ and A $\underset{\sim}{\Pi}^1_n$ A X is Borel in X". Similar improvements for Theorem 43 are possible.

§4. The σ-algebra generated by the abstract rectangles

For any cardinal λ let $R^{\lambda} = \{A \times B: A,B \subseteq \lambda\}$. If $R^{\lambda}_{\omega_1}$ (the σ-algebra generated by R^{λ}) is the set of all subsets of $\lambda \times \lambda$, then $\lambda \leq |2^{\omega}|$ (see [9], [21]).

Theorem 33. If $\alpha_0 < \omega_1$ and there is an $X \subseteq \omega^{\omega}$ such that $|X| = \kappa \geq \omega$ and every subset of X of cardinality less than κ is $\underset{\sim}{\Pi}^0_{\alpha_0}$ in X, then $R^{\kappa}_{\alpha_0} = P(\kappa \times \kappa)$. The same is true if every subset of X of cardinality less than κ is $\underset{\sim}{\Sigma}^0_{\alpha_0}$ in X.

Proof.

Consider $A \subseteq \kappa \times \kappa$ and suppose $(\alpha,\beta) \in A$ implies $\alpha \leq \beta$. It is enough to showsuch sets are in $R^{\kappa}_{\alpha_0}$ since every subset of $\kappa \times \kappa$ can be written as the union of a set above the diagonal and a set below the diagonal. Let T be a normal α_0 tree and $T^* = \{s \in T: |s|_T = 0\}$. For any $y: T^* \to \omega^{<\omega}$ define G^s_y as follows. If $s \in T^*$, then $G^s_y = [y(s)]$, otherwise $G^s_y = \bigcap \{\omega^{\omega} - G^{s\frown n}_y: n < \omega\}$. Let $X = \{x_{\alpha}: \alpha < \kappa\}$ and for each $\beta < \kappa$ choose β so that for all α $((\alpha,\beta) \in A$ iff $x_{\alpha} \in G^{\phi}_{y_{\beta}})$. For $s \in T$ define $B_s \subseteq \kappa \times \kappa$ as follows. If $s \in T^*$, then $B_s = \bigcup \{\{\alpha: t \subseteq x_{\alpha}\} \times \{\beta: y_{\beta}(s) = t\}: t \in \omega^{<\omega}\}$, otherwise $B_s = \bigcap \{(\kappa \times \kappa) - B_{s\frown n}: n < \omega\}$. Clearly $B_{\phi} = A$ and

B_ϕ is "$\underset{\sim}{\Pi}^0_{\alpha_0}$" in R^κ, and so every subset of $\kappa \times \kappa$ is "$\underset{\sim}{\Pi}^0_{\alpha_0}$" in R^κ. Note that $(\kappa \times \kappa) - (A \times B) = ((\kappa - A) \times \kappa) \cup (\kappa \times (\kappa - B))$ and thus if α_0 is even (odd), then $R^\kappa_{\alpha_0}$ is the class of sets "$\underset{\sim}{\Pi}^0_{\alpha_0}$" ("$\Sigma^0_{\alpha_0}$") in R^κ. By passing to complements if necessary we have that $R^\kappa_{\alpha_0} = P(\kappa \times \kappa)$. The second sentence of the theorem is proved similarly. ∎

Corollary. (Kunen [9]; Rao [21]) If there is an $X \subseteq 2^\omega$ such that $|X| = \omega_1$ then $R^{\omega_1}_2 = P(\omega_1 \times \omega_1)$.

The converse of this corollary is also true. Suppose $R \subseteq P(\omega_1)$ is a countable field of sets and $\{(\alpha,\beta): \alpha < \beta < \omega_1\} \varepsilon \{A \times B: A,B \varepsilon R\}_{\omega_1}$. Since this set is antisymetric we conclude that the map given in Proposition 2 is a 1-1 embedding of ω_1 into 2^ω.

Corollary. (Kunen [9]; Silver) (MA) if $\kappa = |2^\omega|$ then $R^\kappa_2 = P(\kappa \times \kappa)$.

Proof.

If X is I-Luzin where I is the ideal of meager sets, then every subset of X of smaller cardinality is $\underset{\sim}{\Sigma}^0_2$ in X (see proof of Theorem 17). ∎

For any $\alpha \leq \omega_1$ $X \subseteq \omega^\omega$ is a Q_α set iff $\text{ord}(X) = \alpha$ and every subset of X is Borel in X.

<u>Theorem</u> 34. If M is countable transitive model of ZFC, $1 \leq \alpha_0 < \omega_1^M$, and $X = M \cap \omega^\omega$, then there is a Cohen extension $M[G]$ such that $M[G] \models$ "X is a Q_{α_0+1} set".

<u>Remark</u>: This shows that the Baire order of the construcible reals can be any countable successor ordinal greater than one. In fact the argument shows that in $M[G]$ for any uncountable $Y \subseteq 2^\omega$ with $Y \in M$ Y is a Q_{α_0+1} set. Thus, for example, if M models $V = L$, then in $M[G]$ there are Π^1_1 Q_{α_0+1} sets. In Theorem 55 we show that it is consistent with ZFC that for every $\alpha < \omega_1$ there is a Q_α set (in that model the continuum is $\aleph_{\omega_1+1}$).

<u>The proof of Theorem 34</u>:

$M[G]$ is gotten by iterated $\underset{\sim}{\Pi}^0_{\alpha_0+1}$-forcing. Let $\kappa = |2^{2^\omega}|$. Suppose we are given $\mathbb{P}^\alpha$ for some $\alpha < \kappa$ and Y_α a term in the forcing language of $\mathbb{P}^\alpha$ for a subset of X ($\phi \Vdash$ "$Y_\alpha \subseteq X$"), then let $\mathbb{P}^{\alpha+1} = \mathbb{P}^\alpha * \mathbb{P}_{\alpha_0+1}(Y_\alpha, X)$. At limit ordinals take direct limits. $\mathbb{P}^\kappa$ may be viewed as a sub-lower lattice of $\sum_\kappa \mathbb{P}_{\alpha_0+1}(\phi, X)$. We may assume that for every set $B \subseteq X$ in $M[G]$ (G $\mathbb{P}^{\kappa}$-generic over M) there exists α such that $Y_{\alpha} = B$. This is because $\mathbb{P}^{\kappa}$ has c.c.c.. It follows from Corollary 6 that $M[G] \models$ "$\mathrm{ord}(X) \leq \alpha_0 + 1$ and every subset of X is Borel in X".

We assume $\mathbb{P}^0 = \mathbb{P}_{\alpha_0+1}(\emptyset, X)$. Let $G_{(0)}$ be one of the $\underset{\sim}{\Pi}^0_{\alpha_0}$ set determined by $G \cap \mathbb{P}^0$. We want to show that $M[G] \models$ "For every K in $\underset{\sim}{\Sigma}^0_{\alpha_0}$, $K \cap X \neq G_{(0)} \cap X$". To this end we make the following definition: For $H \subseteq \omega^{\omega}$, $|p|(H) = \max\{|s|$: there exists $x \notin H$ $(s,x) \in p(\alpha)$ for some $\alpha < \kappa\}$ Let $\mathrm{supp}(p) = \{\alpha < \kappa\colon p(\alpha) \neq \emptyset\}$. Given τ a term in the forcing language of $\mathbb{P}^{\kappa}$ denoting a subset of ω, we can find H included in ω^{ω} and K included in κ with the following properties:

(a) H and K are countable.

(b) for each $n \in \omega$ $\{p \in \mathbb{P}^{\kappa}\colon \mathrm{supp}(p) \subseteq K, |p|(H) = 0\}$, decides "$n \in \tau$".

(c) $\forall x \in H$ $\forall \alpha \in K$ $\{p \in \mathbb{P}^{\kappa}\colon \mathrm{supp}(p) \subseteq K, |p|(H) = 0\}$ decides "$x \in Y_{\alpha}$".

H and K can be found by repeatedly using the c.c.c. of $\mathbb{P}^{\kappa}$.

<u>Lemma</u> 35. If H and K have property (c) then for any $p \in \mathbb{P}^{\kappa}$ and β with $1 \leq \beta < \alpha_0$, there exists $\hat{p} \in \mathbb{P}^{\kappa}$ compatible with p, $|\hat{p}|(H) < \beta + 1$, $\mathrm{supp}(\hat{p}) \subseteq K$, and for

any $q \in \mathbb{P}^{\kappa}$ if $|q|(H) < \beta$ and $\text{supp}(q) \subseteq K$ then [if $\hat{p}$ and q are compatible, then p and q are compatible].

Proof.

The proof of this is like Lemma 8.

Let G be P^{κ}-generic over M with $p \in G$. Choose $\Gamma \subseteq G$ finite with the properties:

(1) $\forall q \in \Gamma(|q|(H) = 0$ and $\text{supp}(q) \subseteq K)$.

(2) If $((n),x) \in p(\alpha)$ for some $n < \omega$, $\alpha \in K$, and $x \in H$ (so $p \upharpoonright \alpha \Vdash$ "$x \notin Y_{\alpha}$"), then there is $q \in \Gamma \cap \mathbb{P}^{\alpha}$ such that $q \Vdash$ "$x \notin Y_{\alpha}$".

(3) If $(s,x) \in p(\alpha)$, $\alpha \in K$, and $|s| = \lambda$ is an infinite limit ordinal, and $|s^\frown i| \leq \beta + 1 < \lambda$ then there is a $j \in \omega$ such that $\{(s^\frown i^\frown j,x)\} \in \Gamma$.

Now let $\hat{p} \in \mathbb{P}^{\kappa}$ be defined by

$\hat{p}(\alpha) = \bigcup\{r(\alpha): r \in \Gamma\} \cup \{(s,x) \in p(\alpha): |s| < \beta + 1$ or $x \in H\}$ when $\alpha \in K$ and $\hat{p}(\alpha) = \phi$ for $\alpha \notin K$.

Note if $((n),x) \in \hat{p}(\alpha)$ then $x \in H$ since $|(n)| = \alpha_0 \geq \beta + 1$. By choice of Γ $\hat{p}$ is a condition and also $|\hat{p}|(H) < \beta + 1$ and is compatible with

p since $\hat{p}$, $p \in G$. It is easily checked as in Lemma 8 that $\hat{p}$ has the required property.

Lemma 36. Let H and K have properties (b) and (c) for τ. Let $B(v)$ be a $\underset{\sim}{\Sigma}^0_\beta$ $(1 \leq \beta \leq \alpha_0)$ predicate with parameters from M and $p \in \mathbb{P}^\alpha$ such that $p \Vdash$ "$B(\tau)$". Then there exists $q \in \mathbb{P}^\alpha$ compatible with p, $|q|(H) < \beta$, $q \Vdash$ "$B(\tau)$", and $\mathrm{supp}(q) \subseteq K$.

Proof.

The proof is by induction on β.

$\beta = 1$:

$p \Vdash$ "$\exists n\, R(n, \tau \upharpoonright n, x \upharpoonright n)$", $x \in M$, and R primitive recursive. Let G be $\mathbb{P}$-generic over M with $p \in G$. There exist $n \in \omega$ and $s \in 2^n$ such that $M[G] \models$ "$R(n, \tau \upharpoonright n, x \upharpoonright n)$ and $\tau \upharpoonright n = s$". By property (b) there exists $q \in G$ such that $q \Vdash$ "$\tau \upharpoonright n = s$", $\mathrm{supp}(q) \subseteq K$, and $|q|(H) = 0$. q does it.

β limit:

$p \Vdash$ "$\exists n\, B_n(\tau)$", $B_n \in \underset{\sim}{\Sigma}^0_{\beta_n}$, $\beta_n < \beta$. Choose $r \geq p$ such that $r \Vdash$ "$B_n(\tau)$" for some n. By induction there exist q such that $q \Vdash$ "$B_n(\tau)$", q is compatible with r (and hence with p), and $|q|(H) < \beta$, $\mathrm{supp}(q) \subseteq K$. q does it.

$\beta + 1$:

If $p \Vdash$ "$\exists n\, B_n(\tau)$" we could extend p to force $B_n(\tau)$ for some particular n. So we may as well assume $p \Vdash$ "$B(\tau)$" where $B(v)$ is $\underset{\sim}{\Pi}^0_\beta$ with parameter in M. Since $1 \leq \beta < \alpha_0$ by

Lemma 35 there is $\hat{p}$ compatible with p, $|\hat{p}|(H) < \beta + 1$, etc. Then $\hat{p} \Vdash$ "$B(\tau)$" because otherwise there is $p_0 \geq \hat{p}$ such that $p_0 \Vdash$ "$\neg B(\tau)$", and so by induction there is q compatible with p_0 (hence with $\hat{p}$) $|q|(H) < \beta$, $\operatorname{supp}(q) \subseteq K$, and $q \Vdash$ "$\neg B(\tau)$". By our assumption on $\hat{p}$, since $\hat{p}$ and q are compatible, p and q are compatible, but $p \Vdash$ "$B(\tau)$". ∎

We now use Lemma 36 to show that for any G $\mathbb{P}^{\kappa}$-generic over M, $M[G] \models$ "For every L a $\underset{\sim}{\Sigma}^0_{\alpha_0}$ set $(L \cap X \neq G_{(0)} \cap X)$" where $G_{(0)}$ is one of the $\underset{\sim}{\Pi}^0_{\alpha_0}$ sets determined by $G \cap \mathbb{P}_{\alpha_0+1}(\phi, X)$. Suppose not; then let τ be a term in forcing language of $\mathbb{P}^{\kappa}$, L a $\underset{\sim}{\Sigma}^0_{\alpha_0}$ set with parameter τ, and $p \in G$ such that $p \Vdash$ "for every $x \in X$ $x \in L$ iff $x \in G_{(0)}$". Choose H and K with properties (a), (b), and (c) with respect to τ and also so that $\operatorname{supp}(p) \subseteq K$ and $|p|(H) = 0$. Since H is countable there exists $x \in X - H$. Let $r = p \cup \{(0,((0),x))\}$ (so $r \Vdash x \in G_{(0)}$). Since $r \Vdash$ "$x \in L$", by Lemma 36 there exists q compatible with r, $|q|(H) < \alpha_0$, and $q \Vdash$ "$x \in L$". Since $|q|(H) < \alpha_0$, $((0),x) \notin q(0)$. Let $\hat{q}$ be defined by:

$$\hat{q}(\alpha) = \begin{cases} p(\alpha) \cup q(\alpha) & \text{if } \alpha > 0. \\ p(0) \cup q(0) \cup \{((0,m),x)\} & \text{otherwise (m sufficiently large so that } \hat{q}(0) \text{ is a condition).} \end{cases}$$

$\hat{q} \Vdash$ "$x \in L$ and $x \notin G_{(0)}$ and ($x \in L$ iff $x \in G_{(0)}$))". This a contradiction and concludes the proof of Theorem 34. ■

<u>Theorem</u> 37. For any α_0 a successor ordinal such that $2 \leq \alpha_0 < \omega_1$, it is relatively consistent with ZFC that $|2^\omega| = \omega_2$ and α_0 is the least ordinal such that $R^{\omega_2}_{\alpha_0} = P(\omega_2 \times \omega_2)$.

<u>Remark</u>: In Theorem 52 we remove the restriction that α_0 is a successor (but the continuum in that model is $\aleph_{\omega+1}$). In [1] it is shown that α_0 cannot be ω_1.

<u>Proof</u>.

Let M be a countable transitive model of "ZFC + $|2^\omega| = |2^{\omega_1}| = \omega_2$". Let $X = \omega^\omega \cap M$ and define $\mathbb{P}^\alpha$ for $\alpha \leq \omega_2$ so that $\mathbb{P}^{\alpha+1} = \mathbb{P}^\alpha * \mathbb{P}_{\alpha_0}(Y_\alpha, X)$ where Y_α is a $\mathbb{P}^\alpha$ term for a subset of X, and at limits take the direct limit. Dovetail so that in $M[G_{\omega_2}]$ for every $Y \subseteq X$ such that $|Y| \leq \omega_1$ there are ω_2 many $\alpha < \omega_2$ such that $Y_\alpha = Y$. By Theorem 33 $R^{\omega_2}_{\alpha_0} = P(\omega_2 \times \omega_2)$.

Now comes the difficulty: we must show some subset of $\omega_2 \times \omega_2$ is not in $R^{\omega_2}_{\alpha_0 - 1}$. For the remainder of the proof let $(A_s: s \in \omega^{<\omega})$ and $(B_s: s \in \omega^{<\omega})$ be fixed terms in the forcing language of $\mathbb{P}^{\omega_2}$ such that for every $s \in \omega^{<\omega}$ $\phi \Vdash$ "$A_s \subseteq X$ and $B_s \subseteq \omega_2$". For $p \in \mathbb{P}^{\omega_2}$ define $\text{supp}(p) = \{\alpha < \omega_2: p(\alpha) \neq \phi\}$ and $\text{trace}(p) = \{x \in X: \exists \alpha \exists t(t,x) \in p(\alpha)\}$. By using the c.c.c. of $\mathbb{P}^{\omega_2}$ choose for each $x \in X$ countable sets $I_x \subseteq X$ and $J_x \subseteq \omega_2$ so that:

(1) for each $s \in \omega^{<\omega}$ $\{p \in \mathbb{P}^{\omega_2}: \text{trace}(p) \subseteq I_x$ and $\text{supp}(p) \subseteq J_x\}$ decides "$x \in A_s$", and

(2) for each $y \in I_x$ and $\alpha \in J_x$ $\{p \in \mathbb{P}^{\omega_2}: \text{trace}(p) \subseteq I_x$ and $\text{supp}(p) \subseteq J_x\}$ decides "$y \in Y_\alpha$".

Similarly for $\alpha < \omega_2$ we can pick countable sets $I_\alpha \subseteq X$ and $J_\alpha \subseteq \omega_2$ having properties (1) and (2) with $\alpha, B_s, I_\alpha, I_\alpha$ in place of x, A_s, I_x, I_x.

For $x \in X$ and $\alpha < \omega_2$ let $L(x,\alpha) = (I_x \times J_x) \cup (I_\alpha \times J_\alpha)$ and define for $p \in \mathbb{P}^{\omega_2}$, $|p|(x,\alpha) = \max\{|s|_{T_{\alpha_0}} : (s,u) \in p(\gamma)$ and $(u,\gamma) \notin L(x,\alpha)\}$.

__Lemma__ 38. Fix $x \in X$ and $\alpha < \omega_2$ and let $|p| = |p|(x,\alpha)$. For any $\beta \geq 1$ and $p \in \mathbb{P}^{\omega_2}$ there is a $\hat{p} \in \mathbb{P}^{\omega_2}$ with $|\hat{p}| < \beta + 1$, $\hat{p}$ compatible with p, and for any $q \in \mathbb{P}^{\omega_2}$ if $|q| < \beta$ and $\hat{p}$ and q are compatible, then p and q are compatible.

Proof.

The proof of this is like that of Lemma 35. Let $p_0 \geq p$ so that if $(s,x) \in p(\gamma)$ with $|s| = \lambda$ a limit ordinal greater than β and $|s^\frown i| \leq \beta + 1$, then there is $j < \omega$ so that $(s^\frown i^\frown j, x) \in p_0(\gamma)$. Let G be $\mathbb{P}^{\omega_2}$-generic with $p_0 \in G$. Choose $\Gamma \subseteq G$ finite so that if $((n),u) \in p_0(\gamma)$ (so $p_0 \restriction \gamma \Vdash$ "$u \notin Y_\gamma$") and $(u,\gamma) \in L(x,\alpha)$, then there is a $q \in \Gamma$ such that $q \Vdash$ "$u \notin Y_\gamma$". Define $\hat{p}$ by $\hat{p}(\gamma) = \bigcup\{q(\gamma): q \in \Gamma\} \cup \{(s,u) \in p_0(\gamma): |s| < \beta + 1$ or $(u,\gamma) \in L(x,\alpha)\}$. ■

For any well founded tree $\hat{T}$ define $C_s(\hat{T})$ for $s \in \hat{T}$ as follows. If $|s|_{\hat{T}} = 0$ then $C_s(\hat{T}) = A_s \times B_s$, otherwise $C_s(\hat{T}) = \bigcup\{(X \times \omega_2) - C_{s^\frown i}(\hat{T}): i < \omega\}$.

Lemma 39. If $x \in X$, $\alpha \in \omega_2$, $\hat{T} \in M$ is a well founded tree, $s \in \hat{T}$ with $|s|_{\hat{T}} = \beta$ where $1 \leq \beta \leq \alpha_0 - 1$, and $p \in \mathbb{P}^{\omega_2}$ such that $p \Vdash$ "$(x,\alpha) \notin C_s(T)$", then there exist q compatible with p, $|q|(x,\alpha) < \beta$, and $q \Vdash$ "$(x,\alpha) \notin C_s(T)$".

Proof.

The proof is by induction on β.

Case 1. $\beta = 1$:

Suppose $p \Vdash$ "$(x,\alpha) \in \bigcup_{i\in\omega}(A_{s^\frown i} \times B_{s^\frown i})$". So there exists $i_0 \in \omega$ and $\hat{p}$ and $\hat{q}$ elements of $\mathbb{P}^{\omega_2}$ so that $(p \cup \hat{p} \cup \hat{q}) \in \mathbb{P}^{\omega_2}$ and using (1) above

$(t,u) \in \hat{p}(\gamma) \to (u,\gamma) \in I_x \times J_x$ and

$(t,u) \in \hat{q}(\gamma) \to (u,\gamma) \in I_\alpha \times J_\alpha$ and

$\hat{p} \Vdash$ "$x \in A_{s^\frown i_0}$", $\hat{q} \Vdash$ "$y \in B_{s^\frown i_0}$". So $\hat{p} \cup \hat{q} = q$ does the job.

Case 2. β a limit ordinal:

Suppose $p \Vdash$ "$(x,\alpha) \in \bigcup_{i\in\omega} C_{s^\frown i}(\hat{T})$ where $|s|_{\hat{T}} = \beta$. Find $q \geq p$ and $i_0 \in \omega$ such that $q \Vdash$ "$(x,y) \in C_{s^\frown i_0}(\hat{T})$. Let $T_0 = \{t \in \hat{T}: s^\frown i_0 \subseteq t$ or $t \subseteq s^\frown i_0\}$. Then $|s|_{T_0} = |s^\frown i|_{\hat{T}} + 1 < \beta$ and $C_s(T_0) = (X \times \omega_2) - C_{s^\frown i_0}(T)$, hence $q \Vdash$ "$(x,\alpha) \notin C_s(T_0)$" where $|s|_{T_0} < \beta$; so by induction hypothesis there exists r compatible with q (and hence with p), $|r|(x,\alpha) < \beta$, and $r \Vdash$ "$(x,\alpha) \in C_{s^\frown i_0}(T)$". r does the trick.

Case 3. $\beta + 1$:

Since $\beta + 1 < \alpha_0$, let q be as from Lemma 38. ■

Define $D \subseteq X \times \omega_2$ by $D = \{(x,\alpha): x \in G^\alpha_{(0)}$ where $G^\alpha_{(0)}$ is one of the $\underset{\sim}{\Pi}^0_{\alpha_0 -1}$ sets created on the α^{th} step. D is $\underset{\sim}{\Pi}^0_{\alpha_0 -1}$ in the rectangles on $X \times \omega_2$. We want to show it is not $\Sigma^0_{\alpha_0 -1}$ in the rectangles on $X \times \omega_2$ in $M[G_{\omega_2}]$.

Define: (x,α) is free (with respect to $(A_s : s \in \omega^{<\omega})$, $(B_s : s \in \omega^{<\omega})$) iff $[x \notin I_\alpha$ and $\alpha \notin J_x]$.

<u>Lemma</u> 40. If $T \subseteq \omega^{<\omega}$ is well founded and $T \in M$, $s \in T$ with $|s|_T \leq \alpha_0 - 1$, (x,α) is free, and $Y_\alpha = \phi$; then for every $p \in \mathbb{P}^{\omega_2}$ such that $|p|(x,\alpha) = 0$ it is not the case that $p \Vdash$ "$(x,\alpha) \in D$ iff $(x,\alpha) \notin C_s(T)$".

<u>Proof</u>.

Let $\hat{p} \geq p$ by defining $\hat{p}(\gamma) = p(\gamma)$ for $\gamma \neq \alpha$ and $\hat{p}(\alpha) = p(\alpha) \cup \{((0),x)\}$. Then $\hat{p} \Vdash$ "$(x,\alpha) \in D$" so by Lemma 39 there exists q compatible with $\hat{p}$, $|q|(x,\alpha) < \alpha_0$, and $q \Vdash$ "$(x,\alpha) \notin C_s(T)$". But (x,α) free implies that $(x,\alpha) \notin L(x,\alpha)$ so q does not say "$x \in G^{y_\alpha}_{(0)}$". Thus for a sufficiently large $m < \omega$ r defined by $r(\gamma) = p(\gamma) \cup q(\gamma)$ for $\gamma \neq \alpha$ and $r(\alpha) = p(\alpha) \cup q(\alpha) \cup \{((0,m),x)\}$ is a member of $\mathbb{P}^{\omega_2}$. But $r \Vdash$ "$(x,\alpha) \notin D$ and $(x,\alpha) \notin C_s(T)$", a contradiction since r extends p. ■

Since the terms $(A_s : s \in \omega^{<\omega})$ and $(B_s : s \in \omega^{<\omega})$ were arbitrary to start with it will complete the proof of the theorem to find lots of (x,α) free.

The next lemma generalizes Kunen [9], p. 74.

<u>Lemma</u> 41. Given $|I_\alpha| < \kappa$ for $\alpha < \kappa^+$, there exists $G \subseteq \kappa^+$ with $|G| = \kappa^+$ and there is S with $|S| \leq \kappa$ so that for any $\alpha, \beta \in G$ if $\alpha \neq \beta$ then $I_\alpha \cap I_\beta \subseteq S$.

<u>Proof</u>.

We can assume $I_\alpha \subseteq \kappa^+$.

Define $\mu_\alpha, z_\alpha < \kappa^+$ for $\alpha < \kappa^+$ nondecreasing so that:

(1) $\mu_\lambda = \sup\{\mu_\alpha : \alpha < \lambda\}$ for λ limit;

(2) z_α's are strictly increasing;

(3) for α a successor and for distinct $\beta, \gamma < \alpha$

$I_{z_\beta} \cap I_{z_\gamma} \subseteq \mu_\alpha$;

(4) if $\mu_{\alpha+1} > \mu_\alpha$ then for any $z > z_\alpha$
$\mu_\alpha \not\supseteq I_z \cap \bigcup\{I_{z_\beta} : \beta \leq \alpha\}$ and $\bigcup\{I_{z_\beta} : \beta \leq \alpha\} \subseteq \mu_{\alpha+1}$.

Let $G = \{z_\alpha : \alpha < \kappa^+\}$ and $S = \sup\{\mu_\alpha : \alpha < \kappa^+\}$.

To see that $S < \kappa^+$ note that for any $\alpha < \kappa^+$
$|\{\beta : \mu_{\beta+1} > \mu_\beta$ and $\beta < \alpha\}| < \kappa$. This is because
$I_{z_\alpha} \cap (\mu_{\beta+1} - \mu_\beta) \neq \phi$ for all $\beta < \alpha$ such that
$\mu_{\beta+1} > \mu_\beta$. ■

<u>Lemma</u> 42. There exists $\Sigma_0 \subseteq X$ and $\Sigma_1 \subseteq \omega_2$ with $|\Sigma_0| = |\Sigma_1| = \omega_2$, for every $\alpha \in \Sigma_1$ $Y_\alpha = \phi$, and for every $(x,\alpha) \in \Sigma_0 \times \Sigma_1$ (x,α) is free.

<u>Proof</u>.

By Lemma 41 there exists $\hat{\Sigma}_0 \subseteq X$ and $S \subseteq \omega_2$ with $|\hat{\Sigma}_0| = \omega_2$ and $|S| < \omega_2$ so that for every distinct

$x,y \in \hat{\Sigma}_0$ $J_x \cap J_y \subseteq S$. Since $\{J_x - S: x \in \hat{\Sigma}_0\}$ is a disjoint family, we can cut down $\hat{\Sigma}_0$ (maintaining $|\hat{\Sigma}_0| = \omega_2$) and find $\hat{\Sigma}_1 \subseteq \omega_2$ so that $|\hat{\Sigma}_1| = \omega_2$, for every $\alpha \in \hat{\Sigma}_1$ $Y_\alpha = \phi$, and for every $x \in \hat{\Sigma}_0$ $J_x \cap \hat{\Sigma}_1 = \phi$. Applying Lemma 41 again find $\Sigma_1 \subseteq \hat{\Sigma}_1$ with $|\Sigma_1| = \omega_2$ and $T \subseteq X$ with $|T| < \omega_2$ so that for every distinct $\alpha,\beta \in \Sigma_1$ $I_\alpha \cap I_\beta \subseteq T$. Since $\{I_\alpha - T: \alpha \in \Sigma_1\}$ are disjoint by cutting down Σ_1 (maintaining $|\Sigma_1| = \omega_2$) we can assume Σ_0 defined to be equal to $\hat{\Sigma}_0 - (T \cup \bigcup\{I_\alpha: \alpha \in \Sigma_1\})$ has cardinality ω_2. Σ_0 and Σ_1 do the job. ∎

Lemma 42 finishes the proof of Theorem 37.

<u>Remark</u>: There is nothing special about ω_2 in the above theorem; we could have replaced it by any larger cardinal κ with $\kappa^{<\kappa} = \kappa$.

Now we turn to a slightly different problem. For X a topological space a set $A \subseteq X^n$ is projective iff it is in the smallest class containing the Borel sets (in the product topology on X^m for any $m \in \omega$) and closed under complementation and projection ($B \subseteq X^m$ is the projection of $C \subseteq X^{m+1}$ iff ($\vec{y} \in B$ iff $\exists x \in X\ x\bar{y} \in C$)).

<u>Theorem</u> 43. If M is a countable transitive model of ZFC then there exists N a c.c.c. Cohen extension of M such that if $M \cap \omega^\omega = X$ then $N \models$ "Every projective set in X is Borel and the Borel hierarchy of X has ω_1 distinct levels ($\mathrm{ord}(X) = \omega_1$)".

This shows the relative consistency of an affirmative answer to a question of Ulam ([31], p. 10). Note that since $X \times X$ is homeomorphic to X (take any recursive coding function), if for every $B \subseteq X \times X$ Borel $\{x: \exists y (x,y) \in B\}$ is Borel in X, then every projective set in X is Borel in X.

<u>Proof</u>.

The proof is slightly simpler if we assume that CH holds in M. We give the proof in that case and then later indicate the necessary modifications. In any case $|2^\omega|^M = |2^\omega|^N$.

Construct a sequence $M = M_0 \subseteq M_1 \subseteq \ldots M_{\omega_1} = N$, by iterated forcing so that $M_{\alpha+1}$ is obtained from M_α by $\underset{\sim}{\Pi}^0_{\alpha+1}$-forcing. On the α^{th} stage we are presented with a term τ_α in the forcing language of $\mathbb{P}^\alpha$ denoting a real. Then letting Y_α be the projective set (over X) determined by τ_α we let $\mathbb{P}^{\alpha+1} = \mathbb{P}^\alpha * \mathbb{P}_{\alpha+1}(Y_\alpha, X)$. What is being done is that at stage α we make Y_α a $\underset{\sim}{\Pi}^0_{\alpha+1}$ set intersected with X. The reason this will work is that after the α^{th} stage our forcing will not interfere with the Borel hierarchy on X up to the α^{th} level. Since this is c.c.c. forcing we can imagine that each X-projective set in N is eventually caught by some τ_α for $\alpha < \omega_1$. So it is clear that $N \models$ "Every X-projective set is Borel in X", for any $N = M[G]$, where G is $\mathbb{P}^{\omega_2}$-generic over M. Define for $H \subseteq X$ and $p \in \mathbb{P}, |p|(H) = \max\{|s|_{T_{\alpha+1}} :$ there exist $\alpha < \omega_1$ and $x \notin H$ $(s,x) \in p(\alpha)\}$. Given τ a term in the forcing language of $\mathbb{P}^\gamma$ denoting a subset of ω $(\gamma < \omega_1)$, there exists $H \subseteq X$ such that:

(a) H is countable.

(b) $\forall n \in \omega$

$\{p \in \mathbb{P}^\gamma : |p|(H) = 0\}$ decides "$n \in \tau$".

(c) $\forall \beta < \gamma$ and $x \in H$

$\{p \in \mathbb{P}^\gamma : |p|(H) = 0\}$ decides "$x \in Y_\beta$".

<u>Lemma</u> 44. (write $|p| = |p|(H)$).

"Exactly statement of Lemma 38" for $\mathbb{P}^\gamma$.

<u>Proof</u>.

Extend $p \leq p_0$ as before. Let G be $\mathbb{P}^\gamma$-generic with

$p_0 \in G$. Choose $\Gamma \subseteq G$ finite so that:

(1) $q \in \Gamma \to |q|(H) = 0$;

(2) if $\langle\langle n\rangle, x\rangle \in p_0(\alpha)$ (so $p \restriction_\alpha \Vdash$ "$x \notin y_\alpha$") then

$\exists\, q \in \Gamma \cap \mathbb{P}^\alpha$ such that $q \Vdash$ "$x \notin y_\alpha$".

Define $\hat{p}(\alpha) = \bigcup\{r(\alpha): r \in \Gamma\} \cup \{\langle s,x\rangle \in p_0(\alpha):$ $|s|_{T_\alpha} < \beta + 1$ or $x \in H\}$. $\hat{p}$ is a condition because if $\langle\langle n\rangle, x\rangle \in p(\alpha)$ and $|\langle n\rangle|_{T_{\alpha+1}} < \beta + 1$, then $\hat{p}\restriction_\alpha \geq p\restriction_\alpha$ (so $\hat{p}\restriction_\alpha \Vdash$ "$x \notin Y_\alpha$" as required).

The $r \in \Gamma$ take care of such requirements about $x \in H$.

The rest of the proof is the same. $\blacksquare$

<u>Lemma</u> 45. If τ, H, γ are as above, $B(v)$ is a $\underset{\sim}{\Sigma}^0_\beta$ predicate for some $\beta \geq 1$ with parameter from M, and $p \in \mathbb{P}^\gamma$ such that $p \Vdash$ "$B(\tau)$", then there is a $q \in \mathbb{P}^\gamma$ compatible with p, $|q|(H) < \beta$ and $q \Vdash$ "$B(\tau)$".

<u>Proof</u>.

The proof is the same as before. $\blacksquare$

We can assume that for unboundedly many $\alpha < \omega_1$ $Y_\alpha = \emptyset$.

Let G_α $(G^\alpha_{(0)})$ be one of the $\underset{\sim}{\Pi}^0_\alpha$ sets determined by $G \cap P_{\alpha+1}(\emptyset, X)$ where $Y_\alpha = \phi$.

<u>Claim</u>: $M[G] \models$ "for any $L \in \underset{\sim}{\Sigma}^0_\alpha$ $(L \cap X \neq G_\alpha \cap X)$".

Proof.

Otherwise let τ be a term for a real in the forcing language $\mathbb{P}^{\gamma}$ for some $\gamma < \omega_1$ such that for some L a $\underset{\sim}{\Sigma}^0_{\alpha}$ set with parameter τ and some $p \in P^{\gamma}$ $p \Vdash$ "$L \cap X = G_{\alpha} \cap X$". Choose H with properties (a), (b), and (c) with respect to τ, and also $|p|(H) = 0$. Let $x \in X - H$. Define $r(\alpha) = p(\alpha) \cup \{((0),x)\}$ and for $\beta \neq \alpha$ $r(\beta) = p(\beta)$. Note that $r \Vdash$ "$x \in G_{\alpha}$" hence $r \Vdash$ "$x \in L$". By Lemma 45 there exists $q \in \mathbb{P}^{\gamma}$ compatible with r, $|q|(H) < \beta$, and $q \Vdash$ "$x \in L$". Since $x \notin H$ we know $((0),x) \notin q(\alpha)$. Define $\hat{q} \in \mathbb{P}^{\omega_1}$ by $\hat{q}(\beta) = p(\beta) \cup q(\beta)$ for $\beta \neq \alpha$ and $\hat{q}(\alpha) = p(\alpha) \cup q(\alpha) \cup \{((0,n),x)\}$ where n is picked sufficiently large so $\hat{q}(\alpha)$ is a condition. But then $\hat{q} \Vdash$ "$x \in L$ and $x \notin G_{\alpha}$ and ($x \in L$ iff $x \in G_{\alpha}$)" and this is a contradiction. This concludes the proof of Theorem 43. ■

When the continuum hypothesis does not hold in M the construction of N still has ω_1 steps but at each step we must take care of all reals in the ground model. That is $\mathbb{P}^{\alpha+1} = \mathbb{P}^{\alpha} * Q_{\alpha}$ where Q_{α} is a term denoting $\Sigma\{\mathbb{P}_{\alpha+1}(H_x,X): x \in \omega^{\omega} \cap M[G_{\alpha}]\}$ for G $\mathbb{P}^{\alpha}$-generic over M. This works since all reals in $N = M[G]$ for G $\mathbb{P}^{\omega_1}$-generic over M are caught at some countable stage.

Remark: It is easy to see that if $V = L$ there is an $X \subseteq \omega^\omega$ uncountable Π^1_1 set such that $X \in L$ and $X \times X$ is homeomorphic to X. Also by absoluteness it is possible to make sure that for every A $\underset{\sim}{\Sigma}^1_2$ in ω^ω, $A \cap X$ is Borel in X. This family of sets includes those obtained by the Souslin operation from Borel sets in X.

Theorem 46. (MA) $\exists X \subseteq 2^\omega$ $\mathrm{ord}(X) = \omega_1$ and $\forall A \in \Sigma^1_1$ in 2^ω $\exists B$ Borel (2^ω) $A \cap X = B \cap X$.

Proof.

Let $\mathbb{B}$ be the c.c.c. countably generated boolean algebra of Theorem 9 with $K(\mathbb{B}) = \omega_1$.

$\mathbb{B} \simeq \mathrm{Borel}(2^\omega)/J$ for some J and ω_1-saturated σ-ideal in the Borel sets.

Lemma 47. If I is an ω_1-saturated σ-ideal in $\mathrm{Borel}(2^\omega)$ then $B_I = \{A \subseteq 2^\omega : \exists B$ Borel $\exists C \in I$ $(A \,\Delta\, B) \subseteq C\}$ is closed under the Souslin operation.

For a proof the reader is referred to [11], page 95.

By Theorem 14 MA implies there is $X \subseteq 2^\omega$ a J-Luzin set. For any $\alpha < \omega_1$ there is A $\underset{\sim}{\Pi}^0_\alpha$ so that for every B $\underset{\sim}{\Sigma}^0_\alpha$, $(A \,\Delta\, B) \notin J$, hence $|(A \,\Delta\, B) \cap X| = |2^\omega|$, so $A \cap X \neq B \cap X$, and thus $\mathrm{ord}(X) = \omega_1$. If A is $\underset{\sim}{\Sigma}^1_1$ then by Lemma 47 there is B Borel and C in J with

$A \Delta B \subseteq C$. Since $|C \cap X| < |2^{\omega}|$ by MA $\exists D \in \mathrm{Borel}(2^{\omega})$ $(A \Delta B) \cap X = D \cap X$. So $A \cap X = (B \Delta D) \cap X$. ■

This suggests the following question:

Can you have $X \subseteq 2^{\omega}$ such that every subset of X is Borel in X and the Borel hierarchy on X has ω_1 distinct levels? The answer is no.

Theorem 48. If $X \subseteq 2^{\omega}$ and every subset of X is Borel in X then $\mathrm{ord}(X) < \omega_1$.

Proof.

Let $X = \{x_\alpha : \alpha < \kappa\}$ and $X_\alpha = \{x_\beta : \beta < \alpha\}$

Lemma 49. If $|X| \leq \kappa$, every subset of X is Borel in X, and $R^{\kappa}_{\omega_1} = P(\kappa \times \kappa)$, then $\mathrm{ord}(X) < \omega_1$.

Proof.

Since every rectangle in $X \times X$ is Borel in $X \times X$ and $R^{\kappa}_{\omega_1} = P(\kappa \times \kappa)$, every subset of $X \times X$ is Borel in $X \times X$. Suppose for contradiction $\forall \alpha < \omega_1$ $\exists H_\alpha \subseteq X$ not $\underset{\sim}{\Pi}^0_\alpha$ in X. Let $H = \bigcup_{\alpha<\omega_1} \{x_\alpha\} \times H_\alpha$. For some $\alpha < \omega_1$, H is $\underset{\sim}{\Pi}^0_\alpha$ in $X \times X$. But then every cross section of H is $\underset{\sim}{\Pi}^0_\alpha$ in X contradiction. ■

The proof of the theorem is by induction on $|X| = \kappa$.

For $\kappa = \omega_1$ it follows from Lemma 49 since
$R_2^{\omega_1} = P(\omega_1 \times \omega_1)$.

For $\mathrm{cof}(\kappa) = \omega$ it is trivial.

For $\mathrm{cof}(\kappa) > \omega_1$:

$\forall \alpha < \kappa$ choose β_α minimal $< \omega_1$ so that every subset of X_α is $\underset{\sim}{\Pi}^0_{\beta_\alpha}$ in X (we can do this since X_α is $\underset{\sim}{\Pi}^0_\beta$ in X some $\beta < \omega_1$). Since $\mathrm{cof}(\kappa) > \omega_1$ there exists $\alpha_0 < \omega_1$ such that for a final segment of ordinal less than κ, $\beta_\alpha = \alpha_0$. By Theorem 33 $R^\kappa_{\omega_1} = P(\kappa \times \kappa)$ so by Lemma 49 $\mathrm{ord}(X) < \omega_1$.

For $\mathrm{cof}(\kappa) = \omega_1$:

Let $\eta_\alpha \uparrow \kappa$ for $\alpha < \omega_1$ be an increasing continuous cofinal sequence.

<u>Lemma</u> 50. $\exists \beta_0 < \omega_1 \; \forall \alpha < \omega_1 \; X_{\eta_\alpha}$ is $\underset{\sim}{\Pi}^0_{\beta_0}$ in X.

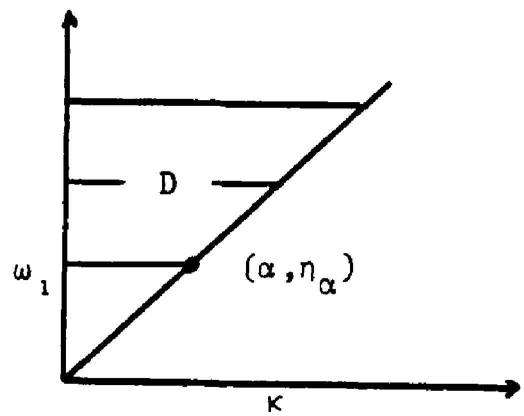
<u>Proof</u>.

If $G \subseteq \kappa \times \kappa$ is the graph of a partial function then $G \in R_2^\kappa$ (Rao [21]). This is because if $f: D \to \kappa$ where $D \subseteq \kappa$ then viewing $X \subseteq$ irrational real numbers we have:

$(f(\alpha) = \beta)$ iff $(\alpha \in D$ and $\forall r \in Q(r < x_{f(\alpha)}$ iff $r < x_\beta))$

where Q is the set of rational numbers.

Then $D = \{(\alpha,\beta): \alpha < \omega_1 \wedge \beta < \eta_\alpha\}$ is the complement in $\omega_1 \times \kappa$ of a countable union of graphs from κ into ω_1.



Hence the set $\bigcup_{\alpha<\omega_1} \{x_\alpha\} \times X_{\eta_\alpha}$ is Borel in $X \times X$. Say it is $\underset{\sim}{\Pi}^0_{\beta_0}$. It follows that each X_{η_α} is $\underset{\sim}{\Pi}^0_{\beta_0}$. ■

For all $\lambda < \omega_1$ let $\beta(\lambda)$ be minimal so that every subset of X_{η_λ} is $\underset{\sim}{\Pi}^0_{\beta(\lambda)}$ in X. If the hypothesis of Theorem 33 fails, then $\exists f: \omega_1 \to \omega_1$ increasing so that for all $\lambda < \omega_1$ $\beta(f(\lambda)) < \beta(f(\lambda + 1))$. So for all $\lambda < \omega_1$ there is some $H_\lambda \subseteq X_{\eta_{f(\lambda+1)}}$ which is not $\underset{\sim}{\Pi}^0_{\beta(f(\lambda))}$ in X. Since every subset of $X_{\eta f(\beta)}$ is $\underset{\sim}{\Pi}^0_{\beta(f(\beta))}$ in X we can assume $H_\lambda \subseteq (X_{\eta_{f(\lambda+1)}} - X_{\eta_{f(\lambda)}})$. Let $H = \bigcup_{\lambda<\omega_1} H_\lambda$. Then H is $\underset{\sim}{\Pi}^0_{\alpha_0}$ in X for some $\alpha_0 < \omega_1$. But for each λ, $H_\lambda = H \cap (X_{\eta_{f(\lambda+1)}} - X_{\eta_{f(\lambda)}})$, so each H is $\underset{\sim}{\Pi}^0_{\max(\alpha_0, \beta_0+1)}$ in X, contradiction.

This ends the proof of Theorem 48. ■

<u>Remark</u>: Kunen has noted that Theorem 48 may be generalized to nonseparable metric spaces. Let $\mathcal{B}$ be a σ-discrete basis for X and assume that every subset of X is Borel in X. By using σ-discreteness it is easily seen that $\exists \mathcal{H} \subseteq \mathcal{B} \; \exists \; \beta < \omega_1$ so that $\mathcal{B} - \mathcal{H}$ is countable and $\forall U \in \mathcal{H}$ $\mathrm{ord}(U) \leq \beta$. But $Y = \{x \in X: \forall U \in \mathcal{B}\; (x \in U \to U \notin \mathcal{H})$ is separable and hence by the theorem $\mathrm{ord}(Y)$ $\mathrm{ord}(Y) < \omega_1$, and so $\mathrm{ord}(X) < \omega_1$.

As a partial converse of Theorem 33 we have:

<u>Theorem</u> 51. If $\kappa = |2^\omega|$, $\kappa^{<\kappa} = \kappa$, and $R^\kappa_{\alpha_0} = P(\kappa \times \kappa)$, then there is $X \subseteq 2^\omega$ with $|X| = \kappa$ and every subset of X of cardinality less than κ is $\underset{\sim}{\Pi}^0_{\alpha_0}$ in X.

<u>Proof</u>.

Let Z_α for $\alpha < \kappa$ be all the subsets of κ of cardinality less than κ. Put $Z = \bigcup_{\alpha<\kappa} \{\alpha\} \times Z_\alpha$ and $W = \{(\alpha,\beta) : \alpha < \beta < \kappa\}$. Let $\{A_n : n < \omega\}$ be closed under finite boolean combinations and $Z, W \in \{A_n \times A_m : n,m < \omega\}_{\alpha_0}$. The map $F: \kappa \to 2^\omega$ defined by $(F(\alpha)(n) = 1$ iff $\alpha \in A_n)$ is 1-1 and the set $X = F''\kappa$ has the required property. ∎

For any cardinal κ let $R(\kappa)$ be the least $\beta < \omega_1$ such that $R^\kappa_\beta = P(\kappa \times \kappa)$ or ω_1 if no such β exists.

<u>Theorem</u> 52. It is relatively consistent with ZFC that $|2^{\omega}| = \omega_{\omega+1}$, for every $n \leq \omega$ $R(\omega_n) = 1 + n$, and $R(\omega_{\omega+1}) = \omega$. This can be generalized to show that for any $\lambda < \omega_1$ a limit ordinal it is consistent with ZFC that $R(|2^{\omega}|) = \lambda$.

<u>Proof</u>.

Let $M \models$ "ZFC + MA + $|2^{\omega}| = \omega_{\omega+1}$" be countable and transitive. Let $\kappa = \omega_{\omega+1}$ and define $\mathbb{P}^{\alpha}$ for $\alpha \leq \kappa$ so that $\mathbb{P}^{\alpha+1} = \mathbb{P}^{\alpha} * \mathbb{P}_{2+\beta+1}(X_{\alpha}, Y_{\alpha})$ where $Y_{\alpha} \subseteq 2^{\omega}$, $Y_{\alpha} \in M$, $|Y_{\alpha}| = \omega_{\beta+1}$, and $\phi \Vdash$ "$X_{\alpha} \subseteq Y_{\alpha}$". At limits take the direct limit. By dovetailing arrange that for any G $\mathbb{P}^{\kappa}$-generic over M, $M[G] \models$ "If $Y \subseteq 2^{\omega}$, $Y \in M$, and $|Y| = \omega_{\beta+1}$ for some $\beta < \omega$, then every subset of Y is $\underset{\sim}{\Pi}^0_{2+\beta+1}$ in Y".

As in the proof of Theorem 34 given any τ a term for a subset of ω, find in M, $H \subseteq 2^{\omega}$, $K \subseteq \kappa$ so that:

(1) $|H| \leq \omega_{\beta_0}$, $|K| \leq \omega_{\beta_0}$.

Let $\{Q = p \in \mathbb{P}^{\kappa}: \text{supp}(p) \subseteq K, |p|(H) = 0\}$

(2) $\forall n \in \omega$ Q decides "$n \in \tau$".

(3) $\forall \beta \in K$ $\forall x \in H$ Q decides "$x \in X_{\beta}$".

(4) If $\alpha \in K$ and $|Y_{\alpha}| \leq \omega_{\beta_0}$ then $Y_{\alpha} \subseteq H$.

Lemma 53. If H,K have property (3), (4) above then for any $p \in \mathbb{P}^{\kappa}$ and β with $1 \leq \beta < 2 + \beta_0$ there is $\hat{p}$ compatible with p, $|\hat{p}|(H) < \beta + 1$, $\text{supp}(\hat{p}) \subseteq K$, and for any q if $|q|(H) < \beta$, $\text{supp}(q) \subseteq H$, and $\hat{p}$ and q are compatible, then p and q are compatible.

Proof.

The proof of this is just like the proof of Lemma 35. To check that the $\hat{p}$ gotten there is an element of $\mathbb{P}^{\kappa}$, note that if $((n),x) \in \hat{p}(\alpha)$ then $x \in H$. Because if $x \notin H$ and $\alpha \in K$, then $|Y_\alpha| \geq \omega_{\beta_0+1}$ because of (4). Say $|Y_\alpha| = \omega_{\gamma+1}$, so $\mathbb{P}^{\alpha+1} = \mathbb{P}^{\alpha} * \mathbb{P}_{2+\gamma+1}(X_\alpha, Y_\alpha)$ and $|(n)|_{T_{2+\gamma+1}} = 2 + \gamma \geq 2 + \beta_0 \geq \beta + 1$, but then it was thrown out, contradiction. ∎

Lemma 54. Suppose H and K have properties (2), (3), and (4) for $\tau \subseteq \omega$. Suppose $1 \leq \beta \leq 2 + \beta_0$ and B(v) is a $\underset{\sim}{\Sigma}^0_\beta$ predicate with parameters from M, $p \in \mathbb{P}^{\kappa}$ and $p \Vdash$ "$B(\tau)$". Then $\exists q \in \mathbb{P}^{\kappa}$ compatible with p, $|q|(H) < \beta$, $\text{supp}(q) \subseteq K$ and $q \Vdash$ "$B(\tau)$".

Proof

This follows from Lemma 53 just as in Theorem 34. ∎

From Lemma 54 we have that:

(A) For any $Y \subseteq 2^\omega$ with $Y \in M$ and n with $1 \leq n \leq \omega$ ($|Y| = \omega_n$ iff Y is a Q_{2+n}-set). We claim that:

(B) For any $n < \omega$ there are $X, Y \subseteq 2^\omega$ with $|X| = |Y| = \omega_{n+2}$ so that if U is the usual $\underset{\sim}{\Pi}^0_{n+2}$ set universal for $\underset{\sim}{\Pi}^0_{n+2}$ sets, then $U \cap (X \times Y)$ is not $\underset{\sim}{\Sigma}^0_{n+2}$ in the abstract rectangles on $X \times Y$.

To prove (B) just generalize the argument of Theorem 37, for $n = 0$ the argument is the same. Let $X \subseteq 2^\omega$ be in M with $|X| = \omega_{n+2}$. Choose $K \subseteq \kappa$, $|K| = \omega_{n+2}$, and $K \in M$, so that for any $\alpha \in K$ $Y_\alpha = X$ and $\phi \Vdash$ "$X_\alpha = \phi$". Let $Y = \{y_\alpha : \alpha \in K\}$ where y_α is the $\underset{\sim}{\Pi}^0_{n+2}$ code (with respect to U) for $G^\alpha_{(0)}$. To generalize the argument allow $I_x, J_x, I_\alpha, J_\alpha$ to have cardinality $\leq \omega_n$ and also whenever $\gamma \in J_x (\gamma \in J_\alpha)$ and $|Y_\gamma| \leq \omega_n$, then $Y_\gamma \subseteq I_x$ $(Y_\gamma \subseteq I_\alpha)$.

In $M[G]$ for any $n < \omega$ $R(\omega_n) = 1 + n$. To see this, let $Y \subseteq 2^\omega$ with $Y \in M$ and $|Y| = \omega_{n+1}$. If $X \subseteq Y$ and $|X| \leq \omega_n$, then there is $Z \in M$ with $|Z| \leq \omega_n$ and $X \subseteq Z$. Because $M \models$ "MA" Z is $\underset{\sim}{\Pi}^0_2$ in Y and since X is $\underset{\sim}{\Pi}^0_{2+n}$ in Z by (A), we have X is $\underset{\sim}{\Pi}^0_{2+n}$ in Y. By Theorem 33 $R^{\omega_{n+1}}_{n+2} = P(\omega_{n+1} \times \omega_{n+1})$. By (B) $n + 2$ is the least which will do.

Thus $R(\omega_\omega) = \omega$. To see that $R(\kappa) = \omega$ let $Y \subseteq 2^\omega$ with $Y \in M$ $|Y| = \kappa$, and every subset $Z \subseteq Y$ such that $|Z| < \kappa$ and $Z \in M$ is $\underset{\sim}{\Sigma}^0_2$ in Y (see (Theorem 17). In $M[G]$ every $Z \subseteq Y$ with $|Z| < \kappa$ is

$\underset{\sim}{\Sigma}^0_\omega$ in Y, so by Theorem 33 $R^\kappa_\omega = P(\kappa \times \kappa)$. ■

Remark: It is easy to generalize Theorem 54 to show that for any $\lambda < \omega_1$ a limit ordinal and $\kappa > \omega$ of cofinality ω, it is consistent that $|2^\omega| = \kappa^+$ and $R(\kappa^+) = \lambda$.

Theorem 55. It is relatively consistent with ZFC that

(a) $|2^\omega| = \omega_{\omega_1+1}$,

(b) for any $\alpha < \omega_1$ there is a Q_α set.

(c) $R(\omega_n) = n + 1$ for $n < \omega$,

(d) $R(\omega_\lambda) = \lambda$ for $\lambda < \omega_1$ a limit ordinal,

(e) $R(\omega_{\lambda+n+1}) = \lambda + n$ for $\lambda < \omega_1$ a limit ordinal and $n < \omega$.

The proof of this is an easy generlization of Theorem 54 and is left to the reader.

A set $U \subseteq 2^\omega \times 2^\omega$ is universal for the Borel sets iff for every $B \subseteq 2^\omega$ there exists $x \in 2^\omega$ such that $B = U_x = \{y: (y,x) \in U\}$.

Theorem 56. It is relatively consistent with ZFC that no set universal for the Borel sets is in the σ-algebra generated by the abstract rectangles in $2^\omega \times 2^\omega$.

Proof.

Let $M \models$ "ZFC + $\neg$CH" and let

$Q = \sum_{\beta<\omega_2} (\Sigma\{\mathbb{P}_\alpha(\phi, 2^\omega \cap M): \alpha < \omega_1\})$. Let G be Q-generic over M, then in $M[G]$ there is no set U universal for the Borel sets in the σ-algebra generated by the rectangles. Suppose G is given by

$(y^\alpha_\beta: T^*_{\alpha+1} \to 2^{<\omega}: \alpha < \omega_1$ **and** $\beta < \omega_2)$ where $T_{\alpha+1}$ is the normal $\alpha + 1$ tree used in the definition of $\mathbb{P}_{\alpha+1}$ and $G^{(0)}_{y^\alpha_\beta}$ are the $\underset{\sim}{\Pi}^0_\alpha$ sets determined by y^α_β. Then as before we can easily get for each $\alpha < \omega_1$ that $V^\alpha = \{(x,\beta): x \in G^{(0)}_{y^\alpha_\beta}\}$ is not $\underset{\sim}{\Sigma}^0_\alpha$ in the abstract rectangles on $(2^\omega \times \omega_2)$. Now suppose such a U existed and were $\underset{\sim}{\Sigma}^0_\alpha$ in the abstract rectangles on $2^\omega \times 2^\omega$. Choose $F: \omega_2 \to 2^\omega$ (necessarily 1-1) so that

$\forall \beta < \omega_2 \ \forall x \in 2^\omega \ ((x,\beta) \in V^\alpha \leftrightarrow (x, f(\beta)) \in U)$.

If U is $\underset{\sim}{\Sigma}^0_\alpha$ in $\{A_n \times B_n: n < \omega\}$ then V^α is $\underset{\sim}{\Sigma}^0_\alpha$ in $\{A_n \times f^{-1}(B_n): n < \omega\}$, contradiction. ■

Remarks:

(1) In [9] Kunen shows that if one adds ω_2 Cohen reals to a model of GCH then no well ordering of ω_2 is in $R^{\omega_2}_{\omega_1}$.

(2) In [1] it is shown that if G is a countable field of sets with Borel$(2^\omega) \subseteq G_{\omega_1}$, the order of G is ω_1.

In the model of Theorem 56 for any countable G and $\alpha < \omega_1$ $\mathrm{Borel}(2^\omega)$ is not included in G_α. This can be seen as follows. Let $G = \{A_n: n < \omega\}$ and let $\{s_n: n < \omega\} = T^*$ where T is a normal α tree. Define for any $y \in \omega^\omega$ and $s \in T$ the set G_y^s as follows. For $s = s_n$ let $G_y^s = A_{y(n)}$, otherwise $G_y^s = \bigcap\{\omega^\omega - G_y^{s\frown n}: n < \omega\}$. If $U = \{(x,y): x \in G_y^\phi\}$ then U is "$\underset{\sim}{\Pi}^0_\alpha$" in the abstract rectangles and universal for all Borel sets, contradicting Theorem 56.

§5 Problems

Show:

(1) If $|X| = \omega_1$ then X is not a Q_ω set.

(2) If $R_\omega^{\omega_2} = P(\omega_2 \times \omega_2)$ then there is $n < \omega$ with $R_n^{\omega_2} = P(\omega_2 \times \omega_2)$.

(3) If there exists a Q_ω set then there exists a Q_n set for some $n < \omega$.

(4) If $R_{\omega_1}^{\omega_2} = P(\omega_2 \times \omega_2)$ and $|2^\omega| = \omega_2$ then $|2^{\omega_1}| = \omega_2$.

(5)* If there is a Q_2 set of size ω_1 then every subset of 2^ω of size ω_1 is a Q_2 set.

(6) If X is a Q_α set and Y is a Q_β set, then $2 \leq \alpha < \beta$ implies $|X| < |Y|$.

Show consistency of:

(7) $\{\alpha: \quad X \subseteq 2^{\omega} \quad \mathrm{ord}(X) = \alpha\} = \{1\} \cup \{\alpha \leq \omega_1 : \alpha$ is even$\}$.

(8) $|2^{\omega}| = \omega_3$ and for any $X \subseteq 2^{\omega}$ if $|X| = \omega_1$ then X is a Q_7 set, if $|X| = \omega_2$ then X is a $Q_{\omega+3}$ set, and if $|X| = \omega_3$ then $\mathrm{ord}(X) = \omega_1$.

(9) For any $\alpha \leq \omega_1$ there is a $\underset{\sim}{\Pi}^1_1$ X with $\mathrm{ord}(X) = \omega_1$.

(10) For any $X \subseteq 2^{\omega}$ if $|X| \geq \omega_1$ then there is an X-projective set not Borel in X.

(11) There is no G countable with $\underset{\sim}{\Sigma}^1_1 \subseteq G_{\omega_1}$. (This is a problem of Ulam, see Fund. Math. 30 (1938), 365.)

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*Answered by William Fleissner in the negative (to appear).

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PART II. VAUGHT'S CONJECTURE FOR THEORIES OF ONE UNARY FUNCTION

Vaught's conjecture [1] is that for any countable first order theory T, $\omega(T) \leq \aleph_0$ or $\omega(T) = 2^{\aleph_0}$ where, $\omega(T)$ is the number of nonisomorphic countable models of T.

Let $\mathfrak{A} = (A, R_n)_{n<\omega}$ where each R_n is k_n-ary relation. Define for $x, y \in A$:

$S(x,y)$ iff $x \neq y \wedge \exists n < \omega \exists x_1, \ldots, x_{kn} \in A(R_n(x_1, \ldots x_{kn}) \bigvee_{i,j}(x = x_i \wedge y = x_j))$. (A,S) is the associated graph of $\mathfrak{A}$ (S is a symmetric, irreflexive binary relation). Define a metric $\delta(x,y)$ on A as follows:

$$\delta(a,b) = \text{least } n \ \exists x_0, \ldots, x_n \, (x_0 = a \wedge x_n = b \bigwedge_{i<n} S(x_i, x_{i+1}))$$

∞ if no such n exists.

Define:

(1) a is connected to b iff $\delta(a,b) = n$ some $n < \omega$.

(2) $C \subseteq A$ is a component if it is a maximal connected subset.

(3) A loop is a set of points $\{x_0, \ldots, x_n\}$ with $n > 1$ such that $\bigwedge_{i<n} S(x_i, x_{i+1}) \wedge S(x_n, x_0) \wedge \bigwedge_{i \neq j} x_i \neq x_j$.

(4) $\omega(\mathfrak{A})$ = number of nonisomorphic elementary substructures of $\mathfrak{A}$.

<u>Theorem</u> A. If $\mathfrak{A} = (A, R_n)_{n<\omega}$ is a countable structure, $G = (A,S)$ the associated graph, and every component of G

contains only finitely many loops then $\omega(\mathfrak{A}) \leq \aleph_0$ or $\omega(\mathfrak{A}) = 2^{\aleph_0}$.

Examples of $\mathfrak{A}$ satisfying the hypothesis are:

(1) $\mathfrak{A} = (A,R)$ where R is a binary relation which is a partial function on A.

(2) $\mathfrak{A} = (A,R_n)_{n<\omega}$ where each R_n is a partial function on A and for each n and m, R_n is equal to R_m on their common domain.

(3) If $\mathfrak{A}$ satisfies the hypothesis then so does any extension of $\mathfrak{A}$ by a countable number of unary predicates.

<u>Theorem</u> B. If T is a complete countable theory such that every countable model of T has the property that every component of its associated graph contains only finitely many loops then $\omega(T) = 1$, $\aleph_0$ or $2^{\aleph_0}$.

Theorem B was proved by myself and Leo Marcus [2] independently. Later M. Rubin pointed out that the fact that ($\omega(T) > 1 \rightarrow \omega(T) \geq \aleph_0$) can be obtained as a corollary of a theorem of Lachlan [3] since every such theory is superstable. The author of this fact is unknown to me. Note that if $M \underset{\sim}{\leq} N \models T$ then for any $a,b \in N - M$ if $\hat{c} \in M$ is the "closest" element of M to a,b then

$$\langle N,a,\hat{c}\rangle \equiv \langle N,b,\hat{c}\rangle \longrightarrow \langle N,a,c\rangle_{c\in M} \equiv \langle N,b,c\rangle_{c\in M}.$$

(This is easily shown by using Lemma 1 and Ehrenfeucht games.) But this shows there are at most $2^{\aleph_0} \cdot |M|$ 1-types over M. Similar argument works for n-types. Note that if a countable theory T fails to have an ω-saturated countable model then $\omega(T) = 2^{\aleph_0}$, hence the rest of Theorem B follows from Theorem A by determining $\omega(\mathfrak{A})$ for $\mathfrak{A}$ countable ω-saturated. It is also not hard to show that the number of non-isomorphic elementary extensions of a model satisfying the hypothesis of Theorem A is 1, $\aleph_0$, $2^{\aleph_0}$.

<u>Theorem</u> C. There is a θ a $PC(L_{\omega_1\omega})$ sentence in one unary operation such that $\omega(\theta) = \aleph_1$.

This disproves the main result of Stanley Burris [4] by showing that the quantifier ranks of Scott sentences of a countable unary operation are arbitrarily high. John Steel [5] has proved Vaught's conjecture for $L_{\omega_1\omega}$ sentences in one unary operation.

Matatyahu Rubin proved Vaught's conjecture for theories of a linear order [8] and more recently for $L_{\omega_1\omega}$ sentences of a linear order [9]. In my abstract [11] I mistakenly stated Theorem C for $PC(L_{\omega\omega})$.

Question: Does there exist a $PC(L_{\omega\omega})$ sentence θ in one unary operation with $\omega(\theta) = \aleph_1$?

For any $(L,<)$ a linear order define the following unary operation (U_L,F_L)

$U_L = \{(a_0,\ldots,a_{n-1}): n < \omega \quad a_0 > a_1 > \ldots > a_{n-1} \quad \forall i < n$
$a_i \in L\}$

$F_L(\langle\rangle) = \langle\rangle$

$F_L(\langle a_0,\ldots,a_n\rangle) = \langle a_0,\ldots,a_{n-1}\rangle$

Claim: If $L = L_1 + L_2$ and $\bar{L} = \bar{L}_1 + \bar{L}_2$ are countable linear orders, L_1 and $\bar{L}_1$ are isomorphic well orders, and either L_2 and $\bar{L}_2$ are both empty or they are both non-empty and have no least element then (U_L,F_L) is isomorphic to $(U_{\bar{L}},F_{\bar{L}})$.

Thus $\theta = \{(U,F): \exists L$ countable linear order $\langle U,F\rangle \cong U_L F_L\}$ is $PC(L_{\omega_1\omega})$ and $\omega(\theta) = \aleph_1$. ■

We only prove Theorem A for $\mathfrak{A} = (A,R,\bar{a})$ where R is binary, symmetric, irreflexive; and $\bar{a}$ is finitely many constants, since it is easy to generalize.

Definition: (1) for $\mathfrak{A}$ having a distinguished constant $\underline{0}$ let $\mathfrak{A}_n = \{a \in A: \delta(a,\underline{0}) \leq n\}$.

(2) $\mathfrak{A} \equiv_n \mathfrak{B}$ iff Player II has a winning strategy in the Ehrenfeucht game of length n [6].

Our main lemma is the following, its proof is on p.88.

<u>Lemma</u> 1. If $\mathfrak{A}$ and $\mathfrak{B}$ are connected with distinguished constants then $(\forall n < \omega \ \mathfrak{A}_n \equiv_n \mathfrak{B}_n) \longrightarrow \mathfrak{A} \equiv \mathfrak{B}$.

<u>Lemma</u> 2. If $\forall \mathfrak{C}$ component of $\mathfrak{A}$ countable structure $(\omega(\mathfrak{C}) \leq \aleph_0$ or $\omega(\mathfrak{C}) = 2^{\aleph_0})$, then $\omega(\mathfrak{A}) \leq \aleph_0$ or $\omega(\mathfrak{A}) = 2^{\aleph_0}$.

<u>Proof</u>:

Note that from Lemma 1 if $\mathfrak{B} \leq \mathfrak{A}$, then the components of $\mathfrak{B}$ are elementary substructures of the corresponding components of $\mathfrak{A}$. If $\omega(\mathfrak{C}) = 2^{\aleph_0}$ some $\mathfrak{C}$ a component of $\mathfrak{A}$ then using Ehrenfeucht games we see $\omega(\mathfrak{A}) = 2^{\aleph_0}$. Otherwise let $\{\mathfrak{C}_n : n < \omega\}$ be pairwise nonisomorphic so that $\forall \mathfrak{B} \precsim \mathfrak{C}$ a component of $\exists\, n\, \mathfrak{C}_n \simeq \mathfrak{B}$. For $k: \omega \to \omega + 1$ let $\mathfrak{A}_k$ be a structure (obtained continuously from k) with exactly $k(n)$ copies of $\mathfrak{C}_n$ for each n and universe subset of ω.

$X = \{k \in (\omega+1)^\omega : \mathfrak{A}_k$ can be elementarily embedded into $\mathfrak{A}\}$

X is a $\underset{\sim}{\Sigma}^1_1$ set and $|X| = \omega(\mathfrak{A})$ so by a classical theorem of descriptive set theory [7] $\omega(\mathfrak{A}) \leq \aleph_0$ or $\omega(\mathfrak{A}) = 2^{\aleph_0}$. ■

Note: $(\forall a \in A\ \omega(\langle\mathfrak{A},a\rangle) \le \aleph_0) \longrightarrow \omega(\mathfrak{A}) \le \aleph_0$.

$(\exists a \in A\ \omega(\langle\mathfrak{A},a\rangle) = 2^{\aleph_0}) \longrightarrow \omega(\mathfrak{A}) = 2^{\aleph_0}$.

If $\mathfrak{A}$ is connected and $Y \subseteq A$ is finite and contains all of $\mathfrak{A}$'s loops then define $\mathfrak{A}\{y\}$ for $y \in Y$

$\mathfrak{A}\{y\} = \{a \in A$: a is connected to y by a path which only intersects Y at $y\}$. By Lemma 1 note that for $\mathfrak{B} \subseteq \mathfrak{A}$ $(\langle\mathfrak{B},y\rangle_{y\in Y} \preceq \langle\mathfrak{A},y\rangle_{y\in Y}$ iff $\mathfrak{B}\{y\} \preceq \mathfrak{A}\{y\})$ for $y \in Y$.

Hence it is enough to count the number of elementary substructures of a tree. Define $\mathfrak{A}$ is a tree iff countable, connected, has no loops, and has a distinguished constant $\underline{0}$. From now on all structures are trees until p. 91.

Examples:

I) Let T_n = 

$n \le \omega$

$T_{n,m}$ = 

$n,m \le \omega$

Consider T = 

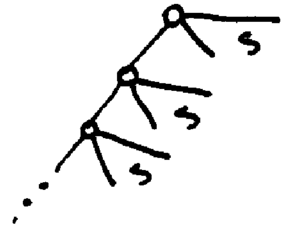
where for each $n,m \le \omega$ infinitely many of the S_i are $T_{n,m}$. $\omega(T) = 2^{\aleph_0}$ is shown by Lemma 3.

II) Let S = 

$\omega(S) = \aleph_0$

a) 

c)



b)

Each of these has $2^{\aleph_0}$ nonisomorphic elementary substructures.

III) To illustrate Lemma 6:

Extend $<$ on ω to $\omega \cup \{\infty\}$ by $n < \infty \ \forall n < \omega$ and $\infty < \infty$. Let $U = \{(a_0, \ldots, a_{n-1}) : n < \omega,$ $a_0 > a_1 > \ldots > a_{n-1},\ a_i \in \omega \cup \{\infty\}\}$. If F is the projection function on U $(F((a_0, \ldots, a_{n+1}))$ $= (a_0, \ldots, a_n))$ then $\omega(F) = \aleph_0$.

<u>Definitions</u>:

(1) a is below b iff b lies on the unique shortest path connecting a to $\underline{0}$.

(2) $\mathfrak{A}(a)$ is the tree with universe $\{b \in A : b$ is below $a\}$ and distinguished constant a.

(3) $P(\mathfrak{A}) = \{a \in A : \delta(a, \underline{0}) = 1\}$ and for $a \in A$ $P(a) = P(\mathfrak{A}(a))$.

(4) for $X \subseteq P(\mathfrak{A})$ $\mathfrak{A}[X]$ is the tree with universe $\underline{0}$ and elements of A below things in X and with

distinguished constant $\underline{0}$.

(5) for $x_n \in P(\mathfrak{A})$ and $y \in P(\mathfrak{A})$($x_n \to y$ iff $x_n \neq x_m$ for $n \neq m$ and $t_p(x_n, \mathfrak{A}) \to typ(y, \mathfrak{A})$, i.e. $\forall\ \psi(v)$ first order $\exists N \forall n \geq N$ $(\mathfrak{A} \models \psi(x_n)$ iff $\mathfrak{A} \models \psi(y))$).

Lemma 3. If $X \dot{\cup} Y = P(\mathfrak{A})$ are disjoint and $\forall y \in Y$ $\exists \langle x_n : n \rangle \in X^\omega$ $x_n \to y$ then $\mathfrak{A}[\mathbf{x}] \leq \mathfrak{A}$.

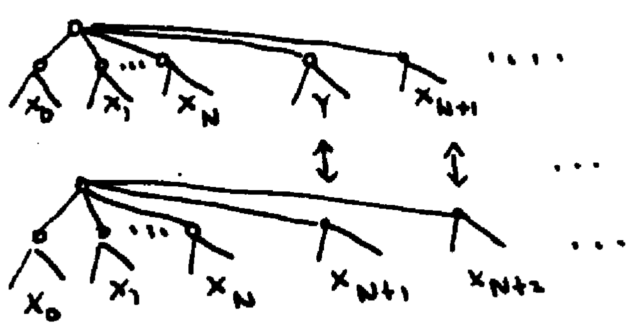
Proof:

It is easy to find $X_y = \{x_n^y : n < \omega\} \subseteq X$ for $y \in Y$ disjoint so that * $x^y \to y$ for each $y \in Y$.

Claim: $\forall n_0 < \omega$ $\forall y \in Y$ $\mathfrak{A}_{n_0}[X_y] \leq \mathfrak{A}_{n_0}[X_y \cup \{y\}]$

Proof:

Let $\mathfrak{B} = \mathfrak{A}_{n_0}$ and $X_y = \{x_n : n < \omega\}$. Clearly * holds for $\mathfrak{B}$ in place of $\mathfrak{A}$, hence we know from the basic lemma on Ehrenfeucht games [6] that $\forall n < \omega\ \exists N < \omega$ $\forall m > N\ \mathfrak{B}(x_m) \equiv_n \mathfrak{B}(y)$. Given $\vec{a} \subseteq \mathfrak{B}[X_y]$ and $n_1 < \omega$, choose N sufficiently large so that $\vec{a} \subseteq \mathfrak{B}[\{x_n : n < N\}]$ and for $m > N$ $\mathfrak{B}(x_m) \equiv_{n_1} \mathfrak{B}(y)$. Now patch together appropriate strategies for Player II as follows:



Claim ■

From Lemma 1 and Claim, $\mathfrak{A}[X_y] \preceq \mathfrak{A}[X_y \cup \{y\}]$ for each $y \in Y$, hence by an easy Ehrenfeucht game argument $\mathfrak{A}[X] \preceq \mathfrak{A}$. ■

Definition: $\mathfrak{A}$ is simple iff $\forall a \in A$ only finitely many nonprincipal types in $Th(\mathfrak{A}(a))$ are realized in $P(a)$.

Note: By using Lemma 3 if $\mathfrak{A}$ is not simple then $\omega(\mathfrak{A}) = 2^{\aleph_0}$.

Definition: Given $(\mathfrak{B}_a : a \in A)$ such that $\mathfrak{B}_a \leq \mathfrak{A}(a)$ for each a the fusion of $(\mathfrak{B}_a : a \in A)$ is the tree $\mathfrak{B}$ with $\underline{0}^{\mathfrak{B}} = \underline{0}^{\mathfrak{A}}$ and universe $\{b$: for all a between $\underline{0}$ and b, $b \in |\mathfrak{B}_a|\}$.

Lemma 4. Given $(\mathfrak{B}_a : a \in A)$ with $\mathfrak{B}_a \preceq \mathfrak{A}(a)$ all a and $\mathfrak{B}$ the fusion then $\mathfrak{B} \preceq \mathfrak{A}$.

proof:

By Lemma 1 we may assume $\mathfrak{A} = \mathfrak{A}_n$ for some $n < \omega$. Now prove it by induction on n. Thus $\mathfrak{B}(b) \preceq \mathfrak{A}(b)$ for all $b \in P(\mathfrak{A})$, hence $\mathfrak{B}(b) \preceq \mathfrak{B}_{\underline{0}}(b)$ $\forall b \in P(\mathfrak{B}_{\underline{0}})$ and by an easy Ehrenfeucht game argument $\mathfrak{B} \preceq \mathfrak{B}_{\underline{0}} \preceq \mathfrak{A}$. ■

Definition: If $\mathfrak{A}$ is simple let $\mathfrak{B}_a^{P_r} = \mathfrak{A}(a)[\{x: tp(x, \mathfrak{A}(a))$ is principal$\}]$ for each $a \in A$, and $\mathfrak{A}^{P_r}$ be the fusion of $\langle \mathfrak{B}_a^{P_r} : a \in A \rangle$. By Lemma 3 $\mathfrak{B}_a^{P_r} \preceq \mathfrak{A}(a)$

and by Lemma 4 $\mathfrak{A}^{p_r} \preceq \mathfrak{A}$.

<u>Lemma</u> 5. If $\mathfrak{A}^{p_r} = \mathfrak{A}$ then $\omega(\mathfrak{A}) = 1$.

The proof is straightforward and left to the reader. $\blacksquare$

<u>Definitions</u>:

$N(a) = \{x \in P(\mathfrak{A}(a)) : tp(x, \mathfrak{A}(a))$ is nonprincipal$\}$

$L = \{a \in A : N(a) \neq \emptyset\}$

$T = \{b \in A : \exists a \in L$ b lies on the unique shortest path connecting a to $\underline{0}\}$.

<u>Lemma</u> 6. If $L = \{a_n : n < \omega\}$ and $\forall n\ N(a_n) = \{b_n\}$ and $a_{n+1} \in \mathfrak{A}(b_n)$ then $\omega(\mathfrak{A}) \leq \aleph_0$.

<u>Proof</u>:

Let for each $n < \omega$ $\mathfrak{B}_n = \mathfrak{A} - \mathfrak{A}(b_n)$ then these are all the nonisomorphic elementary substructures of $\mathfrak{A}$. $\blacksquare$

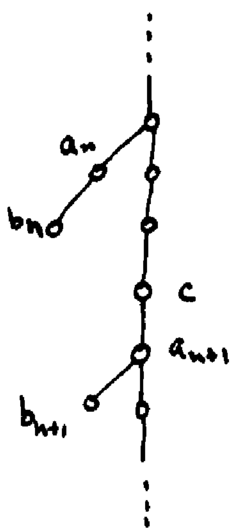
<u>Definition</u>: (1) $[T]$ is the set of infinite branches of T.

(2) $a \in A$ isolates $f \in [T]$ iff $\mathfrak{A}(a)$ is as in the hypothesis of Lemma 6 with $a \in f$.

<u>Lemma</u> 7. If $\mathfrak{A}$ is simple and $\exists f \in [T]$ such that no $a \in A$ isolates f then $\omega(\mathfrak{A}) = 2^{\aleph_0}$.

Proof:

Choose $a_n \in L$ and $b_n \in N(a_n)$ for $n < \omega$ as follows: Having chosen them for $m < n$, let c be any element of f lower than any of the a_m and b_m for $m < n$. Since c does not isolate f $\exists a_n \in \mathfrak{A}(c) \wedge L\ \exists b_n \in N(a_n)(b_n \notin f)$.



Let $B = \{c\colon c$ is strictly between some b_n and $\underline{0}\}$

For $a \notin B$ let $\mathfrak{B}_a = (\mathfrak{A}(a))^{P_r}$

for $a \in B$ let $\mathfrak{B}_a = \mathfrak{A}(a)[X_a]$ where

$X_{a_n} = \{x\colon tp(x, \mathfrak{A}(a_n))$ is principal$\}$ $\cup \{b_n\}$

$X_d = \{x\colon tp(x, \mathfrak{A}(c))$ is principal$\}$ $\cup \{P(\mathfrak{A}(c)) \wedge B\}$ if $d \neq a_n$ any $n < \omega$.

If $\mathfrak{C}$ is the fusion of the $\mathfrak{B}_a$'s then $\mathfrak{C} \leq \mathfrak{A}$. For any $n < \omega$ note that **at most two** $x \in C$ such that $\delta(x,\underline{0}) = \delta(a_n,\underline{0})$ and $N(x)^{\mathfrak{C}} \neq \emptyset$. For any $X \subseteq \omega$ let $\mathfrak{C}_X \leq \mathfrak{C}$ be gotten by fusion so that $\forall n < \omega(b_n \in |\mathfrak{C}_X|$ iff $n \in X)$.

$(X \neq X' \longrightarrow \mathfrak{C}_X \neq \mathfrak{C}_{X'})$. ■

<u>Lemma</u> 8. If $\forall a \in P(\mathfrak{A})\ \omega(\mathfrak{A}(a)) \leq \aleph_0$ or $\omega(\mathfrak{A}(a)) = 2^{\aleph_0}$ then $\omega(\mathfrak{A}) \leq \aleph_0$ or $\omega(\mathfrak{A}) = 2^{\aleph_0}$.

The proof of this is similar to the proof of Lemma 2. ■

Lemma 9. If $\mathfrak{A}$ is a tree then $\omega(\mathfrak{A}) \le \aleph_0$ or $\omega(\mathfrak{A}) = 2^{\aleph_0}$.

Proof:

If $\mathfrak{A}$ is not simple then $\omega(\mathfrak{A}) = 2^{\aleph_0}$ by using Lemma 3. Define $D(T) = \{x \in T\colon x$ does not isolate any $f \in [T]\}$. By Lemma 7 if $D(T)$ is not well founded $([D(T)] \ne \emptyset)$ then $\omega(\mathfrak{A}) = 2^{\aleph_0}$. If $D(T) = \emptyset$ then by Lemma 5 or 6 $\omega(\mathfrak{A}) \le \aleph_0$. Hence we may assume $D(T)$ is well-founded and then the Lemma is proved by induction on the rank of $D(T)$ by using Lemma 8. ■

It remains only to prove Lemma 1. We no longer consider just trees.

Lemma 10. If $\mathfrak{A}$ is connected with distinguished constant then $\forall n < \omega\ \forall \phi(\bar{x},\bar{y})\ \exists N \ge n\ \ N < \omega\ \exists \Gamma$ finite $\forall \vec{a} \in \mathfrak{A} - \mathfrak{A}_N\ \exists \phi^*(\vec{y}) \in \Gamma\ \forall \vec{b} \in \mathfrak{A}_n(\mathfrak{A} \models \phi(\vec{a},\vec{b})$ iff $\mathfrak{A}_N \models \phi^*(\vec{b}))$.

Proof:

The proof is by induction on the logical complexity of $\phi(\vec{x},\vec{y})$. For the atomic case put $N = n + 2$ and $\Gamma = \{T, F, x_1 = x_2, R(x_1,x_2)\}$. On the induction step $\neg$, $\wedge$ are both easy.

$\exists z\ \phi(\vec{x},z,\vec{y})$

By induction $\exists \Gamma_1\ \exists N_1 \geq n$ such that $\forall a\ \vec{a} \in \mathfrak{A} - \mathfrak{A}_{N_1}$ $\exists \sigma(\vec{y}) \in \Gamma_1\ \forall \vec{b} \in \mathfrak{A}_n\ (\mathfrak{A} \models \phi(\bar{a},a,\bar{b})$ iff $\mathfrak{A}_{N_1} \models \sigma(\bar{b})$.

Also by induction $\exists \Gamma_2\ \exists N_2 \geq N_1$ such that $\forall \vec{a} \in \mathfrak{A} - \mathfrak{A}_{N_2}\ \exists \tau(z,\vec{y}) \in \Gamma_2\ \forall b\ \vec{b} \in \mathfrak{A}_{N_1}$ $(\mathfrak{A} \models \phi(\bar{a},b,\bar{b})$ iff $\mathfrak{A}_{N_2} \models \tau(b,\vec{b}))$. Let $N = N_2$ and $\Gamma = \{\bigvee_{\sigma\in F} \sigma^{N_1}(\vec{y}) \vee \exists z \in \mathfrak{A}_{N_1} \tau(z,\vec{y}) : F \subseteq \Gamma_1, \tau \in \Gamma_2\}$. These work since given $\vec{a} \in \mathfrak{A} - \mathfrak{A}_{N_2}$ let $F = \{\sigma(\vec{y}) \in \Gamma_1 : \exists a \in \mathfrak{A} - \mathfrak{A}_{N_1}\ \forall \vec{b} \in \mathfrak{A}_n (\mathfrak{A} \models \phi(\vec{a},a,\bar{b})$ $\leftrightarrow \mathfrak{A}_{N_1} \models \sigma(\vec{b})\}$ and $\tau(z,\vec{y})$ so $\forall b\ \vec{b} \in \mathfrak{A}_{N_1} (\mathfrak{A} \models \phi(\bar{a},b,\bar{b}) \leftrightarrow \mathfrak{A}_{N_2} \models \tau(b,\bar{b}))$. Let $\phi^*(\vec{y}) = \bigvee_{\sigma\in F} \sigma^{N_1}(\vec{y}) \vee \exists z \in \mathfrak{A}_{N_1} \tau(z,\vec{y})$. ■

<u>Remark</u>: Lemma 10 was motivated by the main lemma in Feferman-Vaught [10].

<u>Lemma</u> 11. If $\mathfrak{A}$ is connected with distinguished constant then $\forall \phi(x,\vec{y})\ \forall n < \omega\ \exists N < \omega\ \forall \bar{b} \in \mathfrak{A}_n$ if $\mathfrak{A} \models \exists x \phi(x,\bar{b})$ then $\exists a \in \mathfrak{A}_N\ \mathfrak{A} \models \phi(a,\bar{b})$.

Proof:

Let N_1, Γ be from Lemma 10 for $\phi(x,\vec{y})$ and n. Define: $\phi^*(\vec{y}) \in \Gamma$ is a testing formula for $a \in \mathfrak{A} - \mathfrak{A}_{N_1}$ if $\forall \vec{b} \in \mathfrak{A}_n (\mathfrak{A} \models \phi(a,\bar{b}) \leftrightarrow \mathfrak{A}_{N_1} \models \phi^*(\bar{b}))$. Choose $N \geq N_1$, $N < \omega$ so that $\forall a \in \mathfrak{A} - \mathfrak{A}_{N_1}$ if $\phi^*(\vec{y}) \in \Gamma$ is a testing formula for a then there exists $\exists a' \in \mathfrak{A}_N$ so that

$\phi^*(\bar{y})$ is a testing formula for a'. This N works because $\mathfrak{A} \models \phi(a,\bar{b}) \leftrightarrow \mathfrak{A}_{N_1} \models \phi^*(\bar{b}) \leftrightarrow \mathfrak{A} \models \phi(a',\vec{b})$ some $a' \in \mathfrak{A}_N$ with same testing formula $\phi^*(\bar{y})$ as a. ∎

Lemma 12. If $\mathfrak{A}$ is connected with a distinguished constant and $\mathfrak{A} \equiv \mathfrak{B}$ then $\bigcup_{n\in\omega} \mathfrak{B}_n \leq \mathfrak{B}$.

Proof:

If $\vec{b} \in \mathfrak{B}_n$ and $\phi(x,\bar{y})$ are given then taking $N < \omega$ from Lemma 11, $\mathfrak{A} \models$ " $\forall \bar{y} \in \mathfrak{A}_n (\exists x \phi(x,\bar{y}) \leftrightarrow \exists x \in \mathfrak{A}_N \phi(x,\bar{y}))$". So if $\mathfrak{B} \models \exists x \phi(x,\bar{b})$ then $\exists b \in \mathfrak{B}_N \mathfrak{B} \models \phi(b,\bar{b})$. By Tarski's criterion we are done. ∎

Let $(HC,\varepsilon) \precsim M$ such that ω^M is nonstandard. We assume $\mathfrak{A}, \mathfrak{B} \in HC$. Let $\mathfrak{A}^*$ be the structure determined by M corresponding to $\mathfrak{A}$ and $\mathfrak{A}^*_{st} = \bigcup_{n<\omega} \mathfrak{A}^*_n$. Let $n^* \in \omega^M - \omega$ and $M \models$ "s is a strategy for player II in the Ehrenfeucht game of length n^* played between $\mathfrak{A}^*_{n^*}$ and $\mathfrak{B}^*_{n^*}$". Since n^* is nonstandard the strategy s gives a back and forth property to show $\mathfrak{A}^*_{st} \equiv \mathfrak{B}^*_{st}$ (if player I plays $a \in \mathfrak{A}^*_{st}$ then s must respond with $b \in \mathfrak{B}^*_{st}$). By Lemma 12 $\mathfrak{A}^*_{st} \precsim \mathfrak{A}^*$ and $\mathfrak{B}^*_{st} \leq \mathfrak{B}^*$ and also $\mathfrak{A} \leq \mathfrak{A}^*$ and $\mathfrak{B} \leq \mathfrak{B}^*$ so $\mathfrak{A} \equiv \mathfrak{B}$. ∎

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PART III. THERE ARE NO Q-POINTS IN LAVER'S MODEL FOR THE BOREL CONJECTURE

All ultrafilters are assumed nonprincipal and on ω.

Define:

(1) U q-point (also called rare [C]) iff $\forall (P_n : n < \omega)$ a partition of ω into finite sets $\exists A \in U \forall n |A \cap P_n| \leq 1$.

(2) U p-point iff $\forall (P_n : n < \omega)$ partition of ω either $\exists n\ P_n \in U$ or $\exists A \in U \forall n |A \cap P_n|$ finite.

(3) U is Selective (also called Ramsey) iff $\forall (P_n : n < \omega)$ partition of ω either $\exists n\ P_n \in U$ or $\exists A \in U \forall n |A \cap P_n| \leq 1$.

(4) U is semiselective iff Given $A_n \in U\ \exists f \in \omega^\omega$ $\forall n\ f(n) \in A_n$ and $f''\omega \in U$.

(5) U is semi q-point (also called rapid [C]) iff $\forall f \in \omega^\omega \exists g \in \omega^\omega \forall n\ f(n) < g(n)$ and $g''\omega \in U$.

It is easily seen:

Selective = p-pt + q-pt

semiSelective = p-pt + semi q-pt

Define: $f < g$ iff $\exists n \forall m > n(f(m) < g(m))$

$\mathcal{F} \subseteq \omega^\omega$ is dominant family iff $\forall f \in \omega^\omega \exists g \in \mathcal{F}\ f < g$.

<u>Theorem</u> (1) (Ketonen [Ke]) If $\forall \mathcal{F}$ dominant $|\mathcal{F}| = 2^{\aleph_0}$ then $\exists$ a p-pt.

(2) (Mathias, Taylor) If $\exists \mathcal{F}$ dominant $|\mathcal{F}| = \aleph_1$ then $\exists$ a q-pt.

Kunen [Ku1] showed that adding $\aleph_2$ random reals to a model of ZFC + GCH gives a model with no semiselective ultrafilters. More recently he showed [Ku2] that if you first add $\aleph_1$ Cohen reals (then the random reals) then the resulting model has a p-pt. In either case one has a dominant family of size $\aleph_1$ so there is a q-pt.

The following are equivalent:

(1) U is semi q-pt.

(2) $\forall \langle P_n: n < \omega\rangle (\forall n\ P_n$ finite) $\rightarrow \exists A \in U\, \forall n |A \cap P_n| \leq n$.

(3) $\exists h \in \omega^\omega\, \forall \langle P_n: n < \omega\rangle (\forall n\ P_n$ finite) $\rightarrow \exists A \in U$ $\forall n |A \cap P_n| \leq h(n)$.

Proof.

1) $\Longrightarrow$ 2) Let $f(n) = \sup(\bigcup_{m \leq n} P_m) + 1$. Suppose $g(n) > f(n)$ all n then $P_n \cap g''\omega \subseteq \{g(0), \ldots g(n-1)\}$.

3) $\longrightarrow$ 1) Assume f increasing. Choose $n_0 < n_1 < n_2 < \ldots$ so that $h(k+1) < n_k$. Let $P_k = f(n_k)$ and let $Y \in U$ be so that $|Y \cap P_k| \leq h(k)$. Then for each $m \geq n_0$ $|Y \cap f(m)| < m$, since if $n_k \leq m < n_{k+1}$ then $|Y \cap f(n_{k+1})| \leq h(k+1) < n_k \leq m$. Hence if $g \in \omega^\omega$ enumerates $Y - f(n_0+1)$ in increasing order then $\forall n\ f(n) < g(n)$. ■

Define $U \times V = \{A \subseteq \omega \times \omega: \{n: \{m: (n,m) \in A\} \in V\} \in U\}$. Whilst $U \times V$ is never a p-pt. or a q-pt. nevertheless:

$U \times V$ is semi q-pt. iff V is semi q-pt.

Proof.

upper diagonal in $U \times V$

($\Longrightarrow$)

Given $P_k \subseteq \omega$ finite let $P_k^* = \{\langle n,m\rangle : m \in P_k \wedge n \le m\}$. Choose $Z \in U \times V$ so that $\forall k |Z \cap P_k^*| \le k$. Let $n \in \omega$ so that $Y = \{m \ge n : (n,m) \in Z\} \in V$ then $\forall k |Y \cap P_k| \le k$. (More generally $f_* U = V$ and U semi-q-pt. and f finite to one then V semi q-pt.)

($\Longleftarrow$)

Given $P_k \subseteq \omega \times \omega$ finite choose n_k increasing so that $P_k \subseteq n_k^2$. Let $Y \in V$ so that $\forall k |n_k \cap Y| \le k$. Let $Z = \bigcup_{k<\omega} (\{k\} \times \{m : m \in Y \wedge m \ge n_k\})$ then $Z \cap P_k \subseteq Z \cap n_k^2 \subseteq k \times (n_k \cap Y)$ which has cardinality $\le (k+1)^2$.

Theorem. In Laver's model N for the Borel conjecture [L] there are no semi q-pt's.

Proof.

Some definitions from [L]:

(1) $T \in \mathcal{J}$ iff T subtree of $\omega^{<\omega}$ with the property that

$\exists\, s_T \in T$ (called the stem of T) so that $\forall t \in T$ $t \subseteq s_T$ or $s_T \subseteq t$ and for all $t \geq s_T (t \in T) \longrightarrow$ there are infinitely many $\hat{t} \in T$ immediately below t (t is immediately below $s = (k_0,\ldots,k_n)$ iff $\exists\, k_{n+1}$ $t = (k_0,\ldots,k_n,k_{n+1})$) $\hat{T} \geq T$ iff $\hat{T}$ is a subtree of T.

(2) $T_s = \{t \in T: s \subseteq t \vee t \subseteq s\}$.

(3) $T \,{}^0\!\geq \hat{T}$ iff $T \geq \hat{T}$ and they have the same stem.

<u>Lemma</u> 1. Given $T \in \mathcal{J}$ and for each $s \in T - \{\emptyset\}$ $F_s \subseteq [k_n,k_{n+1}) = \{x: k_n \leq x < k_{n+1}\}$ where $s = (k_0,\ldots,k_n,k_{n+1})$ $(F_{\langle n\rangle} \subseteq [0,n])$ and $\forall s \in T\, \exists N < \omega\, \forall t$ immediately below s in T $|F_t| \leq N$. Then letting $H_{\hat{T}} = \bigcup_{s \in \hat{T}} F_s$ for any $\hat{T} \geq T$, we can find $T_0, T_1 \,{}^0\!\geq T$ so that $H_{T_0} \wedge H_{T_1}$ is finite.

<u>Proof</u>.

We may as well assume stem of T is $\emptyset$.

Given Q any infinite family of sets of cardinality $\leq N < \omega$ there exists G, $|G| \leq N, \exists \hat{Q} \subseteq Q$ infinite so that $\forall F, \hat{F} \in \hat{Q}$ $(F \wedge \hat{F}) \subseteq G$. Now trim T to obtain $\hat{T} \geq T$ and $\forall s \in \hat{T}\, \exists G_s \subseteq [k_n,\omega]$ finite $(s = (k_0,\ldots,k_n))$ and for all $t, \hat{t}$ immediately below s in $\hat{T}$, $(F_t \wedge F_{\hat{t}}) \subseteq G_s$. Build two sequences of finite subtrees of $\hat{T}$:

$T^0_n \subseteq T^0_{n+1} \ldots$

$T^1_n \subseteq T^1_{n+1} \ldots$

so that $[\bigcup_{s\in T_n^0}(F_s \cup G_s)] \cap [\bigcup_{s\in T_n^1}(F_s \cup G_s)] \subseteq G_\phi$

and $\bigcup_{n<\omega} T_n^i = T^i \geq \hat{T}$ for $i = 0,1$

This is done as follows: Suppose we have T_n^0, T_n^1 and we're presented with $s \in T_n^0$ and asked to add an immediate extension of s to T_n^0. Then since

$\{F_t - G_s : t$ immediately below s in $\hat{T}\}$ is a family of disjoint sets and $G_t \subseteq [k_n, \omega]$ where $t = (k_0, \ldots, k_n)$ we can find infinitely many t immediately below s in $\hat{T}$ so that

$$[(F_t - G_s) \cup G_t] \cap [\bigcup_{s\in T_n^1}(F_s \cup G_s)] = \phi$$

■

The above is a double fusion argument.

Some more definitions from [L]:

(1) Fix a natural ω-ordering of $\omega^{<\omega}$ and for any $T \in \mathfrak{J}$ transfer it to $\{t: t \supseteq s_T \wedge t \in T\}$ in a canonical fashion. $\hat{T}^n \geq T$ means the first n elements in this order on $\{t: t \supseteq s_T \wedge t \in T\}$ are still in $\hat{T}$.

(2) The p.o. $\mathbb{P}_{\omega_2}$ is the ω_2 iteration of $\mathfrak{J}$ with

countable support ($p \restriction_\alpha \Vdash$ "$p(\alpha) \in \mathcal{J}^{M[G_\alpha]}$" all α and support $(p) = \{\alpha : p(\alpha) \neq \omega^{<\omega}\}$ is countable).

(3) For K finite and $n < \omega$

$p \overset{n}{\underset{K}{\geq}} q$ iff $[p \geq q \; \forall \alpha \in K \; p \restriction_\alpha \Vdash$ "$p(\alpha) \overset{n}{\geq} q(\alpha)$"].

<u>Lemma</u> 2. Let f be a term denoting the first Laver real and τ any term. If $p \in \mathbb{P}_{\omega_2}$, $p \Vdash$ "$\tau \in \omega^\omega \wedge \forall n \; f(n) < \tau(n) \wedge \tau$ increasing" then $\exists Z_0, Z_1$, $Z_0 \cap Z_1$ finite, $\exists p_0, p_1 \geq p$ and $p_i \Vdash$ "$\tau''\omega \subseteq Z_i$" for $i = 0,1$.

<u>Proof</u>.

Construct a sequence $p \leq^0_{K_0} p_n \leq^0_{K_n} p_{n+1}$ so that $\bigcup_{n<\omega} K_n = \bigcup_{n<\omega} \text{support}(p_n)$. $0 \in K_0$.

Having gotten p_n let $s = (k_0, \ldots, k_m)$ be the n^{th} member of $\{t \in p_n(0) : t \subseteq$ the stem of $p_n(0)\}$ ($s = p_n(0) \langle n \rangle$ in Laver's notation).

Fix $t = (k_0, \ldots, k_m, k_{m+1})$ in $p_n(0)$. Then for each $i \leq m + 1$

$p_t = \langle p_n(0)_t \frown p_n \restriction [1, \omega_2) \rangle \Vdash$ "$\tau(i) \geq k_{m+1} \vee \bigvee_{\ell < k_{m+1}} \tau(i) = \ell$"

Hence by applying Lemma 6 of [L] $m + 2$ many times we can find $q_t \overset{n}{\underset{K_n}{\geq}} p_t$ and $F_t \subseteq [k_m, k_{m+1}]$ such that $|F_t| \leq (m + 2)(n + 1)^{|K_n|}$ and

$q_t \Vdash$ "$\tau''\omega \cap [k_m, k_{m+1}] \subseteq F_t$".

(Note $p_t \Vdash$ "$\forall i \geq m + 1 \; \tau(i) > k_{m+1}$").

Let $p_{n+1}(0) = (p_n(0) - p_n(0)_s) \cup \bigcup \{q_t(0) : t$ immediately below s in $p_n(0)\}$.

Let p_{n+1} $[1,\omega_2)$ be a term denoting

$$q_t \restriction [1,\omega_2) \quad \text{if} \quad q_t(0)$$
$$p_n \restriction [1,\omega_2) \quad \text{if} \quad p_n(0) - \{t: s \leq t\}$$

so $p_{n+1} \overset{n}{K_n} \geq p_n$.

Now let $\hat{p}$ be the fusion of the sequence of p_n (see Lemma 5 [L]).

Then for each $t \in \hat{p}(0)$ if $t = \langle k_0,\ldots,k_m,k_{m+1}\rangle \wedge t \supseteq$ stem $\hat{p}(0)$ then

$\langle \hat{p}(0)_t^{\frown} \hat{p}\restriction[1,\omega_2)\rangle \Vdash$ "τ"$\omega \cap [k_n,k_{n+1}) \subseteq F_t$".

For $t \in \hat{p}(0) \wedge t \subsetneq$ stem $\hat{p}(0)$ let $F_t = k_{m+1}$.

Applying Lemma 1 obtain $T_0,T_1 \geq \hat{p}(0)$ Z_0,Z_1, $Z_0 \cap Z_1$ finite

$\langle T_i^{\frown} p\restriction[1,\omega_2] \rangle \Vdash$ "τ"$\omega \subseteq Z_i$" $i = 0,1$. ■

Proof of the Theorem:

Suppose $M[G_{\omega_2}] \models$ "U is a semi q-pt."

Applying an argument of Kunen's we get $\alpha < \omega_2$

$U \cap M[G_\alpha] \in M[G_\alpha]$.

($M[G_\beta] \models$"CH" all $\beta < \omega_2$ so construct using ω_2-c.c.

$\alpha_\lambda < \omega_2$, $\lambda < \omega_1$ so that $\forall x \in M[G_{\alpha_\lambda}] \cap 2^\omega$, $\mathbb{P}_{\alpha_{\lambda+1}}$ decides "$x \in U$". Let $\alpha = \sup \alpha_\lambda$. Note

$M[G_\alpha] \cap 2^\omega = \bigcup_{\beta<\alpha} M[G_\beta] \cap 2^\omega$ since $\aleph_1$ is not collapsed.)

By Lemma 11 [L] we may assume $U \cap M \in M$. But Lemma 2 clearly implies that for any V ult. in M, $M[G_{\omega_2}] \models$ "no extension of V is a q-pt". ■

Remarks:

1) A similar argument shows that in model gotten by ω_2 iteration of Mathias forcing with countable support there are no semi-q-pt's.

2) In [M] Mathias shows $(\omega \to (\omega)^{\omega}) \longrightarrow$ (no rare filters or non-principal ultrafilters).

3) In neither the Laver or Mathias models are there small dominant families so by Ketenon [Ke] $\exists$ p-pt's. Also it is easily shown no ultrafilter is generated by fewer than $\aleph_2$ sets.

Conjecture: Borel conjecture $\longleftrightarrow \neg \exists$ semi q-point in $\beta N-N$.

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PART IV. MISCELLANEOUS

§1. Universal clopen sets, Wadge degrees, and ω-Boolean operations

Given $B \subseteq \omega^\omega \times \omega^\omega$ define $B_x = \{y: (x,y) \in B\}$. B is said to be universal for clopen sets $(\underset{\sim}{\Delta}^0_1)$ iff

$\forall x \; B_x \in \underset{\sim}{\Delta}^0_1$ and $\forall A \in \underset{\sim}{\Delta}^0_1 \; \exists x (A = B_x)$.

What is the simplest B universal for clopen sets? The reader is obliged to guess before reading on. (For example good choices seem like: open, difference of closed sets, $F_\delta(\underset{\sim}{\Sigma}^0_2)$, etc.)

The complement of a set universal for clopen sets is also. Here is a Π^1_1 definition. Let $A \subseteq \omega^\omega \times \omega^\omega$ be open and universal for open sets.

$(x,y) \in B$ iff $\forall z ((z \in A_{(x)_0} \leftrightarrow z \notin A_{(x)_1}) \wedge y \in A_{(x)_0})$

($(x)_i$ recursive uncodings).

<u>Theorem</u> 1. On the other hand no Borel set is universal for clopen sets.

<u>Proof</u>.

For $C \subseteq \omega^\omega$ and $s \in \omega^{<\omega}$ ($s \in Fr(C)$ iff $\exists y, z$ $y,z \supseteq s (y \in C \wedge z \notin C)$).

$A = \{T \subseteq \omega^{<\omega}: \exists x \forall s \in T \; (s \in Fr(B_x))$ and T closed under subseq.$\}$. If B were Borel and universal for clopen sets then A would be a $\underset{\sim}{\Sigma}^1_1$ set of well founded trees of arbitrarily high rank, contradicting the boundedness theorem. ■

I don't know the answer for sets universal for $\underset{\sim}{\Delta}^0_\alpha$ sets $2 < \alpha < \omega_1$. Harrington has proved Theorem 1 for $\underset{\sim}{\Delta}^0_2$ sets. Similar questions are settled by C.A. Rogers [1] and Kechris-Martin [2].

Define: $A \leq_w B$ for $A \subseteq X$, $B \subseteq Y$, X,Y topological spaces (Wadge [3]) iff $\exists f: X \to Y$ continuous and $f^{-1}(B) = A$.

Given $T \subseteq 2^\omega$ (truth table) define the ω-boolean operation $\Gamma_T: (P(X))^\omega \to P(X)$ for any X by

$$(x \in \Gamma_T((A_n: n < \omega)) \text{ iff } \{n: x \in A_n\} \in T)$$

where we identify 2^ω with $P(\omega)$.

Some examples of ω-boolean operations are countable union, operation $\mathcal{A}$, R-operations of Kolmogorov [5], and the Borel game operations of Burgess [6].

Define $\mathcal{C}_T = \{A \subseteq 2^\omega: \exists(B_n: n < \omega)$ each B_n clopen and $\Gamma_T(B_n: n < \omega) = A$.

<u>Theorem</u> 2. For any $T \subseteq 2^\omega$ $\mathcal{C}_T = \{A \subseteq 2^\omega: A \leq_w T\}$.

<u>Proof.</u>

$(\supseteq)$

Define $A_n \subseteq 2^\omega$ by $\alpha \in A_n$ iff $\alpha(n) = 1$. The A_n's are clopen. Suppose $B \leq_w T$ via continuous function $f: 2^\omega \to 2^\omega$ and $B_n = f^{-1}(A_n)$. Then each B_n is clopen and

$$(\beta \in \Gamma_T < B_n: n < \omega >) \leftrightarrow (\{n: \beta \in B_n\} \in T) \leftrightarrow (\{n: f(\beta) \in A_n\} \in T)$$
$$\leftrightarrow (f(\beta) \in T) \leftrightarrow (\beta \in B), \text{ hence } B \in \sigma_T.$$

($\subseteq$)

Let $(B_n: n < \omega)$ be an ω sequence of clopen sets. We must use the following fact due to Wadge:

Fact: for $A, B \subseteq 2^\omega$, $A \leq_W B$ iff player II has a winning strategy in $G(A,B)$. $G(A,B)$ is the game where player I and player II alternatingly write down 0 or 1 creating two maps $\alpha: \omega \to 2$ and $\beta: \omega \to 2$ respectively. On his moves player II may elect to pass but he must play infinitely often if I does. Player II wins a particular play (α, β) iff ($\alpha \in A$ iff $\beta \in B$).

<u>Claim</u>: II wins $G(\Gamma_T(B_n: n < \omega), T)$.

<u>Proof</u>.

I	II
$\alpha(0)$	$\beta(0)$
$\alpha(1)$	$\beta(1)$
$\vdots$	$\vdots$
$\alpha(n)$	

II waits until either $[\alpha \restriction n] \subseteq B_0$ or $[\alpha \restriction n] \cap B_0 = \phi$ then plays 1 or 0 accordingly. Since B_0 is clopen he will not have to wait indefinitely. If he continues to play in this fashion he produces a play $\beta \in 2^\omega$ so that

$\alpha \in B_n$ iff $\beta(n) = 1$, thus

$\alpha \in \Gamma_T \langle B_n: n < \omega \rangle \leftrightarrow \{n: \alpha \in B_n\} \in T \leftrightarrow \beta \in T$

$\blacksquare$

This theorem was proved by myself and Lon Radon. Other similar questions for ω^ω in place of 2^ω and open in place of clopen are answered by Steel [8] and Van Wesep [9].

The next question I consider is whether or not there is a natural hierarchy for the $\underset{\sim}{\Delta}^1_2$ subsets of ω^ω. The only results known are negative, for example Moschovakis [10]. Note that if $\{T\} \cup \{A_n : n < \omega\} \subseteq \underset{\sim}{\Delta}^1_2$ then $\Gamma_T(A_n : n < \omega) \in \underset{\sim}{\Delta}^1_2$. In general for $\mathcal{C} \subseteq P(\omega^\omega)$ let $\mathcal{C}^*$ be the least contaning $\mathcal{C}$ and if $\{T\} \cup \{A_n : n < \omega\} \subseteq \mathcal{C}^*$ then $\Gamma_T(A_n : n < \omega) \in \mathcal{C}^*$. (Note $((\underset{\sim}{\Delta}^0_1)^* = \underset{\sim}{\Delta}^0_1, (\underset{\sim}{\Sigma}^0_1)^* = \underset{\sim}{\Delta}^1_1)$. Using the method of Kunugie [11] we prove:

<u>Theorem</u> 3. Suppose $\mathcal{C} = \{A : A \leq_W B\}$ where $B \in \underset{\sim}{\Delta}^1_2$ then $\exists\, C \in \underset{\sim}{\Delta}^1_2$ $\mathcal{C}^* \subseteq \{A : A \leq_W C\}$.

pf

Define $U \subseteq \omega^\omega \times \omega^\omega$ by $(x,y) \in U$ iff $f_y(x) \in B$.

$f_y : \omega^\omega \to \omega^\omega$ is the continuous function $\underset{\sim}{\Delta}^1_1$ coded by y.

Then U is $\underset{\sim}{\Delta}^1_2$, and $\forall A \in \mathcal{C}\ \exists y \in \omega^\omega\ U_y = A$.

Define $x = (T,f)$ is a code iff $T \subseteq \omega^{<\omega}$ is a well-founded normal tree and $f : \{s \in T : |s|_T = 0\} \to \omega^\omega$.

Define C_x for $\sigma \in T$ as follows:

$|\sigma|_T = 0$ then $C_x = U_{f(\sigma)}$

$|\sigma|_T > 0$ then $C^\sigma_x = \Gamma_{C_x^{\sigma\frown 0}}(C_x^{\sigma\frown i+1} : i < \omega)$.

Define $P(x,y)$ iff "x is a code and $y \in C^\phi_x$". Clearly $\mathcal{C}^* \subseteq \{A : A \leq_W P\}$, and it is not hard to show that P is $\underset{\sim}{\Delta}^1_2$.

§2 Wadge degrees of orbits

Let $\mathfrak{A}$ be a countable structure. Define $[\mathfrak{A}] = \{(\omega, R_n: n < \omega): \mathfrak{A}$ is isomorphic to $(\omega, R_n: n < \omega)\}$. $[\mathfrak{A}]$ is called the orbit of $\mathfrak{A}$. Scott's Theorem [15] says that for any $\mathfrak{A}$ countable $[\mathfrak{A}]$ is Borel. Recall δ the metric defined in Part II. For any $a \in A$ let $\mathfrak{A}(n,a) = \{b \in A: \delta(a,b) \leq n\}$. $\mathfrak{A}$ has finite valency ([14]) iff for any $a \in A$ and $n < \omega$ $\mathfrak{A}(n,a)$ is finite.

<u>Theorem</u> 4. For every $\mathfrak{A}$ finite valency, countable

$[\mathfrak{A}] \in \underset{\sim}{\Pi}_4^0$

<u>Proof</u>.

For $a \in A$ let

$P_a(v) = \bigwedge\!\!\!\bigwedge_{n<\omega} [\exists x_1, \ldots x_{j_n} \; \theta_n^a(v, x_1, \ldots x_{j_n}) \wedge$
$\neg \exists x_1, \ldots x_{j_n} \; \exists y (\theta_n^a(v, x_1, \ldots x_{j_n}) \wedge \delta(v,y) \leq n \wedge v \neq y \wedge$
$\bigwedge\!\!\!\bigwedge_i x_i \neq y)]$ where $\mathfrak{A}(a,n) = \{a, b_1, \ldots b_{j_n}\}$ and
$\theta_a^n(a, b_1, \ldots b_{j_n})$ is the conjunction of all atomic sentences and negations of atomic sentences involving $a, b_1, \ldots, b_{j_n}$ and some R_m for $m < n$ or equality.

<u>Lemma</u> 1. $\forall \mathfrak{B}$ countable (not necessarily of finite valency) $\forall b \in |\mathfrak{B}|$, $\mathfrak{B} \models P_a(b) \leftrightarrow \mathfrak{A}(\omega, a) \simeq \mathfrak{B}(\omega, b)$.

<u>Proof</u>.

Clearly $\mathfrak{B} \models P_a(b)$ implies there are isomorphisms F_n: $\langle \mathfrak{A}(n,a), R_m \rangle_{m<n} \rightarrow \langle \mathfrak{B}(n,b), R_m \rangle_{m<n}$ each sending a to b. Define a back and forth property $\mathcal{F}$ by:

$((\bar{a}, \bar{b}) \in \mathcal{F} \leftrightarrow$ infinitely many n, $F_n(\bar{a}) = \bar{b})$.
$(a,b) \in \mathcal{F}$ so $\mathcal{F}$ is not empty. Suppose $(\bar{a}, \bar{b}) \in \mathcal{F} \wedge$ $c \in \mathfrak{A}(\omega, a)$. Choose $N < \omega$ so that $\mathfrak{A}(N,a)$ contains $(\bar{a};c)$, then since $|\mathfrak{A}(N,a)|, |\mathfrak{B}(N,b)|$ are finite $\exists d \in |\mathfrak{B}(N,b)|$ such that infinitely many of the F_n sending $\bar{a}$ into $\bar{b}$ send c into d. (Same argument for other direction of the back and forth.) ∎

Choose a_n for $n < N \leq \omega$ so that $\forall \mathcal{C}$ a component (maximally connected) of $\mathfrak{A}$, $\exists n < N$ $\mathcal{C} \cong \mathfrak{A}(\omega, a_n)$.

Define $g: N \to \omega + 1$, $g(n)$ = number of components of $\mathfrak{A}$ isomorphic to $\mathfrak{A}(\omega, a_n)$.

Let $\theta(v,w) = \bigwedge_{n<\omega} \delta(v,\omega) \geq n$.

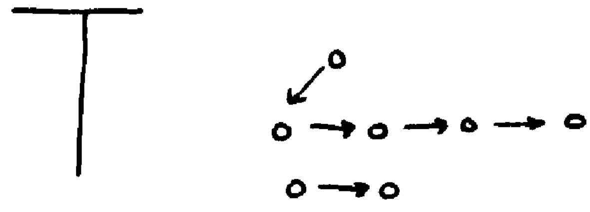
Let $\psi = (\forall x \bigvee_{a \in |\mathfrak{A}|} P_a(x)) \wedge (\bigwedge_{n<N} \bigwedge_{m<g(n)} \exists x_0, \ldots x_m (\bigwedge_{i \neq j} \theta(x_i, x_j) \wedge \bigwedge_{i \leq n} P_{a_n}(x_i))) \wedge (\bigwedge_{\substack{g(n) < \aleph_0 \\ n<N}} \neg \exists x_0, \ldots, x_{g(n)} \bigwedge_{\substack{j, i \leq g(n) \\ i \neq j}} \theta(x_i, x_j) \wedge \bigwedge_{i \leq g(n)} P_{a_n}(x_i))$.

Since θ is $\underset{\sim}{\Pi}^0_1$ and $P_a(v)$ is $\underset{\sim}{\Pi}^0_2$ we have that ψ is $\underset{\sim}{\Pi}^0_4$. Theorem 4 ∎

We show this is best possible:

<u>Theorem</u> 5. If $\mathfrak{A} = (\omega, R)$ $R \subseteq \omega^2$ is the graph of the 1-1 function whose components consist of infinitely many copies of (ω, Sc) and infinitely many copies of (Z, Sc) $(Sc(x) = x + 1)$ then $[\mathfrak{A}] \notin \underset{\sim}{\Sigma}^0_4$.

C_0	C_1	C_2
0	1	1
1	0	1
0	1	1
⋮	⋮	⋮



Play the following game of Solitary: On the n^{th} move you are presented with the n^{th} row of zero's and one's (seemingly at random) in ω columns C_m $m < \omega$. You (eventually to write down a structure $\langle \omega, R \rangle$, $R \subseteq \omega^2$) write down an extension $\mathfrak{A}_n$ of $\mathfrak{A}_{n-1} \subseteq \mathfrak{A}_n$ with universe contained in ω. Let $\mathfrak{A} = \bigcup_{n<\omega} \mathfrak{A}_n$. In order to win this game you must arrange that the universe of $\mathfrak{A}$ is ω and either

$$\mathfrak{A} \simeq \mathfrak{A}_0 = \aleph_0\text{-copies of } \langle \omega, Sc \rangle$$

or

$$\mathfrak{A} \simeq \mathfrak{A}_1 = 1 \text{ - copy of } \langle Z, Sc \rangle + \mathfrak{A}_0 .$$

In addition you must guarantee:

$$\mathfrak{A} \simeq \mathfrak{A}_1 \text{ iff (one of the columns } C_n \text{ has infinitely many one's in it.)}$$

It might be easier for the reader to find his or her own argument. Any finite structure isomorphic to $\langle n, Sc \rangle$ $n < \omega$ will be called a string. Here is a rough description of a winning strategy:

After n moves of the game, $\mathfrak{A}_n$ will consist of finitely many strings labeled $\hat{C}_0, \quad ,\hat{C}_m$ and the rest of the strings will all be labeled G (for garbage). The first thing we do is push each string forward, i.e. given: $\overset{1}{0} \to \overset{2}{0} \to \overset{3}{0}$ we add another element making $\overset{1}{0} \to \overset{2}{0} \to \overset{3}{0} \to \overset{4}{0}$, and we also create a new string $0 \to 0$ and label it G. Next we look to see if a 1 apears in the n^{th} row in any column. If none appears we're done with this move. Otherwise let $k_0 < \omega$ be the least k with 1 appearing in the k^{th} column C_k and n^{th} row. We move the string labeled C_{k_0} back: i.e. $\hat{C}_{k_0} = \overset{1}{0} \to \overset{2}{0} \to \overset{3}{0}$ becomes $\overset{4}{0} \to \overset{1}{0} \to \overset{2}{0} \to \overset{3}{0}$. And we take all strings labeled $\hat{C}_k$ for $k > k_0$ and relabel them G (injured priorities). This is a winning strategy because if none of the columns have infinintely many one's in them, then no copy of $\langle Z, Sc \rangle$ is ever made. So $\mathfrak{A} \simeq \mathfrak{A}_0$.

If k_0 is the least k such that C_k has infinintely many one's in it then at some stage $n_0 < \omega$ none of the columns C_k for $k < k_0$ ever get a one in them. After this point the string labeled $\hat{C}_{k_0}$ is made into $\langle Z, Sc \rangle$ and each $\hat{C}_k$ makes into $\langle \omega, Sc \rangle$ for $k < k_0$ as do things in G. So $\mathfrak{A} \simeq \mathfrak{A}_1$. ■

If $H \in \underset{\sim}{\Pi}^0_2$ then $\exists C: 2^\omega \to P(\omega)$ continuous so that $\forall x (x \in H \leftrightarrow C(x)$ infinite). Hence for any $G \in \underset{\sim}{\Sigma}^0_3$ $\exists \langle G_n : n < \omega \rangle$ cont. so that $(x \in G \leftrightarrow \exists n \; G_n(x)$ infinite).

Let $\bigcap_{n\in\omega} G_n$ be any $\underset{\sim}{\Pi}^0_4$ set $G_n \in \underset{\sim}{\Sigma}^0_3$ and
$G_0 \geq G_1 \geq G_2 \ldots$

Then using above game we have maps $f_n: 2^\omega \to 2^{\omega\times\omega}$ continuous so that $(x \in G_n) \leftrightarrow (f_n(x) \simeq \mathfrak{A}_1) \leftrightarrow (f_n(x) \neq \mathfrak{A}_0)$. So easily we have a continuous map $g: 2^\omega \to 2^{\omega\times\omega}$ $\forall x$ $(x \in G_n) \leftrightarrow (g(x) \simeq \mathfrak{A})$, showing $[\mathfrak{A}]$ is complete $\underset{\sim}{\Pi}^0_4$. ■

<u>Remark</u>: Given $\langle v,E\rangle$ a countable graph $(E \subseteq [v]^2 = \{\{a,b\} \subseteq v: a \neq b\})$ define $\mathfrak{A} = \langle v \cup E, \varepsilon\rangle$ by $(a \,\varepsilon\, b) \leftrightarrow (a \in V \wedge b \in E \wedge b = \{a_0,a_1\} \wedge (a = a_0 \vee a = a_1))$. Then ε on the universe of $\mathfrak{A}$ is a relation with disjoint domain and range (hence a partial order). Furthermore $\omega(Th(\mathfrak{A})) = \omega(Th\langle v,E\rangle)$. Theorem 4 shows that $\mathfrak{A}$ has $\leq \aleph_0$ or $2^{\aleph_0}$ non-isomorphic substructures. The proof is as follows.

Case 1. There are infinitely many $x \in V$ of infinite valency $(|\{y: xEy\}| = \aleph_0)$. Build a distinct sequence $x_n \in V$, $Y_n \subseteq V$ infinite for $n < \omega$ so that $\forall n \forall y \in Y_n(\{x_n,y\} \in E)$ and $\forall n \neq m$ $(Y_n \wedge Y_m = \phi)$. Looking at substructures of $\mathfrak{A}$ allows us in effect not only to drop out vertices (elements of v) but also edges (elements of E). Hence we may "drop" all edges except those connecting each x_n to the elements of Y_n and easily show $\mathfrak{A}$ has $2^{\aleph_0}$ non-isomorphic substructures.

Case 2. There are only finitely many $x \varepsilon v$ of infinite valency. By an easy generalization of Theorem 4 the orbit of every substructure of $\mathfrak{A}$ is $\underset{\sim}{\Pi}^0_8$, hence we conclude that the class of substructures of $\mathfrak{A}$ (note that it is $PC(L_{\omega_1\omega})$) obeys Vaught's conjecture.

Next we characterize the Wadge degrees of well orderings.

<u>Theorem</u> 6. If $\beta = \lambda + m$ where λ is a limit ordinal and $m < \omega$ and $\gamma = \omega^\beta \cdot n + \delta$ where $n < \omega$ and $\delta < \omega^\beta$, then if $n = 1$ then $[(\gamma,<)]$ is $\underset{\sim}{\Pi}^0_{\lambda+2m+1}$ properly and if $n > 1$ then $[(\gamma,<)]$ is $2 - \underset{\sim}{\Pi}^0_{\lambda+2m+1}$ properly.

<u>Proof</u>.

The computation of the upper bound on the complexity will be left to the reader. Now we show that the orbits are properly of the given complexity.

Define $\mathfrak{A} \equiv_\alpha \mathfrak{B}$ iff $\mathfrak{A}$ and $\mathfrak{B}$ model the same $\underset{\sim}{\Pi}^0_\alpha$ sentences. $\mathfrak{A} \leq_\alpha \mathfrak{B}$ is defined similarly.

<u>Lemma</u> 2. If $\beta = \lambda + n$ where λ is a limit ordinal and $n < \omega$, then $(\omega^\beta,<) \leq_{\lambda+2n} (\omega^\beta \cdot \delta,<)$ for any $\delta > 0$.

Proof.

This is essentially what Ehrenfeucht shows in Theorem 12 of [12], except he does not go into transfinite levels, and in his game H_n, player I gets to choose which model he wants to play in each turn.

Instead we should play the following game $K_\alpha(\mathfrak{A},\mathfrak{B})$.

Player I begins by picking $\mathfrak{A}$ or $\mathfrak{B}$ on his 1^{st} move and playing a finite sequence from the model he picks, from then on he must alternate between $\mathfrak{A}$ and $\mathfrak{B}$, on each turn he plays an ordinal β_n with $\beta_{n+1} < \beta_n$. The game is over when I plays zero.

It can be shown that if Player II has winning strategy in $K_\alpha(\mathfrak{A},\mathfrak{B})$, then $\mathfrak{A} \equiv_\alpha \mathfrak{B}$. ■

Now consider $\beta > 0$, $0 < n < \omega$, $\delta < \omega^\beta$, $\beta = \lambda + m$. By the Lemma

$$\underset{\mathfrak{A}_0}{\langle \omega^\beta + \delta, < \rangle} \precsim_{\lambda+2m} \underset{\mathfrak{A}_1}{\langle \omega^\beta \cdot n + \delta, < \rangle} \precsim_{\lambda+2m} \underset{\mathfrak{A}_2}{\langle \omega^\beta \cdot (n+1) + \delta, < \rangle}$$

thus for any $\Sigma^0_{\lambda+2m+1}$ sentence θ:

* if $\mathfrak{A}_0 \models \theta$ then $\mathfrak{A}_1 \models \theta$

** if $\mathfrak{A}_1 \models \theta$ then $\mathfrak{A}_2 \models \theta$

thus $[\langle\omega^\beta + \delta, <\rangle]$ is not $\Sigma^0_{\lambda+2m+1}$ by *

Suppose

$[\langle\omega^\beta \cdot n + \delta, <\rangle]$ were $\mathrm{co}(2 - \underset{\sim}{\Pi}^0_{\lambda+2m+1})$,

then $[\langle\omega^\beta n + \delta, <\rangle]$ would be union of a $\underset{\sim}{\Pi}^0_{\lambda+2m+1}$ and $\underset{\sim}{\Sigma}^0_{\lambda+2m+1}$ hence would be $\Pi^0_{\lambda+2m+1}$ contradicting * or $\underset{\sim}{\Pi}^0_{\lambda+2m+1}$ contradicting **. ■

<u>Lemma</u> 3. If $\mathfrak{A} \underset{\alpha}{\lneq} \mathfrak{B}$, $\mathfrak{A} \neq \mathfrak{B}$ then orbit of $\mathfrak{A}$ is not $\underset{\sim}{\Sigma}^0_{\alpha+1}$ and orbit of $\mathfrak{B}$ is not $\underset{\sim}{\Pi}^0_{\alpha+1}$. Now we give some examples of other orbits.

Define $i,j \le \omega$ $\mathfrak{A}, \mathfrak{B}$ $\mathfrak{p}$-structures then $i \cdot \mathfrak{A} + j \cdot \mathfrak{B}$ is the following $\mathfrak{p} \cup \{\sim\}$ structure.

$|i \cdot \mathfrak{A} + j \cdot \mathfrak{B}| = i$ copies of $|\mathfrak{A}| \cup j$ copies of $|\mathfrak{B}|$

$x \sim y$ iff x, y are in same copy of $|\mathfrak{A}|$ or $|\mathfrak{B}|$

$R\bar{x}$ iff $\bar{x}$ are in same copy of $|\mathfrak{A}|$ or $|\mathfrak{B}|$ and $R\bar{x}$ holds there.

<u>Lemma</u> 4. If $\mathfrak{A} \underset{\alpha}{\lneq} \mathfrak{B}$ then $i \cdot \mathfrak{A} + \omega \cdot \mathfrak{B} \underset{\alpha+1}{\nleq} (i+1) \cdot \mathfrak{A} + \omega \cdot \mathfrak{B}$

<u>Proof</u>.

Easy playing game.

<u>Lemma</u> 5. If $[\mathfrak{A}], [\mathfrak{B}]$ $\alpha \ge 2$ $\underset{\sim}{\Delta}^0_\alpha$ then $\omega \cdot \mathfrak{B}$ is $\underset{\sim}{\Pi}^0_\alpha$ $\wedge$ $1 \cdot \mathfrak{A} + \omega \cdot \mathfrak{B}$ is $2\text{-}\underset{\sim}{\Pi}^0_\alpha$.

<u>Proof</u>.

θ_0 Scott sent for $\mathfrak{A}$

θ_1 Scott sent for $\mathfrak{B}$

For any x, ψ, ψ^x the formula obtained by relativizing the quantifier of ψ to $\{y: y \sim x\}$.

Let $\psi_0 \equiv$ (a) $\bigwedge_{R \in \mathfrak{p}} \forall \bar{x} (R\bar{x} \rightarrow \bigwedge_{i,j} x_i \sim x_j) \wedge$

(b) $\sim$ equivalence relation $\wedge$

(c) $\bigwedge_n \exists x_1, \ldots x_n \bigwedge_{i \neq j} (x_i \nsim x_j)$ $\wedge$

(d) $\forall x\ \theta_1^x$.

ψ_0 is a Scott sentence for $\omega\cdot\mathfrak{B}$.

Let $\psi_1 \equiv (a) \wedge (b) \wedge (c) \wedge$

(e) $\forall\, x(\theta_0^x \vee \theta_1^x) \wedge$

(f) $\forall\, x,y \quad x \not\approx y \to \neg(\theta_0^x \wedge \theta_0^y) \wedge$

(g) $\exists\, x\, \theta_0^x$.

ψ is 2-$\underset{\sim}{\Pi}^0_\alpha$ Scott sentence for $1\cdot\mathfrak{A} + \omega\cdot\mathfrak{B}$. ■

If $\mathfrak{A}, \mathfrak{B}$ ρ-structures L, L' linear orders, then define $\mathfrak{A}\cdot L + \mathfrak{B}\cdot L'$ the $\rho \cup \{\leq\}$-structure as follows.

Let $\mathfrak{A}_\ell, \mathfrak{B}_k$ be copies of $\mathfrak{A}, \mathfrak{B}$ for each $\ell \in L$, $k \in L'$

$|\mathfrak{A}\cdot L + \mathfrak{B}\cdot L'| = \bigcup_{\ell\in L} |\mathfrak{A}_\ell| \cup \bigcup_{k \in L'} |\mathfrak{A}_k|$

$x \leq y$ iff $[\exists \ell\, \exists k\, (x,y \in \mathfrak{A}_\ell)$ or $(x,y \in \mathfrak{B}_k)$ or $(x \in \mathfrak{A}_\ell \wedge y \in \mathfrak{B}_k)$ or $(\ell \leq k \wedge x \in \mathfrak{A}_\ell \wedge y \in \mathfrak{A}_k)$ or $(\ell \leq k \wedge x \in \mathfrak{B}_\ell \wedge y \in \mathfrak{B}_k)]$.

$R\bar{x} \leftrightarrow [\bar{x}$ in one copy of $\mathfrak{A}$ or $\mathfrak{B}$ and $R\bar{x}$ holds there].

Lemma 6. If $\mathfrak{A} \leq_\alpha \mathfrak{B}$ then $\mathfrak{B}\cdot\eta + \mathfrak{A}\cdot\eta \leq_{\alpha+2} \mathfrak{B}\cdot\eta + \mathfrak{A}(1+\eta)$.

Proof.

Easy using games--the extra copies of $\mathfrak{A}$ on left correspond to some $\mathfrak{A}_s$ $s \in \eta$ on left. ■

Lemma 7 ($\alpha \geq 2$). If $[\mathfrak{A}], [\mathfrak{B}] \in \underset{\sim}{\Delta}^0_\alpha$ then $[\mathfrak{B}\cdot\eta + \mathfrak{A}(1+\eta)] \in \underset{\sim}{\Sigma}^0_{\alpha+1}$.

Proof.

Define $x \sim y$ iff $x \leq y \wedge y \leq x$. Let

θ_0 Scott sentence for $\mathfrak{A}$ and

θ_1 Scott sentence for $\mathfrak{B}$. Then the conjunction of

(a) $\leq/\sim$ has order type η

(b) $\bigwedge_{R \in p} R\bar{x} \rightarrow \bigwedge_{i,j} x_i \sim x_j$

(c) $\forall x \forall y (\theta_0^x \wedge \theta_1^y \rightarrow x > y)$

(d) $\forall x (\theta_0^x \vee \theta_1^x)$

(e) $\exists x\, \theta_0^x \wedge \forall y < x \quad \theta_1^y$

is a $\underset{\sim}{\Sigma}^0_{\alpha+2}$ Scott sentence for $\mathfrak{B} \cdot \eta + \mathfrak{A} \cdot (1+\eta)$. ■

Theorem 7. For each α, $0 < \alpha < \aleph_1$ there are orbits which are properly:

$\underset{\sim}{\Pi}^0_1$, $2 - \underset{\sim}{\Pi}^0_1$, $\underset{\sim}{\Pi}^0_\alpha$, $2 - \underset{\sim}{\Pi}^0_{\alpha+1}$, and $\underset{\sim}{\Sigma}^0_{\alpha+2}$.

Proof.

The ordinals give examples of $\underset{\sim}{\Pi}^0_{\lambda+2n+1}$, $2\text{-}\underset{\sim}{\Pi}^0_{\lambda+2n+1}$ orbits for $(\lambda > 0$ limit $0 \leq n < \omega)$ or $(\lambda = 0 \wedge 1 \leq n < \omega)$. For λ a limit > 0, choose $\alpha_n \uparrow \omega^\lambda$. It is easy to see that the orbit $[\langle \omega^\lambda, <, p \rangle]$ where $p = \{\alpha_n : n < \omega\}$ is $\underset{\sim}{\Pi}^0_\lambda$ and not $\underset{\sim}{\Sigma}^0_\lambda$.

Now let $\mathfrak{A} = \langle \omega^{\lambda+n}, < \rangle \not\leq_{\lambda+2n} \langle \omega^{\lambda+n} \cdot 2, < \rangle = \mathfrak{B}$.

By Lemma 4 $(\omega \cdot \mathfrak{B}) \not\leq_{\lambda+2n+1} (1 \cdot \mathfrak{A} + \omega \cdot \mathfrak{B}) \not\leq_{\lambda+2n+1} (2 \cdot \mathfrak{A} + \omega\, \mathfrak{B})$.

Since $\mathfrak{A}, \mathfrak{B}$ have $\underset{\sim}{\Delta}^0_{\lambda+2n+2}$ orbits by Lemma 5

$\omega \cdot \mathfrak{B}$ is $\underset{\sim}{\Pi}^0_{\lambda+2n+2}$ and $1 \cdot \mathfrak{A} + \omega \cdot \mathfrak{B}$ is $2\text{-}\underset{\sim}{\Pi}^0_{\lambda+2n+2}$.

They are properly so by Lemma 2.

Thus we have examples of proper $\underset{\sim}{\Pi}^0_\alpha$, $2\text{-}\underset{\sim}{\Pi}^0_\alpha$, $\alpha \geq 3$ orbits.

In fact $\forall \alpha$, $2 \leq \alpha < \aleph_1$ we have structures $\mathfrak{A} \preceq_\alpha \mathfrak{B}$ such that

orb($\mathfrak{A}$), orb($\mathfrak{B}$) are $\underset{\sim}{\Delta}^0_{\alpha+2}$. By Lemma 6

$\mathfrak{B} \cdot \eta + \mathfrak{A} \cdot \eta \preceq_{\alpha+2} \mathfrak{B} \cdot \eta + \mathfrak{A}(1+\eta)$.

Hence by Lemma 3 $\mathfrak{B} \cdot \eta + \mathfrak{A}(1+\eta)$ is not $\underset{\sim}{\Pi}^0_{\alpha+3}$.

By Lemma 7 $[\mathfrak{B} \cdot \eta + \mathfrak{A}(1+\eta)]$ is $\underset{\sim}{\Sigma}^0_{\alpha+3}$.

Now let $\mathfrak{A}, \mathfrak{B}$ be the following structures in one relation symbol $\sim$.

$\sim$ equivalence relation one equivalence class.

$\sim$ equivalence relation two equivalence classes, one of which has size 1.

It is easy to see that $[\mathfrak{A}]$ is $\underset{\sim}{\Pi}^0_1$,

$\wedge$ $[\mathfrak{B}]$ is $2\text{-}\underset{\sim}{\Pi}^0_1$.

Since $\mathfrak{A} \subseteq_0 \mathfrak{B}$ we have $[\mathfrak{B} \cdot \eta + \mathfrak{A}(1+\eta)]$ is $\underset{\sim}{\Sigma}^0_3$.

Now let $\mathfrak{A} = \langle \mathbb{Z}, Sc \rangle$, $\mathfrak{B} = \langle \mathbb{Z} + \mathbb{Z}, Sc \rangle$,

then $\mathfrak{A}$ is $\underset{\sim}{\Pi}^0_2$ $\mathfrak{B}$ is $2\text{-}\underset{\sim}{\Pi}^0_2$, and $\mathfrak{A} \preceq_1 \mathfrak{B}$ so by above $\mathfrak{B} \cdot \eta + \mathfrak{A}(1+\eta)$ is $\underset{\sim}{\Sigma}^0_4$.

This gives examples of all orbits promised except for $\underset{\sim}{\Sigma}^0_{\lambda+2}$ λ limit $\lambda > 0$; which we now provide: (keep in mind the strucutre $\langle \mathbb{Q}, C_n, \leq \rangle$ where C_n are strictly increasing to 0).

Suppose we have $\mathfrak{p}$-structures $\mathfrak{A}_n$, and $\mathfrak{A}$, then $\mathfrak{B}$ is the following $\mathfrak{p} \cup \{\leq\}$-structure.

$|\mathfrak{B}| = \{\langle r,a\rangle : r \in Q\ ,\ a \in |\mathfrak{A}_n|$ if $r = c_n$ for some $n < \omega$

$a \in |\mathfrak{A}|$ otherwise$\}$.

$(\langle r,a\rangle \leq \langle s,b\rangle) \leftrightarrow (r \leq s)$.

$R\overline{x} \leftrightarrow (\exists r \quad x = \langle\langle r,a\rangle, \langle r,a_2\rangle, \ldots, \langle r,a_n\rangle\rangle$ and

$Ra_1,\ldots,a_n$ holds in appropriate structure).

$\hat{\mathfrak{B}} = \mathfrak{B} - \{\langle 0,a\rangle : a \in |\mathfrak{A}|\}$.

<u>Lemma</u> 8. Suppose $\alpha_n \uparrow \lambda$ $\mathfrak{A}_n \leq_{\alpha_n} \mathfrak{A}$ then $\hat{\mathfrak{B}} \not\leq_{\lambda+1} \mathfrak{B}$

<u>Proof</u>.

Easy using game criterion.

<u>Lemma</u> 9. Orbit of $\mathfrak{B}$ is $\Sigma^0_{\lambda+2}$.

<u>Proof</u>.

Just write it all down.

Note: If $\alpha_n \uparrow \lambda$ then $(\omega^{\alpha_n},<) \not\leq_{\alpha_n} (\omega^\lambda,<)$.

This concludes proof of Theorem 7. ■

<u>Remark</u>: An immediate corollary of D. Miller's invariant difference hierarchy theorem [13] is that if $[\mathfrak{A}] \in \underset{\sim}{\Delta}^0_{\alpha+1}$ then there are invariant $\underset{\sim}{\Pi}^0_\alpha$ sets A and B so that $[\mathfrak{A}] = A \wedge \sim B$. Also a theorem of Vaught [19] says that a $\underset{\sim}{\Pi}^0_\alpha$ set B is invariant iff B is the set of countable models of some $\underset{\sim}{\Pi}^0_\alpha$ sentence of $L_{\omega_1\omega}$. Thus if $[\mathfrak{A}] \subseteq B$ where B is an invariant

$\underset{\sim}{\Sigma}^0_\omega$ set then $[\mathfrak{A}] \subseteq B' \subseteq B$ where B' is an invariant Π^0_n set some $n < \omega$. The following diagram summerizes the content of these remarks and Theorems 7 and 8. The only open question is: Are there any $\underset{\sim}{\Sigma}^0_{\lambda+1}$ orbits for $\lambda > 0$ a limit?

Yes $\underset{\sim}{\Pi}^0_1$		$\underset{\sim}{\Sigma}^0_1$ No
	$2\text{-}\underset{\sim}{\Pi}^0_1$ Yes	
Yes $\underset{\sim}{\Pi}^0_2$		$\underset{\sim}{\Sigma}^0_2$ No
	$2\text{-}\underset{\sim}{\Pi}^0_2$ Yes	
Yes $\underset{\sim}{\Pi}^0_3$		$\underset{\sim}{\Sigma}^0_3$ Yes
	⋮	
Yes $\underset{\sim}{\Pi}^0_\omega$		$\underset{\sim}{\Sigma}^0_\omega$ No
	$2\text{-}\underset{\sim}{\Pi}^0_\omega$ No	
Yes $\underset{\sim}{\Pi}^0_{\omega+1}$		$\underset{\sim}{\Sigma}^0_{\omega+1}$?
	$2\text{-}\underset{\sim}{\Pi}^0_{\omega+1}$ Yes	
Yes $\underset{\sim}{\Pi}^0_{\omega+2}$		$\underset{\sim}{\Sigma}^0_{\omega+2}$ Yes
	$2\text{-}\underset{\sim}{\Pi}^0_{\omega+2}$ Yes	

In [13] D. Miller proves that in the topology generated by first order formulas there are no $\underset{\sim}{\Sigma}^0_2$ orbits. Next we show in the usual topology that such orbits are impossible.

Theorem 8. Proper $\underset{\sim}{\Sigma}{}^0_2$ orbits are impossible.

Proof.

Suppose $\mathfrak{A} = \langle A, \bar{R} \rangle$, $\|A\| = \aleph_0$, $\bar{R}$ countable similarity type containing only relation symbols. Suppose

$\theta_0 = \exists x_0, \ldots, x_{n-1} \bigwedge_{m<\omega} \psi_m(x_0, \ldots, x_{n-1})$

where $\psi_m(\bar{x})$ Π^0_1 formula of 1^{st} order logic.

Suppose

$* \ \forall \mathfrak{B} \quad \|\mathfrak{B}\| = \aleph_0 (\mathfrak{B} \cong \mathfrak{A} \leftrightarrow \mathfrak{B} \models \theta_0).$

Lemma 10. $\mathfrak{A}$ is ω-saturated (in fact $Th(\mathfrak{A})$ is $\aleph_0$-categorized).

Proof.

$\mathfrak{A}$ is weakly saturated. To see this let Σ be a type consistent with $Th(\mathfrak{A})$. Let $\mathfrak{B} \succ \mathfrak{A}$ be countable realizing Σ. Since $\mathfrak{B} \models \theta_0$, $\mathfrak{B} \cong \mathfrak{A}$. So $\mathfrak{A}$ is weakly saturated. Thus $Th(\mathfrak{A})$ has only countably many n-types each n. So there exists $\mathfrak{B}$ countable ω-saturated model of $Th(\mathfrak{A})$. Since $\mathfrak{A} \prec \mathfrak{B}$, $\mathfrak{B} \models \theta_0$ so $\mathfrak{B} \cong \mathfrak{A}$. $\blacksquare$

Define $\theta(x_0, \ldots, x_{n-1}) = \bigwedge_{i \neq j} (x_i \neq x_j) \wedge \bigvee_{f \in n!} \bigwedge_{m<\omega} \psi_m(x_{f(0)}, \ldots, x_{f(n-1)})$ where

n! is the symmetric group on n.

Thus $\theta^{\mathfrak{A}} = \{x \in [A]^n : \mathfrak{A} \models \theta(x)\}$ partitions the n-element subset of A ($[A]^n$).

By Ramsey's theorem $\exists v \subseteq A$ inf. so that

$[v]^n \subseteq \theta^{\mathfrak{A}}$ or $[v]^n \subseteq [A]^n - \theta^{\mathfrak{A}}$ $(\equiv \sim \theta^{\mathfrak{A}})$.

If the first happens then we have $\langle v, \bar{R} \rangle \models \forall \bar{x}\ \theta(\bar{x})$ and hence by * $\forall \bar{x}\ \theta(\bar{x})$ is a $\underset{\sim}{\Pi}^0_1$ Scott sentence for $\mathfrak{A}$.

So we assume $[v]^n \subseteq \sim \theta^{\mathfrak{A}}$.

Choose $B \varepsilon \theta^{\mathfrak{A}}$ and throw out of v any part of B.

By repeatedly applying Ramsey's theorem we obtain $\hat{v} \subseteq v$ infinite so that

$\forall F \subseteq B$ either (a) $\forall G \varepsilon [\hat{v}]^{n-||F||}$ $F \cup G \varepsilon \theta^{\mathfrak{A}}$

or (b) $\forall G \varepsilon [\hat{v}]^{n-||F||}$ $F \cup G \not\varepsilon \theta$

Choose $F \subseteq B$ of minimal cardinality so that (a) happens (it always exists since $F = B$ will do).

Let $F = A_0$, $||A_0|| = n_0$.

Note by * $\langle A_0 \cup \hat{v}, \bar{R} \rangle \approx \mathfrak{A}$ so we assume $A_0 \cup \hat{v} = A$.

<u>Lemma</u> 11. $\forall B \varepsilon [A]^{n_0} [B = A_0 \leftrightarrow \forall C \varepsilon [A-B]^{n-n_0}$ $B \cup C \varepsilon \theta^{\mathfrak{A}}]$.

<u>Proof</u>.

$\rightarrow$ By definition of A_0.

$\leftarrow$ Suppose $B \neq A$. Choose $C \varepsilon [A - (B \cup A_0)]^{n-n_0}$.

Since $B \cap A_0 \subseteq A_0$ has smaller cardinality (b) happens, and hence $B \cup C =$

$(A_0 \cap B) \cup [(B-A_0) \cup C] \varepsilon \sim \theta^{\mathfrak{A}}$.

∎

<u>Lemma</u> 12. $\forall B \leq \hat{V}$ infinite there is an isomorphism $F: \langle B \cup A_0, \bar{R} \rangle \to \langle \hat{V} \cup A_0, \bar{R} \rangle = \mathfrak{A}$ which sends A_0 into A_0.

<u>Proof</u>.

The fact that there is an isomorphism follows from *

that it sends A_0 into A_0 is immediate from Lemma 11. ∎

Recall

$$\theta(\bar{x}) = \bigwedge_{i \neq j} x_i \neq x_j \bigvee_{f \in n!} \bigwedge_{m<\omega} \psi_m(f(\bar{x})).$$

This is equivalent to

$$\bigwedge_{m<\omega} (\bigwedge x_i \neq x_j \wedge \bigvee_{f \in n!} \bigwedge_{k} \psi_k(f(\bar{x})). = \bigwedge_{m<\omega} \sigma_m(\bar{x})$$

Define $\forall k < \omega$

$$\tau_k(x_0, \ldots, x_{n_0 - 1}) \equiv \bigwedge_{\substack{i \neq j \\ < n_0}} x_i \neq x_j \wedge \bigwedge_{m<k} (\bigwedge_{\substack{i \neq j \\ < n}} x_i \neq x_j \to \sigma_m(\bar{x}))$$

$$\tau(\bar{x}) \equiv \bigwedge_{k<\omega} \tau_k(\bar{x}).$$

Thus each τ_k Π^0_1 formula and

** $\forall \mathfrak{B} \quad (\|\mathfrak{B}\| = \aleph_0 (\mathfrak{B} \simeq \mathfrak{A} \leftrightarrow \exists x \in [B]^{n_0} \mathfrak{B} \models \tau(x)))$

$\forall x \in [A]^{n_0} (\mathfrak{A} \models \tau(\bar{x}) \leftrightarrow \bar{x} = A_0).$

<u>Lemma</u> 13. $\exists N < \omega \quad (N \geq 3n_0) \quad \forall H \in [A]^N$ there is at most one $B \in [H]^{n_0}$ such that $\langle H, \bar{R} \rangle \models \tau_N(B)$

<u>Proof</u>.

Let T be the following theory:

(a) $\exists x_0, \, , x_k \quad \bigwedge_{i \neq j} x_i \neq x_j$ for $k < \omega$;

(b) $\tau_k(\{b_0,\ldots,b_{n_0-1}\})$ for $k < \omega$;

(c) $\tau_k(\{c_0,\ldots,c_{n_0}-1\})$ for $k < \omega$; and

(d) $\{b_0 \quad b_{n_0-1}\} \neq \{c_0 \quad c_{n_0-1}\}$.

This theory must be inconsistent thus

$\exists\, N < \omega$ as required. ■

Define $\bar{a}, \bar{b} \in A^m$, $\bar{a} \sim \bar{b}$ iff $\forall i,j (a_i = a_j \leftrightarrow b_i = b_j)$

$\Delta = \{\phi(\bar{x}) : \phi$ is quantifier free formula with parameters from $A_0\}$.

<u>Lemma</u> 14. $\hat{V}$ is a Δ-indisc set over A_0 in $\mathfrak{A}$. (that is $\forall \theta \in \Delta\ \forall \bar{b}, \bar{a} \in \hat{V}^m$

$\bar{a} \sim \bar{b} \rightarrow (\theta(\bar{a}) \leftrightarrow \theta(\bar{b}))$

<u>Proof</u>.

Consider $Th(\langle \mathfrak{A}, a\rangle_{a\in A_0}) = T$ for any linear order $\langle X, < \rangle$, $\|X\| = \aleph_0$. There exists $\mathfrak{B} \models T$, $X \subseteq |\mathfrak{B}|$ $\|\mathfrak{B}\| = \aleph_0$, $\langle X, < \rangle$ $<$-indiscernible over $\mathfrak{B}$.

Let $\mathfrak{B} = \langle \mathfrak{B}_0, b_a \rangle_{a\in A_0}$, by $*$, $\mathfrak{B}_0 \cong \mathfrak{A}$ it is clear that $\{b_a\}_{a\in A_0} = A_0$.

Let $\langle X_1, < \rangle$ have order type $\omega + \omega$.

$X_1 = \{b_i : i < \omega + \omega\}$.

<u>Claim</u>: $\forall \theta \in \Delta\ \forall i_1 < i_2 < \ldots < i_m < k < \ell < j_1 < j_2 < \ldots < j_{m_2}$

$\theta(b_{i_1}, \ldots b_{i_{m_1}}, b_k, b_\ell, b_{j_1}, \ldots b_{j_{m_2}}) \leftrightarrow \theta(\bar{b}_{\bar{i}}, b_\ell, b_k, \bar{b}_{\bar{j}})$

Proof.

Suppose not and define

$p(x) \leftrightarrow [\bigwedge_{a\in A_0} x \neq a \wedge \bigwedge_{i<m_1} x \neq b_i \wedge \bigwedge_{k<m_2} x \neq b_{\omega+k}]$

$\psi(x,y) \leftrightarrow [\theta(b_0,\ldots,b_{m_1-1},x,y,b_\omega, \quad ,b_{\omega+(m_2-1)}) \wedge p(x) \wedge p(y) \wedge x \neq y]$.

Let $B_0 = \{b_i : i < \omega + (m_2-1)\}$ by Lemma 12,

$\mathfrak{B} = \langle B_0 \cup A_0, \bar{R}\rangle \simeq \mathfrak{A}$ sending A_0 into A_0.

But by indiscernibility

$\langle P^{\mathfrak{B}}, \psi^{\mathfrak{B}}\rangle \simeq \langle \omega, <\rangle$ contradicting ω-saturation of $\mathfrak{A}$.

or $\simeq \langle \omega^*, <\rangle$ proves Claim.

Define $P \subseteq S_m$ by

$\sigma \in P \leftrightarrow \forall \theta \in \Delta \ \forall x_1 < x \ < \ldots < x_m \in X$

$[\theta(x_1, x_2, \ldots, x_m) \leftrightarrow \theta(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(m)})]$.

$(\sigma,\tau \in P \rightarrow \sigma \bullet \tau \in P)$ and P contains all 2-cycles of the form $(i,i+1)$ by Claim. But these generate $m!$ so $P = m!$

Now since $\langle A_0 \cup X, \bar{R}\rangle \simeq \mathfrak{A}$ the lemma follows. ■

Lemma 15. Let Q be any bijection of $\hat{V}$ into itself. Then the map F_Q defined on $A_0 \cup \hat{V}$ by the identity on A_0 and Q on V is an automorphism of $\langle A_0 \cup \hat{V}, \bar{R}\rangle$.

Proof.

This is immediate from Lemma 14. ■

<u>Lemma</u> 16. (For N in claim 4) $\forall H \in [A]^N$
$\forall B \in [H]^{n_0} (\langle H,\bar{R}\rangle \models$ "$\tau_N(B)$" $\leftrightarrow B = A_0)$

<u>Proof</u>.

($\leftarrow$) is obvious since $\mathfrak{A} \models$ "$\tau_N(A_0)$" and τ_N is Π^0_1 .

($\rightarrow$) If $B \neq A_0$ suppose $\langle H,\bar{R}\rangle \models$ "$\tau_N(B)$". Let $C = B - A_0$. Let $D \in [H - (A_0 \cup B)]^{||C||}$. Define $Q: \hat{V} \to \hat{V}$ so that Q exchanges C and D and is the identity everywhere else. By lemma 15 F_Q is an automorphism of $\mathfrak{A}$, and since F_Q maps H into H it is an automorphism of H. Hence we have $\langle H,\bar{R}\rangle \models$ "$\tau_N(F_Q(B))$", $F_Q(B) \neq B$ contradicting Lemma 13. ■

To prove the theorem just note that Σ^0_1 sentence
$\exists H \in [A]^N \exists B \in [H]^{n_0}$ "$\langle H,\bar{R}\rangle \models \tau_N(B)$" together with the $\underset{\sim}{\Pi}^0_1$ sentence:

$\forall H \in [A]^N \forall B \in [H]^{n_0}($"$\langle H,\bar{R}\rangle \models \tau_N(B)$"$) \rightarrow \tau(B)$

is a Scott sentence for $\mathfrak{A}$. Theorem 8 ■

It also is not hard to show that $\forall f \in \omega^\omega$ $[\langle\omega,f\rangle] \in \underset{\sim}{\Sigma}^0_2$ implies $[\langle\omega,f\rangle] \in \underset{\sim}{\Delta}^0_2$. But the most general statement remains open.

§3. Reduction of Vaught's Conjecture to Π^0_2 sentences in one binary relation

Theorem 9. $\exists$ a map $:\sigma \to \sigma^*$ (effective) from first order sentences to $\underset{\sim}{\Pi}^0_2$ sentences in one binary relation such that $\forall\kappa \geq \omega$ $\kappa(\sigma) = \kappa(\sigma^*)$ ($\kappa(\psi)$ = number of non-isomorphic models of ψ of size κ). Using same procedure it is easily shown that Vaught's conjecture for sentences of $L_{\omega_1\omega}$ reduces to $\underset{\sim}{\Pi}^0_2$ sentences in one binary relation.

Description of map:

First replace σ by one having only relation symbols. Next reduce σ to Π^0_2 as follows: for each subformula of σ add a relation symbol and add axioms:

$$\forall \bar{x}(R_{\exists y\,\theta(\bar{x})}(y,\bar{x}) \leftrightarrow \exists y\; R_{\theta(y,\bar{x})}(y\,\bar{x}))$$

$$\forall \bar{x}(R_{\neg\theta(x)}(\bar{x}) \leftrightarrow \neg R_{\theta(x)}(\bar{x}))$$

$$\forall \bar{x}(R_{\theta(\bar{x})\wedge\psi(\bar{x})}(\bar{x}) \leftrightarrow R_{\theta(\bar{x})}\bar{x} \wedge R_{\psi(\bar{x})}\,\bar{x})$$

R_σ

Thus we obtain σ_0 $\underset{\sim}{\Pi}^0_2$ containing only relation symbols and $\forall\kappa$ $\kappa(\sigma) = \kappa(\sigma_0)$.

Next: let $R_i(x_i - x_{\kappa_i})$, $i < n$ be the relation symbol of σ_0.

Suppose

$$\sigma_0 \equiv \forall \bar{x} \exists \bar{y} \mathcal{M}(R_i, R_i) \quad \text{where}$$

$\mathcal{M}$ positive boolean. Let R, U, P_i^j, Q_i^j be new symbols.

R binary relation

U unary relation

P_i^j $i < n$ $j < \kappa_i$ unary relations

Q_i^j " " " "

We next construct σ_1 in this new language σ_1 is $\underset{\sim}{\Pi}{}_2^0$ and $\kappa(\sigma_0) = \kappa(\sigma_1)$ $\forall \kappa \geq \omega$.

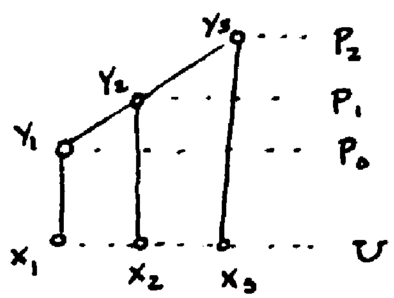
σ_1 will be conjunction of (1) → (7)

(1) R is symmetric $\wedge$ irreflexive.

(2) U, P's, Q's are all disjoint and everything is in one of them.

(3) $\bigwedge \forall x,y\ (Sx \wedge Sy) \to \neg Rxy$:
$S \in \{U, P_i^j, Q_i^j : i < n,\ j < \kappa_i\}$.

Now we describe an interpretation



for $R(x_1, x_2, x_3)$

$$\theta_{R_i}(\bar{x}) \equiv \exists \bar{y} \Big(\bigwedge_{i \neq j} y_i \neq y_j \wedge \bigwedge_j P_i^j(y_j) \wedge \bigwedge_i R(x_i, y_i)$$

$$\bigwedge_i R(y_i, y_{i+1}) \Big)$$

$\theta_{\neg R_i}(\bar{x}) \equiv$ same except Q's in place of P's.

Write $\theta_{R_i}(\bar{x}) \equiv \exists \bar{y}\, \psi_{R_i}(\bar{x},\bar{y})$ for short.

(4) $\forall \bar{x} \exists \bar{y}\ M(R_i/\theta_{R_i}, \neg R_i/\theta_{\neg R_i})$.

(5) $\bigwedge \forall \bar{x}(\theta_{R_i}(\bar{x}) \leftrightarrow \neg \theta_{\neg R_i}(\bar{x}))$

(6) $$\bigwedge_{i<n} \bigwedge_{j<\kappa_i} \left[\begin{array}{l} \forall y\ P_i^j(y) \to \exists \bar{x} \exists \bar{y}(\psi_{R_i}(\bar{x},\bar{y}) \wedge y_j = y) \\ \forall y\ Q_i^j(y) \to \exists \bar{x} \exists \bar{y}(\psi_{\neg R_i}(\bar{x},\bar{y}) \wedge y_j = y) \end{array}\right]$$

This says everything not in U is being used to code.

(7) $$\bigwedge_{i<n} \left(\begin{array}{l} \forall \bar{x} \forall \bar{y} \forall \bar{z}[(\psi_R(\bar{x},\bar{y}) \wedge \psi_R(\bar{x},\bar{z})) \to \bar{y} = \bar{z}] \quad \text{and} \\ \forall \bar{x} \forall \bar{y} \forall \bar{z}[(\psi_{\neg R}(\bar{x},\bar{y}) \wedge \psi_{\neg R}(\bar{x},\bar{z})) \to \bar{y} = \bar{z}] \end{array}\right)$$

This says codes are unique.

Thus σ_1 is Π_2^0 in language with one binary relation and finite number of unary predicates and

$\forall \kappa \geq \omega\ (\kappa(\sigma) = \kappa(\sigma_1))$.

Relabel the language of σ_1 so that it is

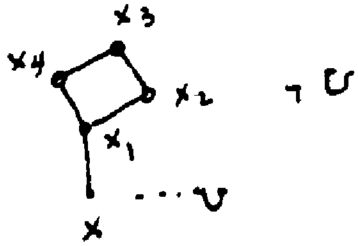
$\{S, P_n : n < N\}$ S binary, P_n unary. Then let

$\sigma_1 \equiv \forall \bar{x} \exists \bar{y}\ \hat{M}(S, P_n, \neg P_n)$ where

$\hat{M}$ position boolean in $(P_n, \neg P_n)$.

Now we describe σ_2 in language R binary and

U unary.



For $n < 2N$ let

$$\tau_n(x) = U(x) \wedge \exists x_1,\ldots,x_n[\bigwedge_{i \neq j} x_i \neq x_j \wedge \bigwedge_i \neg U(x_i)$$

$$\wedge \bigwedge\{R(x_i,x_j): i \equiv j \pm 1 \mod(n)\}$$

$$\wedge \bigwedge\{\neg R(x_i,x_j): i \not\equiv j \pm 1 \mod(n)\}$$

$$\wedge \quad R(x,x_1)].$$

$\tau_n(x) \equiv \exists \bar{x}\ \psi_n(x,\bar{x})$

$\theta(x,y) \equiv U(x) \wedge U(y) \wedge R(x,y)$

σ_2 conjunction of 1) → 5)

(1) R is symmetric and irreflexive.

(2) $\forall \bar{x} \exists \bar{y}\ \hat{M}(S/\theta,\ P_i/\tau_i,\ \neg P_i/\tau_{N+i})$

(3) $\bigwedge_{i<N} \forall x\ U(x) \rightarrow [\tau_i(x) \leftrightarrow \neg \tau_{N+i}(x)]$.

(4) $\forall z(\neg U(z) \rightarrow [\bigvee_{n<2N} \exists x\ \bar{x}\ \psi_n(x,\bar{x}) \wedge \ (\bigvee_{1 \leq i \leq n} x_i = z])$

(says everything not in U is being used to code).

(5) $\bigwedge_n \forall x\ \bar{x}\ \bar{y}[(\psi_n(x,\bar{x}) \wedge \psi_n(x,\bar{y})) \Longrightarrow \bar{x}'' = \bar{y}'']$

To get σ^* use reflexitivity of R to code U,

$U = \{x: R(x,x)\}$ and

$\neg U = \{x: \neg R(x,x)\}$. ■

<u>Remark</u>: Vaught's conjecture for $\bigvee_n \exists \bar{x}_n \bigwedge_m \forall \bar{y}_{nm} \theta_{nm}$ where θ_{nm} quantifier free reduces to univeral theories, since we

can reduce to

$\bigwedge_{m<\omega} \forall \bar{y}_{n_0 m} \theta(\bar{y}_{n_0 m}, \bar{c}_n)$ $\bar{c}_{n_0}$ constants.

William Hanf [16] shows Vaught's conjecture for any countable first order theory reduces to complete first-order theories in the language of one binary relation. Combined with above it easily reduces it to Π_2^0 axiomatizable complete theories in one binary relation.

§4. The number of countable rigid models and the Barwise compactness theorem.

In the author's abstract [18] two theorems were claimed. Unfortunately there was a mistake in the proof (pointed out by S. Shelah). Here is what remains:

Theorem B ($\neg$CH). For $\kappa = \aleph_1$ if $L \models$ "L_κ is Σ_1-compact (L_κ is Σ_1-compact)" then $\forall\, \theta \in L_{\omega_1 \omega} \wedge L$ (θ first order sentence) if θ has exactly $\aleph_1$-rigid models then θ has an uncountable rigid model.

L is the constructible sets of Gödel. An admissable set M is Σ_1-compact if for every T a Σ_1 definable subset of $L_{\omega_1\omega} \cap M$, if every $\Delta \in M$ included in T has a model then T has a model. Σ_1 means Σ_1 without parameters.

Lemma 3 is due to Mansfield (19)

Lemma 3. Suppose A is $\underset{\sim}{\Sigma}_1$ over (HC,ε) (the hereditarily countable sets) with constructible parameter. If $A - L \neq \theta$ then $|A| = 2^{\aleph_0}$.

$\equiv_{\infty,\omega}$, $\equiv^{\alpha}$ for α an ordinal

$\sigma^{\alpha}_{\bar{a}}(\bar{x})$ for $\bar{a} \in |\mathfrak{A}|^{<\omega}$

$Sr(\mathfrak{A})$ Scott rank of $\mathfrak{A}$

are defined in Barwise [20], p. 297-303.

Definition: $\mathfrak{A}$ is ∞-rigid iff $\forall a,b \in |\mathfrak{A}|$ $(\langle \mathfrak{A},a\rangle \equiv_{\infty,\omega} \langle \mathfrak{A},b\rangle \to a = b)$. Note that $\mathfrak{A}$ ∞-rigid $\Longrightarrow$ $\mathfrak{A}$ is rigid and vice versa if $\mathfrak{A}$ is countable. They are not equivalent since AC allows us to find a dense $A \subseteq \mathbb{R}$ such that $(A,<)$ is rigid.

Definition: $T_{\alpha}(\mathfrak{A})$ for α an ordinal.

$T_0(\mathfrak{A}) = \{(\bar{a},\bar{b}): (\bar{a},\bar{b}) \in \bigcup_{n<\omega} A^n \times A^n$ and $a_i \to b_i$ is a partial isomorphism$\}$.

$$T_{\alpha+1}(\mathfrak{A}) = \{(\bar{a},\bar{b}) \in T_{\alpha}(\mathfrak{A}): \forall a\, \exists b\, \langle \bar{a}^\frown a,\bar{b}^\frown b\rangle \in T_{\alpha}(\mathfrak{A})$$
$$\forall b\, \exists a\, \langle \bar{a}^\frown a,\bar{b}^\frown b\rangle \in T_{\alpha}(\mathfrak{A})\}$$

$T_{\alpha}(\mathfrak{A}) = \bigcap_{\beta<\alpha} T_{\beta}(\mathfrak{A})$ for α limit.

Lemma 4. Suppose $Sr(\mathfrak{A}) = \alpha$ then the following are equivalent:

(1) $\mathfrak{A}$ is ∞-rigid.

(2) $\forall a,b \in |\mathfrak{A}|$ $((\mathfrak{A} \models \sigma^{\alpha}_{a}(b))$ iff $a = b)$

(3) $\forall a,b \in |\mathfrak{A}|$ $(\langle a,b\rangle \in T_\alpha(\mathfrak{A})$ iff $a = b)$

Proof.

(1) iff (2) is just 6.3 of Barwise [20], p. 298 and the definition of $Sr(\mathfrak{A})$.

(1) iff (3) is proved by showing by induction on β that $\forall \bar{a},\bar{b}$ $\langle \mathfrak{A},\bar{a}\rangle \equiv_\beta \langle \mathfrak{A},\bar{b}\rangle$ iff $\langle \bar{a},\bar{b}\rangle \in T_\beta(\mathfrak{A})$

since $\langle \mathfrak{A},\bar{a}\rangle \equiv^\alpha \langle \mathfrak{A},\bar{b}\rangle$ iff $\langle \mathfrak{A},\bar{a}\rangle \equiv_{\infty,\omega} \langle \mathfrak{A},\bar{b}\rangle$

the result follows. ⊠

The idea behind the proof of the next lemma was suggested to me by Charles Gray.

Lemma 5. If $2^{\aleph_0} > \aleph_1$ and θ has exactly $\aleph_1$ rigid models all of which are countable then $\exists \mathfrak{A}_\alpha$ $\alpha < \aleph_1$, such that $\mathfrak{A}_\alpha \in L$, $|\mathfrak{A}_\alpha| = \lambda_\alpha$, λ_α's are strictly increasing and less than $\aleph_1$ and $\forall \alpha$ $\mathfrak{A}_\alpha$ is an ∞-rigid model of θ.

pf

For $\mathfrak{A}$ ∞-rigid define $\mathfrak{B}(\mathfrak{A})$ the canonical model isomorphic to $\mathfrak{A}$. Let $\alpha = Sr(\mathfrak{A})$

$|\mathfrak{B}(\mathfrak{A})| = \{\sigma_a^\alpha(v_1): a \in |\mathfrak{A}|\}$.

$R^{\mathfrak{B}(\mathfrak{A})}(\sigma_a^\alpha(v_1),\ldots,\sigma_{a_n}(v_1)) \leftrightarrow R^{\mathfrak{A}}(a_1,\ldots,a_n)$.

Note that by Lemma 4, part 2) for $\mathfrak{A}$ ∞-rigid $\mathfrak{B}(\mathfrak{A}) \simeq \mathfrak{A}$ and $\forall \mathfrak{A}'$ ∞-rigid $(\mathfrak{A}' \simeq \mathfrak{A}$ iff $\mathfrak{B}(\mathfrak{A}) = \mathfrak{B}(\mathfrak{A}'))$.

Define $A = \{\mathfrak{B}: (HC,\varepsilon) \models$ "$\exists \mathfrak{A}, \mathfrak{A} \models$"$\theta$"$\wedge$ $\mathfrak{A}$ ∞-rigid $\wedge$ $\mathfrak{B} = \mathfrak{B}(\mathfrak{A})$"$\}$. A is a $\underset{\sim}{\Sigma}_1$ HC set without parameters and has the same cardinality as the number of countable rigid models of θ. Since $2^{\aleph_0} > \aleph_1$, by Lemma 3 every member of A is constructible. The existence of the sequence described is immediate since $|A| = \aleph_1$. ■

We now write a theory T Σ_1 over (L_κ, ε) without parameters.

Let $\bar{R}$ be the similarity type of θ, then the language of T is: ε, c_a for $a \in L_\kappa$, $\bar{R}$, λ (new individual constant).

T will say the following:

1) "ZFC^-"
2) for each $a \in L_\kappa$ "$\forall x \quad x \in c_a \leftrightarrow \bigvee_{b \in a} x = c_b$"
3) "λ is an ordinal" and for each $\alpha < \kappa$ "$\lambda > c_\alpha$"
4) "$(\lambda, R) \models \theta$"
5) "(λ, R) is ∞-rigid".

Note that (L_κ, ε) is essentially uncountable from the view point of L, thus by Σ_1 compactness and Lemma 5 and Theorem 9.5, p. 359 of Barwise [20] T has a well-founded model M. Since $M \models ZFC^-$, $\alpha = Sr((\lambda, \bar{R})) \varepsilon M$ and so is $(T_\beta(\lambda, \bar{R}) : \beta \leq \alpha)$; hence we get an uncountable ∞-rigid model of θ. Theorem B ■

Remarks:

a) Using the fact that there are only countably many first order formulas it's easy to see that there are regular Σ_1

compact cardinals less than $\aleph_{\omega_1}$. However κ $\underset{\sim}{\Sigma}_1$-compact implies that κ is inaccessible, limit of inaccessibles, etc., see Barwise [20].

Questions:

(a) If $V = L$ does there exist T complete first order such that $\{\alpha \in OR : \langle L_\alpha, \varepsilon \rangle \models T\}$ is an unbounded subset of ω_1?

(b) Does $L \models$ "$L_{\aleph_1}$ Σ_1-compact" $\wedge$ $2^{\aleph_0} > \aleph_1$ imply that every Π^1_1 sentence with exactly $\aleph_1$ countable models has an uncountable model?

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