Special sets of reals [1]

Arnold W. Miller

Introduction

This is a survey paper which will update Miller [1984b]. It is concerned with totally imperfect subsets of the reals. A set of reals is totally imperfect iff it does not contain an uncountable closed set.

Examples of such sets are: Luzin sets, Sierpiński sets, concentrated sets, strong measure zero sets, universal measure zero sets, perfectly meager sets, strong first category sets, σ-sets, λ-sets, Q-sets, γ-sets, s_0-sets, and C'-sets. We have concentrated this survey on answering the questions which appeared in the earlier one.

Forcing Axiom

(Q1)Is $\neg CH$ consistent with either of:

(P1) For every ccc poset $\mathbb{P}$ of cardinality less than or equal to the continuum $\mathfrak{c}$, there exists a Luzin set of filters, i.e., $\langle G_\alpha : \alpha < \omega_1\rangle$ such that each G_α is a $\mathbb{P}$-filter and for every dense set $D \subseteq \mathbb{P}$ for all but countably many α we have $G_\alpha \cap D \neq \emptyset$.

(P2) For all ccc σ-ideals I in the Borel subsets of $\mathbb{R}$ there exists an I-Luzin set, i.e., an uncountable set $X \subseteq \mathbb{R}$ such that for every $A \in I$ we have $A \cap X$ is countable.

These two properties seem related because of the following result of Martin and Solovay [1970].

Theorem 1 *(Martin, Solovay) MA is equivalent to the statement that for every ccc σ-ideal I in the Borel subsets of $\mathbb{R}$ and for every $J \in [I]^{<\mathfrak{c}}$ we have $\bigcup J \neq \mathbb{R}$.*

[1] in **Set Theory of the Reals**, ed Haim Judah, Israel Mathematical Conference Proceedings, vol 6 (1993), 415-432, American Math Society.

Corollary 2 *MA is equivalent to the statement that* $\mathfrak{c}$ *is regular and for every ccc* σ*-ideals* I *in the Borel subsets of* $\mathbb{R}$ *there exists a* $\mathfrak{c}$*-*I*-Luzin set, i.e.,* $X \in [\mathbb{R}]^{\mathfrak{c}}$ *such that for every* $A \in I$ *we have* $|A \cap X| < \mathfrak{c}$.

In Miller-Prikry [1984a] it is shown that (P2) is consistent with $\neg CH$. It is also shown that the regularity of $\mathfrak{c}$ is necessary in Corollary 2. Surprisingly Todorcevic [1991] has shown that (P1) implies CH.

Theorem 3 *(Todorcevic) The continuum hypothesis is equivalent to the proposition that every ccc poset* $\mathbb{P}$ *of cardinality less than or equal to the continuum* $\mathfrak{c}$ *has a Luzin set of filters.*

proof:

That this is true assuming CH is easy since such a poset contains only ω_1 maximal antichains. To prove the other direction we will use the following four lemmas. First define for $f, g \in \omega^\omega$,

$f \leq g$ iff for all $n \in \omega$ we have $f(n) \leq g(n)$, and

$f \leq^* g$ iff for all but finitely many $n \in \omega$ we have $f(n) \leq g(n)$.

Lemma 4 *Suppose* $F \subseteq \omega^\omega$ *are increasing functions well-ordered by* $\leq^*$ *and unbounded. Then there exists distinct* $f, g \in F$ *such that* $f \leq g$.

proof:

Since F is unbounded its order type under $\leq^*$ cannot have countable cofinality. Consequently, we can find $g_0 \in F$ such that $\{f \in F : f \leq^* g_0\}$ is dense in F. Let $F' \subseteq F$ be unbounded such that there exists n_0 such that for every $f \in F'$ and $n \geq n_0$ we have $g_0(n) \leq f(n)$. Let

$$T = \{s \in \omega^{<\omega} : \{f \in F' : f \supseteq s\} \text{ unbounded}\}.$$

It is easy to see that T cannot be finitely branching. Consequently, there exists $s \in T$ with $|s| \geq n_0$ and for which there exists infinitely many n such that $s\hat{\ }n \in T$. Let $f \supset s$ with $f \leq^* g_0$ and $f \in F$. For some $k > n_0$ we have that $g_0(l) \geq f(l)$ for all $l \geq k$. Let $g \in F'$ be such that $g \supseteq s\hat{\ }n$ where $n > f(k)$. Then $f \leq g$ because:

- $f(l) = g(l) = s(l)$ for $l \leq |s|$,

- $f(l) \leq f(k) \leq g(|s|) = n \leq g(l)$ for $|s| \leq l \leq k$ (by choice of n and since they are increasing),
- $f(l) \leq g_0(l) \leq g(l)$ for $l > k$ (by choice of k and definition of F').

■

Lemma 5 *Suppose $F \subseteq \omega^\omega$ are increasing functions ordered by $\leq^*$ in type ω_1 and unbounded. Suppose $\mathbb{P}$ is the poset of finite pairwise $\leq$-incomparable subsets of F. Then $\mathbb{P}$ has ccc.*

proof:

First note that for every $A, B \subseteq F$ unbounded there exists unbounded $A' \subseteq A$ and $B' \subseteq B$ such that for every $f \in A'$ and $g \in B'$ we have f and g are $\leq$-incomparable. This is easy to see since if we let

$$T_A = \{s \in \omega^{<\omega} : \{f \in A : f \supseteq s\} \text{ unbounded}\}$$

$$T_B = \{s \in \omega^{<\omega} : \{f \in B : f \supseteq s\} \text{ unbounded}\}$$

then these are both infinite branching trees and so we can choose $s \in T_A$ and $t \in T_B$ with $s(k) < t(k)$ and $t(l) < s(l)$ for some k and l.

To see that $\mathbb{P}$ has ccc let $p_\alpha \in \mathbb{P}$ be given for $\alpha < \omega_1$. By applying the Δ-system lemma we may assume they are disjoint and also all of the same size, say $p_\alpha = \{f^i_\alpha : i < n\}$. Now repeatedly apply the note to the families $\{f^i_\alpha : \alpha < \omega_1\}$ and $\{f^j_\alpha : \alpha < \omega_1\}$ for each pair i and j.

■

Lemma 6 *Suppose $F \subseteq \omega^\omega$ are increasing functions ordered by $\leq^*$ in type ω_1 and unbounded. Then there exists $F' =^* F$, i.e., $F' = \{f' : f \in F\}$ where each $f' =^* f$, such that for every $p \in \mathbb{P}_{F'}$ there exists unboundedly many $f \in F'$ such that $p \cup \{f\} \in \mathbb{P}_{F'}$.*

proof:

This is a dovetailing argument. Let $F = \langle f_\alpha : \alpha < \omega_1 \rangle$ and let $\langle A_\alpha : \alpha < \omega_1 \rangle$ list $[\omega_1]^{<\omega}$ with $A_\alpha \subseteq \alpha$ and ω_1 repetitions. Construct $f'_\alpha =^* f_\alpha$ so that if $\{f'_\beta : \beta \in A_\alpha\} \in \mathbb{P}_{F'}$, then $(\{f'_\beta : \beta \in A_\alpha\} \cup \{f'_\alpha\}) \in \mathbb{P}_{F'}$.

■

Lemma 7 *(P2) There exists $F \subseteq \omega^\omega$ with $|F| = \omega_1$ which is dominating with respect to $\leq^*$.*

proof:

Let $\mathbb{H} = \{(f,n) : n \in \omega, f \in \omega^\omega\}$ and $(f,n) \leq (g,m)$ iff $n \geq m$, $f \geq g$, and $f \restriction m = g \restriction m$. This is the usual order for forcing a dominating real. Now if $\langle G_\alpha : \alpha < \omega_1 \rangle$ is a Luzin set of filters, they will determine the set F which was needed.

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Finally, we give the proof of Theorem 3. Let $h_\alpha : \alpha \to \omega_1$ be one-to-one for each $\alpha < \omega_2$. Use Lemma 7 and Lemma 6 to obtain $F = \{f_\alpha : \alpha < \omega_1\} \subseteq \omega^\omega$ which is unbounded (in fact dominating) and for every $p \in \mathbb{P}_F$ there exists unboundedly many $f \in F$ such that $p \cup \{f\} \in \mathbb{P}_F$.

Define the poset $\mathbb{P}$ as follows: $(p, A) \in \mathbb{P}$ iff

- $p \in \mathbb{P}_F$,
- $A \in [\omega_2]^{<\omega}$, and
- for all $\alpha, \beta \in A$ if $\alpha < \beta$ then there exists $\gamma > h_\beta(\alpha)$ such that $f_\gamma \in p$.

Order $\mathbb{P}$ by inclusion on each coordinate.

Claim. $\mathbb{P}$ has ccc.

proof:

If $p \cup p' \in \mathbb{P}_F$, then by Lemma 6

$$(p \cup p' \cup \{f_\gamma\}, A \cup A') \leq (p, A),\ (p', A')$$

for all sufficiently large γ.

■

Note that $\{(p,A) : \gamma \in A\}$ is dense for each $\gamma \in \omega_2$. If there is a Luzin set of filters, then one of them must meet at least ω_2 of these dense sets. Consequently there exists $\mathbb{P}$-filter G with $|G| = \omega_2$. Now look at $X = \bigcup\{A : \exists p\ (p,A) \in G\}$. Since X has cardinality ω_2, it has an ω_1^{th} element and so the set $\bigcup\{p : \exists A\ (p,A) \in G\}$ has cardinality ω_1. But this contradicts Lemma 4. This proves Theorem 3.

■

Note that this proof also gives a counterexample to Theorem 2.3 of Miller and Prikry [1984a]. The mistake in the proof occurs in the first sentence where it is stated that "without loss of generality we may assume $\mathbb{P}$ is a boolean algebra". To correct that theorem we must change it to read:

Theorem 8 *Suppose GCH and $\mathbb{P}$ is ccc of cardinality ω_2, then there exists a countably closed poset $\mathbb{Q}$ with ω_2-cc and cardinality ω_2 such that in $V^{\mathbb{Q}}$ there exists $\langle G_\alpha : \alpha < \omega_1 \rangle$ centered subsets of $\mathbb{P}$ such that for every dense $D \subseteq \mathbb{P}$ for all but countably many α we have $G_\alpha \cap D \neq \emptyset$.*

Now, of course, in a boolean algebra centered subsets generate filters, but in a partial order this may not be the case. This seems to be one of the ingredients of Todorcevic's example. The following example sheds more light on this.

Theorem 9 *(Juhasz-Kunen) There exists a poset $\mathbb{P}$ which is σ-centered but not σ-filtered.*

proof:

Let $\mathbb{P} = \{(C, F) : C \subseteq 2^\omega \text{ clopen}, \emptyset \neq F \in [C]^{<\omega}, \mu(C) \leq \frac{1}{|F|}\}$. Order $\mathbb{P}$ by: $(C_0, F_0) \leq (C_1, F_1)$ iff $C_0 \subseteq C_1$ and $F_0 \supseteq F_1$. To see that $\mathbb{P}$ is σ-centered, note that there are only countably many clopen sets and any finite set of conditions with the same clopen set have a lower bound. To see that it cannot be written as the union of countably many filters, for any filter G define

$$F_G = \bigcup\{F : \exists C\ (F, C) \in G\}$$

and

$$C_G = \bigcap\{C : \exists F\ (F, C) \in G\}.$$

Note that $F_G \subseteq C_G$ and either F_G is finite or C_G has measure zero. So in either case F_G has measure zero. Consequently, given any family $\{G_n : n \in \omega\}$ of filters there exists $x \in 2^\omega \setminus \bigcup_{n\in\omega} F_{G_n}$. Since $(2^\omega, \{x\}) \in \mathbb{P}$ we have $\mathbb{P} \neq \bigcup_{n\in\omega} G_n$.

■

Strong first category

(Q2)(Galvin) Does every Sierpinski set have strong first category?

An uncountable set of reals is a Sierpinski set iff it meets every measure zero set in a countable set. A set has strong first category iff it can be translated away from every measure zero set. This problem remains open although there are some partial results.

Theorem 10 *(Bartoszynski-Judah [1990]) (CH) Every Sierpinski set is the union of two sets of sets of strong first category.*

proof:

Let $X = \{x_\alpha : \alpha < \omega_1\}$ be any Sierpinski set. Construct X_i^α countable for $\alpha < \omega_1$ and $i < 2$ so that

- $X_i^\alpha \subseteq X_i^\beta$ if $\alpha < \beta$,
- $X_0^\alpha \cap X_1^\alpha = \emptyset$,
- $X_0^\alpha \cup X_1^\alpha \subseteq X$, and
- $x_\alpha \in X_0^{\alpha+1} \cup X_1^{\alpha+1}$.

Afterwards $X_i = \bigcup_{\alpha<\omega_1} X_i^\alpha$ will give us our partition $X = X_0 \cup X_1$ into sets of strong first category. Let $\langle G_\alpha, i_\alpha\rangle$ for $\alpha < \omega_1$ list all pairs $\langle G, i\rangle$ for G a G_δ-set of measure zero and $i \in \{0,1\}$. Let $X_i^0 = \emptyset$ and for limit ordinals α let $X_i^\alpha = \bigcup_{\beta<\alpha} X_i^\beta$. At stage $\alpha+1$ proceed as follows. Let $i = i_\alpha$ and $G = G_\alpha$. First find z such that

$$(z + G) \cap X_i^\alpha = \emptyset.$$

This is easy to do because the set X_i^α is countable, so any z not in the measure zero set $X_i^\alpha - G$ will do. Now let

$X_i^{\alpha+1} = X_i^\alpha$ and let

$X_{1-i}^{\alpha+1} = X_{1-i}^\alpha \cup ((z + G) \cap X) \cup \{x_\alpha\}$ (where $\{x_\alpha\}$ is added only if it is not in $X_i^{\alpha+1}$.

So $(z + G) \cap X \subseteq X_{1-i}$ and therefore $(z + G) \cap X_i = \emptyset$.

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(Q3)It is also an open question of Galvin if the union of two strong first category sets must have strong first category.

Theorem 11 *(Bartoszynski-Judah [1990]) If ZFC is consistent, then so is ZFC + there exists a Sierpinski set + every Sierpinski set has strong first category.*

This model is obtained by starting with a model of MA+$\neg$CH and adding ω_1 random reals.

Note that by a Lowenheim-Skolem argument, if there exists a Sierpinski set which fails to have strong first category, then there exists an inner model of CH in which there exists a Sierpinski set which fails to have strong first category. Woodin [to appear] has shown that Σ^2_1 statements are absolute between models of CH, assuming there is a Woodin cardinal. He has noted that the existence of a Sierpinski set which fails to have strong first category is a Σ^2_1 statement, consequently, it is unlikely to be independent of CH. Note that in the model of Theorem 11 CH fails.

Theorem 12 *(Jasinski-Weiss [1991]) If S is a Sierpinski set and $\bigcup_{n<\omega} C_n$ is a union of compact sets of measure zero, then there exists x such that $(x + \bigcup_{n<\omega} C_n) \cap S = \emptyset$.*

proof:

In fact, $\{x : (x + \bigcup_{n<\omega} C_n) \cap S = \emptyset\}$ is comeager. The following is copied from Miller [1990/1].

Let the C_n be increasing and compact. Let $\bigcap_{n<\omega} U_n$ be decreasing, open, $\mu(U_n) < \frac{1}{n}$, and $\mathbb{Q} + \bigcup_{n<\omega} C_n \subseteq \bigcap_{n<\omega} U_n$.

Claim. $Y = \{x : x + \bigcup_{n<\omega} C_n \subseteq \bigcap_{n<\omega} U_n\}$ is a dense G_δ and hence comeager.
proof:

Since Y contains the rationals $\mathbb{Q}$ it is enough to see that it is a G_δ. But note that for each n

$$\{x : x + C_n \subseteq U_n\}$$

is open, since C_n is compact and U_n is open. Since

$$Y = \{x : x + \bigcup_{n<\omega} C_n \subseteq \bigcap_{n<\omega} U_n\} = \bigcap_{n<\omega} \{x : x + C_n \subseteq U_n\}$$

it is G_δ and the Claim is proved.

To prove the theorem let $X = \bigcap_{n<\omega} U_n \cap S$ and let $Z = Y \setminus (X - \bigcup_{n<\omega} C_n)$. Since X is countable and so $X - \bigcup_{n<\omega} C_n$ is meager, it is enough to see that

$(z+\bigcup_{n<\omega} C_n)\cap S=\emptyset$ for every $z\in Z$. But $z\in Y$ implies $z+\bigcup_{n<\omega} C_n\subseteq \bigcap_{n<\omega} U_n$ implies $(z+\bigcup_{n<\omega} C_n)\cap S\subseteq X$ implies (if nonempty) that $z+c=x$ where $c\in\bigcup_{n<\omega} C_n$ and $x\in X$ so finally $z\in X-\bigcup_{n<\omega} C_n$.
■

The referee has informed me that Theorem 12 was also proved by Tim Carlson.

The following is an open question.

(Q4)Is it consistent to have every strong measure zero set countable (Borel conjecture) and every first category zero set countable (Dual Borel conjecture)?

Carlson's theorem that in the Cohen real model the Dual Borel conjecture holds has been extended by Judah-Shelah [1989] to show that it is consistent with MA(σ-centered) that the Dual Borel conjecture holds. This argument has been improved by Pawlikowski [1990] to show that any model obtained by the finite support iteration of σ-centered forcing the dual Borel conjecture holds. Judah-Shelah-Woodin [1990] have shown that the Borel conjecture is consistent with the continuum arbitrarily large. See Bartoszynski-Shelah [to appear] or Bartoszynski-Judah [to appear] for a correction to that argument.

Q-sets and strong measure zero

A set of reals is a Q-set iff every subset is a relative G_δ. A set of reals X has strong measure zero iff for any sequence of positive reals $\langle\epsilon_n : n\in\omega\rangle$ the set X can be covered by sequence of sets $\langle I_n : n\in\omega\rangle$ where each I_n has diameter less than ϵ_n.

(Q5)(Fleissner) Does every Q-set have strong measure zero?

This question was answered by the following theorem.

Theorem 13 *(Judah-Shelah [1988]) If we add one Mathias or Laver real f to a model M of MA+$\neg$CH, then in $M[f]$ every uncountable set of reals $X\in M$ remains a Q-set and furthermore does not have strong measure zero.*

Judah-Shelah [to appear] constructed a Q-set in a model $V[G]$ where V is a model of GCH and the extension satisfies the Sack's property: for every

$f \in \omega^\omega \cap V[G]$ there exists $\langle H_n : n \in \omega \rangle \in V$ such that for every n $f(n) \in H_n$ and $|H_n| \leq 2^n$.

Strong measure zero sets are also denoted as C-sets. Rothberger defined C' and C'' as follows. A set of reals X is C'' iff for any sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of X there exists $\langle U_n : n \in \omega \rangle$ with each $U_n \in \mathcal{U}_n$ such that $X \subseteq \bigcup_{n\in\omega} U_n$. The definition of C' is the same except the open covers $\mathcal{U}_n$ are also required to be finite.

(Q6)(Rothberger) Does every set with property C' have property C''?

This question is answered in the negative in Miller-Fremlin [1988] where it is shown that assuming CH that there is a C' set which is not C''.

Another question about Q-sets was answered by Knight [to appear] where it is shown to be consistent to a have a Δ-set which is not a Q-set. A Δ-set is a set of reals X with the property that for every sequence $X_n \subseteq X$ with the property that $\bigcap_{n\in\omega} X_n = \emptyset$ there exists open sets U_n with $X_n \subseteq U_n$ and $(\bigcap_{n\in\omega} U_n) \cap X = \emptyset$. For more about Δ-sets see Tanaka [1986], [1990].

Hierarchy order

If X is a separable metric space, then define $ord(X)$ (Borel rank or Baire order) to be the smallest $\alpha < \omega_1$ such that every Borel set in X is Σ^0_α in X. A set A is analytic in X iff $A = \bigcup_{f\in\omega^\omega} \bigcap_{n\in\omega} B_{f\restriction n}$ for some Borel sets B_s in X for $s \in \omega^{<\omega}$.

(Q7)(Mauldin) Is it consistent to have a space X where every $ord(X) < \omega_1$ but not every analytic set is Borel?

This question remains open. A related result is proved in Miller [1990a]: assuming CH there exists a separable metric space X such that every analytic in X set is Borel in X but there exists a set analytic in X^2 set which is not Borel in X^2. This doesn't answer the question since the space X^2 has Borel subsets of arbitrarily large rank while X has bounded Borel rank.

The following is another open question along similar lines.

(Q8)Is it consistent to have a space X where $ord(X) > 3$ but the difference hierarchy inside the Δ^0_3 sets is bounded?

The paper Miller [1990a] contains some related results about projective hierarchies.

Universal measurable sets

A set of reals is universally measurable iff it is measurable with respect to any measure on the real line.

(Q9)(Mauldin) Is it consistent to have only $\mathfrak{c}$ many universally measurable sets?

This question remains open. It is theorem of Laver that it is consistent that there are only $\mathfrak{c}$ universal measurable zero sets. This holds, for example, in the random real model (see Miller [1983])

The following theorem of Grzegorek and Ryll-Nardzewski seems relevant.

Theorem 14 *(Grzegorek, Ryll-Nardzewski [1980]) There are universally measurable sets which are not equal to a Borel set modulo a universal measure zero set.*

proof:

Let W be the Π^1_1 set of well-orderings. Then W is universally measurable. On the other hand if B is any Borel set then $W \Delta B$ cannot be universally measurable.

$$W \Delta B = (W \setminus B) \cup (B \setminus W)$$

The set $B \setminus W$ is Σ^1_1, hence if it is uncountable it would contain a perfect set and not be universal measure zero. So without loss we may assume $B \subseteq W$. But then by the boundedness theorem it is easy to get a perfect subset of $W \setminus B$ and so it doesn't have universal measure zero.

■

Perfectly meager

A set of reals P is perfect iff it is homeomorphic to 2^ω. A set of reals X is perfectly meager iff for every perfect set P the set $X \cap P$ is meager in P.

(Q10)(Marczewski) Is the product of perfectly meager sets perfectly meager?

This was answered by Reclaw.

Theorem 15 *(Reclaw [to appear]) If there exists a Luzin set, for example assuming CH, then there exists X and Y are perfectly meager such that $X \times Y$ is not perfectly meager.*

proof:

Let $\langle C_s : s \in \omega^{<\omega} \rangle$ be a family of perfect subsets of 2^ω such that

- $C_{\langle\rangle} = 2^\omega$,
- for each $s \in \omega^{<\omega}$ $\langle C_{s\hat{}n} : n \in \omega \rangle$ is a disjoint family of nowhere dense subsets of C_s,
- for each $s \in \omega^{<\omega}$ every $t \in 2^{<\omega}$ if $C_s \cap [t] \neq \emptyset$, then for some $n \in \omega$ we have $C_{s\hat{}n} \subseteq [t]$.

The Luzin function $f : \omega^\omega \to 2^\omega$ is defined by $\{f(x)\} = \bigcap_{n\in\omega} C_{x\restriction n}$. We will use the following forcing Lemma about f. Let $\mathbb{P} = \omega^{<\omega}$, that is Cohen forcing for ω^ω.

Lemma 16 *Let (x, y) be the pair of Cohen reals added by forcing with $\mathbb{P}^2$. For every perfect $P \subseteq (2^\omega)^2$ and $p \in \mathbb{P}$ there exists $q \leq p$ and $C \subseteq P$ closed nowhere dense in P such that $q \Vdash$ "$(x, f(y)) \in C$ or $(x, f(y)) \notin P$".*

proof:

Let $p = \langle p_o, p_1 \rangle$.

Case 1. $\{x\} \times C_{p_1} \not\subseteq P$.

Since P is closed there exists $t \in 2^{<\omega}$ such that $[t] \cap C_{p_1} \neq \emptyset$ and $q_0 \leq p_0$ such that

$q_0 \Vdash$"$(\{x\} \times [t]) \cap P = \emptyset$".

For some $m \in \omega$ we have that $C_{p_1\hat{}m} \subseteq [t]$. Let $q_1 = p_1\hat{}m$. Then $q \Vdash$"$(x, f(y)) \notin P$".

Case 2. $\{x\} \times C_{p_1} \subseteq P$.

Let $q_0 \subseteq x$ with $q_0 \leq p_0$ such that $q_0 \Vdash$"$\{x\} \times C_{p_1} \subseteq P$". Let

$$H = \{u \in 2^\omega : \{u\} \times C_{p_1} \subseteq P\}.$$

It is easy to see that H is closed. Since there is a comeager in $[q_0]$ set of Cohen reals in $H \cap [q_0]$ it must be that $[q_0] \subseteq H$ and thus $[q_0] \times C_{p_1} \subseteq P$. If we let $q_1 = p_1\hat{}n$ for an arbitrary n and $C = [q_0] \times C_{q_1}$, then $C \subseteq P$ is nowhere dense in P and $q \Vdash$"$(x, f(y)) \in C$".

■

Finally we show how the lemma proves Theorem 15. Suppose

$$K = \{(x_\alpha, y_\alpha) : \alpha < \kappa\}$$

is a Luzin set. This means that κ is uncountable, but for every countable transitive model M of a fragment of set theory, all but countably many of the (x_α, y_α) are $\mathbb{P}^2$ generic over M. It follows from Lemma 16 that

$$K_1 = \{(x_\alpha, f(y_\alpha)) : \alpha < \kappa\}$$

is perfectly meager. To see this, let $P \subseteq (2^\omega)^2$ be any perfect set. Let M be a countable transitive model of a partial fragment of set theory which contains a code for P. Let $\{C_n : n \in \omega\}$ be all closed nowhere dense in P sets which are coded in M. By Lemma 16 for all but countably many α we have that $(x_\alpha, f(y_\alpha)) \notin P$ or $(x_\alpha, f(y_\alpha)) \in \bigcup_{n\in\omega} C_n$. Hence $K_1 \cap P$ is meager in P. Similarly

$$K_2 = \{(f(x_\alpha), y_\alpha) : \alpha < \kappa\}$$

is perfectly meager. However consider

$$H = \{\langle(x_\alpha, f(y_\alpha)), (y_\alpha, f(x_\alpha))\rangle : \alpha < \kappa\}.$$

$H \subseteq K_1 \times K_2$ and H is homeomorphic to the Luzin set K since it is essentially just the graph of the function $f \times f : K \to 2^\omega$. It follows that H is Luzin in its closure and so $K_1 \times K_2$ is not perfectly meager.

■

A generalization of this result appears in Pawlikowski [1989].

σ-sets

A set of reals is a σ-set iff every relative Borel set is a relative F_σ-set, i.e. $ord(X) = 2$.

(Q11)Can you map a σ-set continuously onto the reals?

This was answered by Reclaw.

Theorem 17 *(Reclaw) If a set of reals X can be mapped continuously onto 2^ω, then $ord(X) = \omega_1$.*

proof:

For a countable $H \subset P(Y)$ define the Borel hierarchy generated by H as follows. Let

$$\Sigma_0(H) = H$$

and for $\alpha < \omega_1$ let

$$\Sigma_\alpha(H) = \{\cup_{n\in\omega}(Y \setminus A_n) : A_n \in \Sigma_{\beta_n}(H), \beta_n < \alpha\}.$$

Let Borel(H) be the union of all $\Sigma_\alpha(H)$ for $\alpha < \omega_1$. We will need the following theorem which generalizes the classical theorem of Lebesgue.

Theorem 18 *(Bing, Bledsoe, Mauldin [1974]) Suppose $H \subseteq P(2^\omega)$ is a countable family such that every clopen set is in Borel(H). Then $ord(H) = \omega_1$.*

proof:

To prove this theorem we will need the following two lemmas. Given a countable $H \subseteq P(2^\omega)$ let

$$R = \{C \times A : A \in H, C \subseteq 2^\omega \text{ is clopen}\}.$$

Lemma 19 *(Universal sets) For each α with $1 \leq \alpha < \omega_1$ there exists $U \in \Sigma_\alpha(R)$ which is universal for $\Sigma_\alpha(H)$ sets, i.e., for every $A \in \Sigma_\alpha(H)$, there exists $x \in 2^\omega$ such that $A = \{y : (x, y) \in U\}$.*

proof:

For $\alpha = 1$: Let $H = \{A_n : n \in \omega\}$ let

$$U = \bigcup_{n\in\omega} \{x : x(n) = 1\} \times (2^\omega \setminus A_n).$$

For $\alpha > 1$: Let $x \mapsto \langle x_n : n \in \omega\rangle$ be a nice recursive coding taking $2^\omega \to (2^\omega)^\omega$. Let β_n for $n \in \omega$ be cofinal in α, and $U_n \in \Sigma_{\beta_n}(R)$ be universal for $\Sigma_{\beta_n}(H)$ sets. Define U'_n by $(x, y) \in U'_n$ iff $(x_n, y) \in U_n$. It is easy to check that U'_n is also $\Sigma_{\beta_n}(R)$ and universal for $\Sigma_{\beta_n}(H)$. But now taking

$$U = \bigcup_{n\in\omega} (2^\omega \setminus U'_n)$$

gives us a set in $\Sigma_\alpha(R)$ which is universal for $\Sigma_\alpha(H)$ sets.

■

Lemma 20 *(Diagonalization) Suppose that every clopen set is in Borel(H). Then for every* $B \in$*Borel(R) the set* $\{x : (x, x) \in B\}$ *is in Borel(H).*

proof:

For $B = C \times A$ where $A \in H, C \subseteq 2^\omega$ is clopen. Note that $\{x : (x, x) \in B\} = C \cap A$. Since the by assumption $C \in$Borel(H), we have the result for elements of R. To do Borel(R) is an easy induction.

■

Now we give a proof of Theorem 18. Suppose Borel$(H) = \Sigma_\alpha(H)$. By Lemma 19 there exist U in Borel(R) which is universal for $\Sigma_\alpha(H)$ and hence Borel(H). By Lemma 20 the set

$$D = \{x : (x, x) \notin U\}$$

is in Borel(H). But this means that for some x that $D = \{y : (x, y) \in U\}$. But then $x \in D$ iff $x \notin D$.

■

Now we give a proof of Theorem 17. Suppose that $f : X \to 2^\omega$ is onto and continuous. Let $\mathcal{C}$ be a a countable clopen basis for X and let $H = \{f''C : C \in \mathcal{C}\}$. Since it is clear that H contains all clopen sets, by Theorem 18, the $ord(H) = \omega_1$. But the map f takes the Borel hierarchy of X directly over the hierarchy on Borel(H), so $ord(X) = \omega_1$.

■

An extension of Reclaw's result to Souslin (operation A) sets appears in Miller [1990a].

λ-sets and λ'-sets

A set of reals X is a λ-set iff every countable subset of X is a relative G_δ in X. A set of reals $X \subseteq Y$ is a λ'-set with respect to Y iff for every countable $F \subseteq Y$ the set $X \cup F$ is a λ-set. It is a theorem of ZFC that there are λ' sets of cardinality ω_1.

This section will clear up some confusion about a remark (p 266) in Reed [1983] which is attributed to me.

The cardinal $\mathfrak{b}$ is the minimum cardinality of a set $F \subseteq \omega^\omega$ such that F is unbounded, i.e. there does not exist $g \in \omega^\omega$ such that $f \leq^* g$. In Reed [1983] this is referred to as the least α for which $M(\alpha)$ holds.

The cardinal $\mathfrak{d}$ is the minimum cardinality of a set $F \subseteq \omega^\omega$ such that F is a dominating family, i.e., for every $g \in \omega^\omega$ there exists a $f \in F$ such that $g \leq^* f$. Obviously $\mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$.

Theorem 21 *(Rothberger [1939]) There exists a λ-set X of cardinality $\mathfrak{b}$, in fact, X is a λ'-set with respect to the irrationals.*

proof:
Let $F = \{f_\alpha : \alpha < \mathfrak{b}\} \subseteq \omega^\omega$ be an unbounded family such that $\alpha \leq \beta$ implies $f_\alpha \leq^* f_\beta$. We claim that for every countable $H \subseteq \omega^\omega$ the set $F \cup H$ is a λ-set. It is enough to see that for every countable $H \subseteq \omega^\omega$ H is a relative G_δ in $F \cup H$. Since F is unbounded there exists $\alpha < \mathfrak{b}$ such that for every $h \in H$ we have $f_\alpha \not\leq^* h$. This also holds for every $\beta > \alpha$ since $f_\beta \geq^* f_\alpha$. So if

$$K = \{g \in \omega^\omega : f_\alpha(n) \not\leq^* g\}$$

then K is a G_δ such that $H \subset K$ and $|K \cap (X \cup H)| < \mathfrak{b}$. So it now suffices to note the following:

Claim: Any set of reals Y of cardinality less than $\mathfrak{b}$ is a λ-set.

Let $Y = \{x_n : n < \omega\} \cup \{y_\alpha : \alpha < \kappa\}$. Let $\{U_n(x_m) : n \in \omega\}$ be a decreasing neighborhood base for each x_m. For each $\alpha < \kappa$ define $g_\alpha \in \omega^\omega$ so that $y_\alpha \notin U_{g_\alpha(n)}(x_n)$. Since $\kappa < \mathfrak{b}$ there exists $g \in \omega^\omega$ such that $g_\alpha \leq^* g$ for all $\alpha < \kappa$. Let $U = \bigcap_{n<\omega}(\bigcup_{m>n} U_{g(n)}(x_n))$. Then U is a G_δ-set such that $U \cap Y = \{x_n : n \in \omega\}$.
■

Theorem 22 *(Miller [1983]) In the Cohen real model there are no λ-sets of cardinality greater than ω_1.*

proof:
It is shown that every set of reals of cardinality ω_2 contains the one-to-one continuous image of a Luzin set. But such a set cannot be a λ-set. If L is Luzin and D a countable dense subset of L, then D cannot be relatively G_δ in L. But if $f : L \to X$ is one-to-one and continuous, then $f''D$ cannot be G_δ in X.
■

In the Cohen real model $\mathfrak{b} = \omega_1$ and $\mathfrak{d} = \mathfrak{c}$.

Theorem 23 *In the Laver model for the Borel conjecture ($\mathfrak{b} = \mathfrak{d} = \mathfrak{c} = \omega_2$) there does not exist a λ'-set with respect to the reals of cardinality ω_2.*

proof:

Lemma 14 of Laver [1976] implies the following lemma.

Lemma 24 *Suppose $p \Vdash$ "$\tau \in [0,1]$", then there exists $q \leq p$ and a finite set U_s for each splitting node s of $q(0)$, such that for all $\epsilon > 0$ and all but finitely many n with $s\hat{\ }n \in q(0)$*

$$q(0)_{s\hat{\ }n}\hat{\ }q \restriction [1, \omega_2) \Vdash \exists u \in U_s |u - \tau| < \epsilon$$

Lemma 25 *Given the above q, τ, and $U = \bigcup_{s \in q(0)} U_s$, for any G_δ set $G \supseteq U$ there exists $r \leq q$ such that $r \Vdash$ "$\tau \in G$".*

proof:

This is an easy fusion argument on $q(0)$. If $G = \bigcap_{n \in \omega} G_n$ where G_n is open, then make sure that everything in the n^{th} forces that $\tau \in G_n$.

■

Let $V[f_\beta : \beta < \omega_2]$ be the Laver model. To prove Theorem 23 suppose that $X \subseteq [0,1]$ is a λ'-set in $V[f_\beta : \beta < \omega_2]$. Via a Lowenheim-Skolem type argument there exists $\alpha < \omega_2$ such that

$$(X \cap V[f_\beta : \beta < \alpha]) \in V[f_\beta : \beta < \alpha]$$

and a function $f \in V[f_\beta : \beta < \alpha]$ such that for every countable $D \subseteq [0,1]$ in $V[f_\beta : \beta < \alpha]$ we have $f(D)$ is a G_δ code for a set G such that

$$G \cap (X \cup D) = D.$$

But now Lemma 25 (applied with $V[f_\beta : \beta < \alpha]$ as the ground model) implies that $X \subseteq V[f_\beta : \beta < \alpha]$. But since $V[f_\beta : \beta < \alpha]$ satisfies CH, the set X has cardinality ω_1.

■

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Woodin, H., to appear.

Department of Mathematics
University of Wisconsin
480 Lincoln Drive
Madison, WI 53706, USA
miller@math.wisc.edu.