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ON SQUARES OF SPACES AND F_σ-SETS

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Abstract. We show that the continuum hypothesis implies there exists a Lindelöf space X such that X^2 is the union of two metrizable subspaces but X is not metrizable. This gives a consistent solution to a problem of Balogh, Gruenhage, and Tkachuk. The main lemma is that assuming the continuum hypothesis there exist disjoint sets of reals X and Y such that X is Borel concentrated on Y, i.e., for any Borel set B if $Y \subseteq B$ then $X \setminus B$ is countable, but $X^2 \setminus \Delta$ is relatively F_σ in $X^2 \cup Y^2$.

In Balogh, Gruenhage, and Tkachuk [1] the following question is asked:

Question 4.1. Let X be a regular paracompact space X such that $X \times X$ is the union of two metrizable subspaces. Must X be metrizable? What if X is Lindelöf?

Theorem 1. *Assume the continuum hypothesis. Then there exists a nonmetrizable regular Lindelöf space X such that X^2 is the union of two metrizable subspaces.*

We first prove the following Lemma.

Lemma 2. *(CH) There are uncountable disjoint sets $X, Y \subseteq 2^\omega$ such that*

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(1) *X is Borel concentrated on Y, i.e. every Borel set in 2^ω containing Y contains all but countably many elements of X,*
(2) *$Y^2 \setminus \Delta$ is F_σ in $X^2 \cup Y^2$, and*
(3) *$X^2 \setminus \Delta$ is F_σ in $X^2 \cup Y^2$.*

Here $\Delta = \{(x,x) : x \in 2^\omega\}$.

Proof.
We identify the Cantor space 2^ω with the power set $P(\omega)$ of ω. We use $[\omega]^\omega$ to stand for the infinite subsets of ω. Define for $y \in [\omega]^\omega$

$$[y]^{*\omega} = \{x \in [\omega]^\omega : x \subseteq^* y\}$$

where $\subseteq^*$ stands for inclusion mod finite. Let $\langle B_\alpha : \alpha < \omega_1\rangle$ be all Borel subsets of $[\omega]^\omega$. We construct y_α for $\alpha < \omega_1$ so that

(1) $\alpha < \beta$ implies $y_\beta \subseteq^* y_\alpha$ and $y_\beta \neq^* y_\alpha$ and
(2) either $y_\alpha \notin B_\alpha$ or $[y_\alpha]^{*\omega} \subseteq B_\alpha$.

These conditions are easy to get. Given y_β for $\beta < \alpha$ and B_α let $y \in [\omega]^\omega$ be arbitrary with $y \subseteq^* y_\beta$ but $y_\beta \neq^* y$ for each $\beta < \alpha$. If $[y]^{*\omega}$ is not a subset of B_α, then simply take $y_\alpha \in [y]^{*\omega} \setminus B_\alpha$, otherwise take $y_\alpha = y$.

Let

$$X = \{y_\alpha \setminus y_{\alpha+1} : \alpha < \omega_1\} \text{ and } Y = \{y_\alpha : \alpha < \omega_1\}$$

Iff B is any Borel set containing Y, then choose α so that $B = B_\alpha$. At stage α of the construction it must have been that $[y_\alpha]^{*\omega} \subseteq B_\alpha$. But this means that $x_\beta \in B_\alpha$ for all $\beta \geq \alpha$. So X is Borel concentrated on Y.

If we define

$$F = \{(u,v) \in P(\omega) \times P(\omega) : (u \subseteq^* v \text{ or } v \subseteq^* u) \text{ and } u \neq v\}$$

then F is an F_σ set and

$$F \cap (X^2 \cup Y^2) = (Y^2 \setminus \Delta)$$

Also if we define

$$H = \{(u,v) \in P(\omega) \times P(\omega) : u \cap v =^* \emptyset\}$$

then H is an F_σ set and

$$H \cap (X^2 \cup Y^2) = (X^2 \setminus \Delta)$$

This finishes the proof of the Lemma.

QED

Now define the following Michael-line like topology. Suppose that M is a topological space and $X \subseteq M$. Then $M(X)$ is the topological space on the same set but with the following topology. For $x \in X$ we make x an isolated point, i.e., add $\{x\}$ to the topology of $M(X)$. For any point $y \in M \setminus X$ neighborhoods in M form a neighborhood basis for y in $M(X)$. It is easy to see that $M(X)$ is regular for any regular space M and subset $X \subseteq M$.

The following is Exercise 5.5.2 from Engelking [2]:

Proposition 3. *Suppose M is a metric space and $X \subseteq M$. Then $M(X)$ is metrizable iff X is an F_σ set in M.*

Our example is $M(X)$ where X and Y are from the Lemma and $M = X \cup Y$ has its usual (separable metric) topology as a subspace of 2^ω. It follows from the Proposition that $M(X)$ is not metrizable.

Claim 1. $M(X)$ is a Lindelöf space.

Take any open cover $\mathcal{U}$ of $M(X)$. Open sets in $M(X)$ have the form $U \cup Z$ where U is open in M and $Z \subseteq X$ is arbitrary. Then since Y has its standard topology, countably many elements of $\mathcal{U}$ will cover Y, say

$$\{(U_n \cup X_n : n < \omega\} \subseteq \mathcal{U}$$

where each U_n open in M and $X_n \subseteq X$. But since X is Borel concentrated on Y we have that $X \setminus \cup\{U_n : n < \omega\}$ is countable, so we need only add countably many more elements of $\mathcal{U}$ to cover all of $M(X)$.

Claim 2. $M(X)^2$ is the union of two metrizable subspaces.

Let

$M_1 = (X^2 \setminus \Delta) \cup Y^2$ and

$M_2 = (X \times Y) \cup (Y \times X) \cup (X^2 \cap \Delta)$.

Note that M_1 is $N(X^2 \setminus \Delta)$ where $N = (X^2 \setminus \Delta) \cup Y^2$ in its separable metric topology as a subspace of $2^\omega \times 2^\omega$. By the Lemma we have that $X^2 \setminus \Delta$ is relatively F_σ in N and so by Proposition 3 M_1 is metrizable.

To see that M_2 is metrizable use the Bing Metrization Theorem:

> A topological space is metrizable iff it is regular and has a σ-discrete basis.

A family $\mathcal{B}$ of subsets of X is discrete iff every point of X has a neighborhood meeting at most one element of $\mathcal{B}$. σ-discrete means the countable union of discrete families.

Note that for each $x \in X$ the sets $\{x\} \times Y$ and $Y \times \{x\}$ are open in M_2. Let $\mathcal{B}$ be a countable open basis for Y. Then

$$\mathcal{C} = \{U \times \{x\},\ \ \{x\} \times U,\ \ \{(x,x)\}\ :\ \ x \in X, U \in \mathcal{B}\}$$

is an open basis for M_2. It is σ-discrete. The family $\{\{(x,x)\} : x \in X\}$ is discrete in M_2 since $X^2 \cap \Delta$ is closed in M_2. And for each fixed $U \in \mathcal{B}$ the family $\{U \times \{x\}\ : x \in X\}$ is discrete in M_2. (For $(x,x) \in X$ use the neighborhood $\{x\} \times \{x\}$. For (y,x) with $y \in Y$ and $x \in X$ use the neighborhood $Y \times \{x\}$ and for (x,y) use the neighborhood $\{x\} \times Y$.) Similarly, for each $U \in \mathcal{B}$ the family $\{\{x\} \times U\ :\ x \in X\}$ is discrete in M_2. Since $\mathcal{B}$ is countable, M_2 has a σ-discrete basis and is therefor metrizable.

This proves Theorem 1.

QED

The next Theorem is an easy generalization of Theorem 1 using the tower cardinal $\mathfrak{t}$ which is defined as follows. $\mathfrak{t}$ is the minimum cardinality of a set $T \subseteq [\omega]^\omega$ which is linearly ordered by $\subseteq^*$ but there does not exist $z \in [\omega]^\omega$ with $z \subseteq^* y$ for every $y \in T$. Martin's axiom implies that $\mathfrak{t} = \mathfrak{c}$.

Theorem 4. *Suppose $\mathfrak{t} = \mathfrak{c}$. Then there exists a nonmetrizable regular paracompact space X such that X^2 is the union of two metrizable subspaces.*

Proof.
The main Lemma changes to:

Lemma 5. *($\mathfrak{t} = \mathfrak{c}$) There are disjoint sets $X, Y \subseteq 2^\omega$ of cardinality $\mathfrak{c}$ such that*

1. *X is Borel $\mathfrak{c}$-concentrated on Y, i.e., for every Borel set B in 2^ω, if $Y \subseteq B$ then $|X \setminus B| < \mathfrak{c}$,*
2. *$Y^2 \setminus \Delta$ is F_σ in $X^2 \cup Y^2$, and*
3. *$X^2 \setminus \Delta$ is F_σ in $X^2 \cup Y^2$.*

The proof is similar. The space $M = X \cup Y$ is the same. Since X is not relatively Borel in M we have by Proposition 3 that $M(X)$ is not metrizable. But $M(X)$ is regular and paracompact for any $X \subseteq M$ and metric M, see example 5.1.22 Engelking [2].

QED

Remark. The Michael line is the topological space $M(X)$ where M is the unit interval, $[0,1]$, and X the irrationals in $[0,1]$. Michael Granado in unpublished work has shown that the square of the Michael line is not the union of two metrizable subspaces.

Question 6. *(Using just ZFC) Do there exist disjoint sets of reals X and Y such that X is not F_σ in $X \cup Y$ but $X^2 \setminus \Delta$ is F_σ in $X^2 \cup Y^2$?*

References

1. Balogh, Zoltan; Gruenhage, Gary; Tkachuk, Vladimir; *Additivity of metrizability and related properties*, Proceedings of the International Conference on Set-theoretic Topology and its Applications, Part 2 (Matsuyama, 1994). Topology Appl. **84** (1998), no. 1-3, 91–103.
2. Engelking, Ryszard; *General topology.* Translated from the Polish by the author. Second edition. Sigma Series in Pure Mathematics, 6. Heldermann Verlag, Berlin, 1989. viii+529 pp. ISBN: 3-88538-006-4.

The appendix is not intended for final publication but for the electronic version only.

Appendix

> Suppose M is a metric space and $X \subseteq M$. Then $M(X)$ is metrizable iff X is an F_σ in M. (Engelking 5.5.2)

Proof.
Suppose X is not F_σ in M, then $Y = M \setminus X$ is closed in $M(X)$ (since the points of X are isolated, X is open). But Y is not G_δ in $M(X)$. To see, this suppose that $Y = \cap_{n\in\omega} U_n$ were each U_n is open in $M(X)$. Then there would exists V_n open in M and $X_n \subseteq X$ with $U_n = V_n \cup X_n$. But then $Y = \cap_{n\in\omega} V_n$ which contradicts Y is not G_δ in M.

For the converse, suppose X is F_σ in M and write it as the union of closed sets $X = \cup_{n<\omega} C_n$. $M(X)$ is regular so it is enough by the Bing Metrization Theorem to check that it has a σ-discrete base. Let $\mathcal{B}$ be a σ-discrete base for M. We claim that

$$\mathcal{B} \cup \{\{x\} : x \in X\}$$

which is a basis for $M(X)$ is σ-discrete in $M(X)$. $\mathcal{B}$ is σ-discrete in M so it is also σ-discrete in $M(X)$.

$$\{\{x\} : x \in X\} = \cup_{n<\omega} \mathcal{C}_n \text{ where } \mathcal{C}_n = \{\{x\} : x \in C_n\}$$

shows that it is σ-discrete, since for any n if $x \notin C_n$ then $M \setminus C_n$ is a neighborhood of x missing all elements of $\mathcal{C}_n$.

> $M(X)$ is regular paracompact, whenever M is metric. (Engelking 5.1.22)

Proof.
Regular: Given $p \in M$ if $p \in X$ then it is has the clopen neighborhood $\{p\}$, if $p \notin X$, then the neighborhoods of p in M are also a neighborhood basis in $M(X)$.

Paracompact: Let $\mathcal{U}$ be an open cover of basic open sets in $M(X)$. We may assume it has the form:

$$\mathcal{U} = \mathcal{V} \cup \{\{x\} : x \in Z\}$$

where $\mathcal{V}$ is a family of basic open sets in M and $Z = X \setminus \cup\mathcal{V}$. Since metric spaces are hereditarily paracompact, there exists a locally finite refinement $\mathcal{W}$ of $\mathcal{V}$ with $\cup\mathcal{V} = \cup\mathcal{W}$. But then $\mathcal{W} \cup \{\{x\} : x \in Z\}$ is a locally finite refinement of $\mathcal{U}$.

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