CHAPTER 5

Special Subsets of the Real Line

Arnold W. MILLER

The University of Texas,
Austin, TX 78712, USA

Contents

HANDBOOK OF SET-THEORETIC TOPOLOGY
Edited by K. Kunen and J.E. Vaughan

1. Introduction

The purpose of this chapter is to discuss some peculiar sets of real numbers and some of the methods for obtaining them. The first such set was constructed by Bernstein in 1908. He constructed a set of reals of cardinality the continuum which is neither disjoint from nor contains an uncountable closed set. His construction used transfinite induction and the fact that every uncountable closed set has cardinality the continuum. The kinds of singular sets which we will discuss can almost all be found in KURATOWSKI (1966, §40) "Totally imperfect spaces and other singular spaces".

We are going to be concerned with the topological notion of first category set or meager set and the fundamental properties of Lebesgue measure. Recall that a set is nowhere dense iff its closure has no interior. A set of reals is meager iff it is the countable union of nowhere dense sets. A set of reals is comeager iff it is the complement of a meager set. The Baire category theorem says that no complete metric space is meager in itself.

Let us establish some of the terminology to be used. $\mathbb{R}$ denotes the real line, $\mathbb{Q}$ the set of rationals, and $[0, 1]$ the closed unit interval. The symbol ω stands for the set $\{0, 1, 2, \ldots\}$ and 2 for the set $\{0, 1\}$. We will use ω_1 to denote the first uncountable cardinal and c to denote the cardinality of the continuum. The space ω^ω (Baire space) consists of the set of all functions from ω to ω. It is given the product topology where ω is given the discrete topology. This topology is most conveniently described as follows. Let $\omega^{<\omega}$ be the set of finite sequences of elements of ω. For any $s \in \omega^{<\omega}$ let

$$[s] = \{f \in \omega^\omega : s \subseteq f\},$$

and for any $n < \omega$ let $s\hat{\ }n$ be the finite sequence which begins like s and ends in n. The topology on ω^ω is given by letting $\{[s] : s \in \omega^{<\omega}\}$ be the basic open sets. Similarly 2^ω (Cantor space) is the space of all functions from ω to 2 given the product topology. The space 2^ω also has on it the product measure which is determined by declaring that for each $s \in 2^n$, $[s]$ has measure $(\frac{1}{2})^n$. Of course the space 2^ω is homeomorphic to Cantor's "middle thirds" set which is the set of all $x \in [0, 1]$ whose ternary expansion

$$x = \sum_{n=1}^{\infty} \frac{i_n}{3^n}$$

has only $i_n = 0$ or $i_n = 2$. There is a useful continuous map $\rho: 2^\omega \to [0, 1]$ defined by

$$\rho(x) = \sum_{n=0}^{\infty} \frac{x(n)}{2^{n+1}}.$$

This map is onto, one-to-one except on a countable set (the set of $x \in 2^\omega$ which are eventually one or eventually zero), and takes each $[s]$ onto an interval of the same measure. Furthermore if we identify (via characteristic functions) 2^ω with $P(\omega)$ (the set of all subsets of ω), then $X \subseteq Y$ implies $\rho(X) \leq \rho(Y)$. Let $[\omega]^\omega$ be the subspace of $P(\omega) = 2^\omega$ of infinite subsets of ω. Then ω^ω is homeomorphic to $[\omega]^\omega$ via the natural embedding $\sigma: \omega^\omega \to [\omega]^\omega$ where $\sigma(g)$ is the set contained in ω whose characteristic function is given by the sequence of zeros and ones:

$$0^{g(0)}1\, 0^{g(1)}1\, 0^{g(2)}1 \cdots$$

($g(0)$ zeros, then a one, then $g(1)$ zeros, then a one, etc.).

The space ω^ω is also homeomorphic to the set of irrationals. Let $\mathbb{Z}$ be the set of integers. Then clearly $\mathbb{Z}^\omega$ is homeomorphic to ω^ω. Construct a family of open intervals I_s for $s \in \mathbb{Z}^{<\omega}$ as follows. For each $n \in \mathbb{Z}$ let $I_{\langle n\rangle} = (n, n+1)$. Suppose we have already found I_s for some $s \in \mathbb{Z}^{<\omega}$. Let $\{I_{s^\frown n}: n \in \mathbb{Z}\}$ be a family of disjoint open subintervals of I_s ordered like $\mathbb{Z}$ and lying next to each other (i.e. the right hand end point of $I_{s^\frown n}$ is the left hand end point of $I_{s^\frown(n+1)}$), with union dense in I_s, and each having diameter less than $\frac{1}{2}$ the diameter of I_s. If we define $\tau: \mathbb{Z}^\omega \to R$ by

$$\{\tau(g)\} = \bigcap\{I_{g\restriction n}: n < \omega\},$$

then it is not hard to check that τ is a homeomorphism of $\mathbb{Z}^\omega$ and $R - H$ where H is the countable dense set of end points of the I_s's. If we give $\mathbb{Z}^\omega$ the lexicographical order, then τ is order preserving. Since H is order isomorphic to $\mathbb{Q}$ there is an order isomorphism

$$f: \mathbb{R} \to \mathbb{R}$$

taking H to $\mathbb{Q}$. I think this argument is roughly equivalent to the classical one using continued fractions (see ALEXANDROFF and URYSOHN (1927)).

2. Luzin and Sierpiński sets

Arguments using transfinite induction to construct singular sets of reals are certainly the most plentiful. Until Cohen's method of forcing arrived on the scene this was practically the only method used. Most of the time such a construction requires the continuum hypothesis (CH) or at least Martin's Axiom (MA). Most of the time I have refrained from pointing out the obvious generalization of an argument or definition under CH to one that works under MA.

In 1914 Luzin constructed, using the continuum hypothesis, an uncountable set of reals having countable intersection with every meager set. The same construction had been published in 1913 by Mahlo. But (as is not unusual in mathematics) such a set has become universally known as a Luzin set.

2.1. THEOREM (MAHLO (1913), LUZIN (1914)). *Assuming the continuum hypothesis there is a set of reals of cardinality the continuum which has countable intersection with every meager set.*

PROOF. Let $\{C_\alpha : \alpha < \omega_1\}$ be the set of all closed nowhere dense sets. Inductively choose x_α a real not in:

$$\{x_\beta : \beta < \alpha\} \cup \bigcup \{C_\beta : \beta < \alpha\} .$$

We can always make such a choice by the Baire category theorem. Then

$$X = \{x_\alpha : \alpha < \omega_1\}$$

is the required set. □

A similar construction using Borel sets of measure zero results in a set called a Sierpiński set. For some applications of Luzin and Sierpiński sets to topology see VAN DOUWEN, TALL, and WEISS (1977), TALL (1976), and TALL (1978).

2.2. THEOREM (SIERPIŃSKI (1924)). *Assuming the continuum hypothesis there is a set of reals of cardinality the continuum which has countable intersection with every measure zero set.*

HUREWICZ (1932) also used this argument to construct (assuming the continuum hypothesis) an uncountable set $X \subseteq \mathbb{R}^\omega$ with the property that every uncountable subset of X has infinite dimension. See also WALSH (1979).

A modern day construction of a Luzin set is to note that the generic set of reals in a Cohen extension is a Luzin set. Similarly the generic set of reals in Solovay's random real extension is a Sierpiński set. See Kunen's chapter for the details on this.

Assuming MA + $\neg$CH (Martin's axiom plus the failure of the continuum hypothesis) neither Luzin nor Sierpiński sets exist. This is because under MA any set of reals of cardinality less than the continuum is both meager and has measure zero (MARTIN–SOLOVAY (1970) see also SHOENFIELD (1975) and RUDIN (1977)).

KUNEN (1976) generalized this to show that assuming MA + $\neg$CH, if Y is a Hausdorff space without isolated points, then there are no Luzin sets in Y. It is easy to show that it is consistent with $\neg$CH that there exists a Luzin set and a Sierpiński set. One way to do this is to start with a model of $\neg$CH and iteratively with finite support forcing, add a Cohen real and a random real. In the resulting model the Cohen reals will be a Luzin set and the random reals will be a Sierpiński set. Another way is to note that a Luzin or Sierpiński set remains such when Sacks reals (perfect set forcing) are added. Already in 1938, ROTHBERGER knew that ω_1 was the best one could do.

2.3. THEOREM (ROTHBERGER (1938)). *Suppose X is a Luzin set and Y is a Sierpiński set. Then*

$$|X| = |Y| = \omega_1 .$$

PROOF. The following lemma is necessary to prove Theorem 2.3.

LEMMA. *If X is not meager (not measure zero) and X has cardinality κ, then the real line is the union of κ many measure zero sets (meager sets).*

PROOF. Let G be a comeager set of measure zero (see OXTOBY (1971), Corollary 1.7). Then the real line is covered by the family:

$$\{x + G : x \in X\} .$$

Because if z is not in $x + G$ for any $x \in X$, then $z - G$ is disjoint from X. But $z - G$ is comeager. A similar argument works for the dual statement. □

Since every uncountable subset of a Luzin set is a Luzin set, and Luzin sets are not meager, it follows that the existence of a Luzin set implies that the real line is the union of ω_1 many measure zero sets. Thus any Sierpiński set must have cardinality ω_1. Dually the existence of a Sierpiński set implies any Luzin set must have cardinality ω_1. □

It is possible to generalize the construction of Luzin sets as follows. Suppose I is a countably additive proper ideal of subsets of the real line containing all singletons. We call a set of reals I-Luzin iff it is uncountable and it has countable intersection with every element of I. A family $J \subseteq I$ generates I iff for all $X \in I$ there is a $Y \in J$ with $X \subseteq Y$. Assuming the continuum hypothesis and that the ideal I is generated by a family J of cardinality $\leqslant \omega_1$, it is easy to inductively construct an I-Luzin set. For example, if I is the ideal of meager sets (which is generated by the meager F_σ sets), then an I-Luzin set is just what Luzin constructed. Similarly if I is the ideal of measure zero sets (which is generated by the G_δ sets of measure zero), then an I-Luzin set is a Sierpiński set. An interesting extension of Luzin's construction is given by TALL (19··). He shows that assuming Baumgartner's axiom and the continuum hypothesis an I-Luzin set can be constructed for any ideal I generated by fewer than 2^{ω_1} sets.

In MILLER (1976b) I give a result of Kunen and myself that for each α, $2 < \alpha < \omega_1$, there is an ideal I_α (generated by its Borel members) such that any I_α-Luzin set has Baire order α.

If we assume Martin's axiom and the failure of the continuum hypothesis it is often the case that in order to generalize results proved under the continuum hypothesis you must replace 'countable' by 'less than continuum' (see KUNEN

(1968)). Thus we might define a c-I-Luzin set as a set of reals X of cardinality the continuum meeting each member of I in a set of cardinality less than the continuum. In MILLER (1979a) I show that assuming Martin's axiom a c-I_α-Luzin set has Borel order between α and $\alpha+2$.

Clearly under Martin's axiom c-Luzin sets and c-Sierpiński sets exist. In MARTIN–SOLOVAY (1970) it is shown that Martin's axiom is equivalent to saying that for any σ-ideal I generated by its Borel members and satisfying the countable chain condition, the real line cannot be covered by fewer than continuum many elements of I. Thus Martin's axiom is equivalent to saying for any ideal I as above there is a c-I-Luzin set (assuming c is regular).

What should be the complete negation of Martin's axiom? More specifically consider the following question.

Question. Is the failure of the continuum hypothesis consistent with either of the following?

(1) For any partial order $\mathbb{P}$ with the countable chain condition and cardinality $\leq c$ there exists $\langle G_\alpha: \alpha < \omega_1\rangle$ a sequence of $\mathbb{P}$-filters such that for any dense $D \subseteq \mathbb{P}$ all but countably many G_α meet D.

(2) For any non-trivial ideal I in the Borel sets with the countable chain condition there is an I-Luzin set of cardinality ω_1.

This question was motivated by the definable forcing axiom (DFA) of VAN DOUWEN–FLEISSNER (19··). The models of BELL–KUNEN (1981) and STEPRĀNS (1982) in which the continuum is $\aleph_{\omega_1}$ may be relevant. Recently BAUMGARTNER (19··) has shown that the following sentence is true in the side-by-side Sacks model. For every partial order $\mathbb{P}$ with the countable chain condition there are ω_1 dense sets in $\mathbb{P}$ such that no filter in $\mathbb{P}$ meets them all.

3. Concentrated sets and sets of strong measure zero

A set of reals X has strong measure zero iff given any sequence $\varepsilon_n > 0$ for $n < \omega$, X can be covered by a sequence of sets X_n each having diameter less than ε_n (BOREL (1919)). A set of reals X is concentrated on a set D iff for any open set G if $D \subseteq G$, then $X - G$ is countable (BESICOVITCH (1934)). BOREL (1919) conjectured that every strong measure zero set is countable. This conjecture is now known to be independent.

3.1. THEOREM. (i) (SZPILRAJN (MARCZEWSKI) (1938b)). *A set of reals X is a Luzin set iff X is uncountable and concentrated on every countable dense set of reals.*

(ii) (SIERPIŃSKI (1928)). *A set of reals X concentrated on a countable set has strong measure zero.*

PROOF. Part (i) is easy to prove since X is a Luzin set iff for every dense open G, $X - G$ is countable.

Part (ii) is proved as follows. Suppose X is concentrated on the set $D = \{d_N: n < \omega\}$. Given any sequence $\varepsilon_n > 0$ for $n < \omega$ let I_n be an open interval about d_n of diameter less than ε_{2n}. Let:

$$G = \bigcup\{I_n: n < \omega\}.$$

We have that $X - G$ is countable. Use the ε_{2n+1} for $n < \omega$ to cover $X - G$. □

3.2. THEOREM (LAVER (1976)). *It is consistent that every strong measure zero set is countable.*

Thus special set theoretic axioms or models of set theory must be used to construct uncountable strong measure zero sets, concentrated sets, etc. Assuming Martin's axiom c-Luzin sets exist and it is not hard to see that c-Luzin implies c-concentrated implies strong measure zero. The axiom $V = L$ implies the continuum hypothesis, thus Luzin and Sierpiński sets exist. In fact, since there is a Δ^1_2-well-ordering of the reals in L it is easy to show that there are Δ^1_2 Luzin sets and Δ^1_2 Sierpiński sets. See MOSCHOVAKIS (1980) Chapter 5 for descriptive set theory in L. Since a Sierpiński set is not measurable and a Luzin set does not have the property of Baire, neither of these sets can be $\boldsymbol{\Sigma}^1_1$ or $\boldsymbol{\Pi}^1_1$. An uncountable concentrated set cannot be $\boldsymbol{\Sigma}^1_1$ since it cannot contain a perfect subset. However the following is true.

3.3. THEOREM (ERDÖS, KUNEN, MAULDIN (1981)). *If $V = L$, then there is an uncountable Π^1_1 set which is concentrated on the rationals.*

Sierpiński asked whether every strong measure zero set is concentrated on some countable set. This was answered by BESICOVITCH (1942).

3.4. THEOREM (BESICOVITCH (1942)). *Assuming the continuum hypothesis there is a set of reals which has strong measure zero but is not concentrated on any countable set.*

PROOF. Construct (as in the construction of a Luzin set) a sequence P_α for $\alpha < \omega_1$ of disjoint nowhere dense perfect sets (perfect = closed, nonempty, and no isolated points) with the property that for any meager set M, M meets at most countably many of the P_α. For each $\alpha < \omega_1$ let $E_\alpha \subseteq P_\alpha$ be a (relativized to P_α) Luzin set. Then:

$$E = \bigcup\{E_\alpha: \alpha < \omega_1\}$$

has strong measure zero, but is not concentrated on any countable set. For any sequence $\varepsilon_n > 0$ for $n < \omega$ let G be an open dense set which is the union of intervals I_n each of diameter less than ε_{2n}. Then there is an $\alpha < \omega_1$ such that:

$$\bigcup\{E_\beta : \alpha < \beta < \omega_1\} \subseteq \bigcup\{P_\beta : \alpha < \beta < \omega_1\} \subseteq G .$$

But $\bigcup\{E_\beta : \beta \leq \alpha\}$ being the countable union of strong measure zero sets can be covered by a union of intervals J_n each of diameter less than ε_{2n+1}. E is not concentrated anywhere since for any countable D there is an α such that $P_\alpha \cap D = \emptyset$, and so the complement of P_α is an open set containing D but disjoint from the uncountable set E_α. □

GARDNER (1979) generalized Besicovitch's argument and studies a hierarchy of sets which are not concentrated on any countable set, yet which do have strong measure zero.

An interesting characterization of strong measure zero sets was found by GALVIN, MYCIELSKI, and SOLOVAY (1979).

3.5. THEOREM (GALVIN, MYCIELSKI, and SOLOVAY (1979)). *A set X has strong measure zero iff for any meager set G there exists a real x such that $(x + X) \cap g = \emptyset$.*

The implication from right to left is trivial. Given any sequence of $\varepsilon_n > 0$ let Q be an open dense set which is a union of intervals I_n of length ε_n. By assumption there exists x such that $x + X \subseteq Q$ and therefore $X \subseteq \bigcup_{n<\omega}(I_n - x)$. The implication from left to right is harder. For simplicity let's assume that $X \subseteq [0, 1]$. We need the following lemma.

3.5.1. LEMMA. *For any closed nowhere dense C and closed interval J there exists an $\varepsilon > 0$ and a finite family F of closed subintervals of J such that for any interval $I \subseteq [0, 1]$ of length ε there exists $J' \in F$ such that $(J' + I) \cap C = \emptyset$.*

PROOF. Since C is nowhere dense, for any $x \in [0, 1]$ there exists I_x with $x \in I_x$ and $J_x \subseteq J$ such that

$$(J_x + I_x) \cap C = \phi .$$

By compactness finitely many I_x cover $[0, 1]$. Thus for some sufficiently small $\varepsilon > 0$ every interval I of length ε is contained in some I_x (i.e. ε smaller than the length of overlap of any two I_x in the finite cover will do). □

Now we prove Theorem 3.5.

Suppose $G = \bigcup_{n<\omega} C_n$ where each C_n is closed nowhere dense and $C_n \subseteq C_{n+1}$. Using the Lemma construct a finitely branching tree $T \subseteq \omega^{<\omega}$ (ordered by inclusion) along with J_s and ε_s for each $s \in T$ satisfying:

(1) $J_{s\hat{}n}$ is a closed subinterval of J_s;

(2) $\varepsilon_s > 0$; and

(3) for any $s \in \omega^n \cap T$ if $I \subseteq [0,1]$ has length less than ε_s, then for some $s\hat{}m \in T$

$$(J_{s\hat{}m} + I) \cap C_n = \emptyset.$$

For $n < \omega$ define $\delta_n = \min\{\varepsilon_s : s \in \omega^n \cap T\}$. Since T is finitely branching $\delta_n > 0$. Since X has strong measure zero there exists I_n of length less than δ_n such that

$$X \subseteq \bigcap_{m<\omega} \bigcup_{n>m} I_n$$

By the construction of T there exists $f \in \omega^\omega$ such that for all n

$$(J_{f \restriction (n+1)} + I_n) \cap C_n = \emptyset\,.$$

Letting $x \in \bigcap_{n<\omega} J_{f \restriction n}$ we get that

$$(x + \bigcap_{m<\omega} \bigcup_{n>m} I_n) \cap \bigcup_{n<\omega} C_n = \emptyset\,. \quad \square$$

This result suggests we define X to have strong first category or to be strongly meager iff for every set H of measure zero there exists a real x such that

$$(x + X) \cap H = \emptyset\,.$$

The consistency of the dual Borel conjecture has been shown by Carlson

3.6. THEOREM (CARLSON (19· ·a)). *It is consistent that every strong first category set is countable.*

In fact, he shows this holds in the Cohen real model. Although it is not difficult to show that some uncountable Sierpiński sets have strong first category, I don't know if all do.

Question (Galvin). Does every Sierpiński set have strong first category?

4. σ-sets and Q-sets

A set of reals X is a σ-set iff every subset of X which is a relative G_δ is also a relative F_σ.

4.1. THEOREM (SZPILRAJN (MARCZEWSKI) (1930)). *If X is a Sierpiński set, then X is a σ-set.*

PROOF. Suppose A is any Borel (indeed measurable) set. Then there is an F_σ set $F \subseteq A$ such that $A - F$ has measure zero (see ROYDEN (1971), Chapter 3. Section 3, Proposition 15). Since X is a Sierpiński set: $(A - F) \cap X$ is countable, and therefore an F_σ set F_0. Thus we have:

$$A \cap X = (F \cup F_0) \cap X$$

and so A is a relative F_σ. □

A natural generalization of a σ-set is the notion of Q-set. A set of reals X is a Q-set iff every subset of X is a relative F_σ. Q-sets are also studied because of their connection with the normal Moore space problem (see Fleissner's chapter and also FLEISSNER (1978) and PRZYMUSIŃSKI (1977)).

4.2. THEOREM (SILVER, see MARTIN–SOLOVAY (1970) or RUDIN (1977)). *Assuming Martin's axiom every set of reals of cardinality less than the continuum is a Q-set.*

4.3. THEOREM (MILLER (1979b), Theorem 22). *It is consistent that there are no uncountable σ-sets, in fact, it is consistent that every uncountable set of reals has Baire order ω_1.*

Under MA + ¬CH + $\omega_1^V = \omega_1^L$ there are Π_1^1 sets of cardinality ω_1, therefore it is consistent to have uncountable Π_1^1 sets which are Q-sets. Also, it is easy to show that it is consistent with the continuum hypothesis that there are uncountable Π_1^1 sets which are σ-sets.

While a σ-set need not have measure zero (e.g. a Sierpiński set is not measurable), Q-sets must have measure zero.

4.4. THEOREM (LUZIN, see FLEISSNER (1978)). *Every Q-set has universal measure zero.*

However σ-sets are meager (see the next section).

Question (FLEISSNER (1978)). Do all Q-sets have strong measure zero?

For any countable ordinals α, we say that a set of reals X has Baire order $\leq\alpha$ iff for every Borel set C there is a Borel set B of rank $\leq\alpha$ such that $X \cap C = X \cap B$. If there is no such countable α, then X has Baire order ω_1. POPROUGÉNKO (1930) showed that the Baire order of a Luzin set is 3. A Q-set is a Q_2-set. A Q_α-set is an uncountable set of reals of Baire order α with the property that every subset is a relative Borel set.

4.5. THEOREM (MILLER (1979a)). (i) *It is consistent that for every* α, $2 \leq \alpha < \omega_1$, *there is an uncountable* Q_α*-set.*

(ii) *If* X *is a set of reals such that every subset of* X *is relatively Borel, then* X *is a* Q_α*-set for some* $\alpha < \omega_1$.

In HANSELL (1980) a Q_A set is defined to be any set of reals such that every subset is (relatively) analytic. He, in fact, is concerned there with arbitrary topological spaces. For further work in this direction see BALOGH–JUNNILA (19··) and FREMLIN–HANSELL–JUNNILA (19··).

In MILLER (1981a) I show that it is consistent to have a Q_A set which has Baire order ω_1. Theorem 4.3 of MILLER (1979b) shows that it is consistent to have a set of reals X of Baire order ω_1 such that every relatively analytic set is relatively Borel. One natural question here is:

Question (Mauldin). Is it consistent to have a set of reals X of bounded Baire order, but not every relatively analytic set is relatively Borel?

In BROWN (1977) it is shown that the continuum hypothesis implies that there is a set of reals concentreated on the rationals with Baire order ≥ 4. In FLEISSNER–MILLER (1980) we show that it is consistent to have an uncountable Q-set which is concentrated on the rationals. Using a similar argument we can show for any $\alpha \leq \omega_1$, that it is consistent with the continuum hypothesis to have a set of reals of Baire order α which is concentrated on the rationals. In fact, the continuum hypothesis implies there is an uncountable σ-set concentrated on a countable set. (See Theorem 5.7.)

5. Universal measure zero sets, perfectly meager sets, λ, λ' and s_0-sets, and Hausdorff gaps

In this section we consider the weakest properties of Luzin and Sierpiński sets. A set of reals X has universal measure zero iff for all measures μ on the Borel sets there is a Borel set of μ-measure zero covering X. By measure we always mean a countably additive, atomless (i.e. points have measure zero), finite measure. Alternatively let $\mathscr{B}$ be the σ-algebra of (relative) Borel subsets of a set X. Then X has universal measure zero iff for any measure μ on $\mathscr{B}$, $\mu(X) = 0$. The following result is classical.

5.1. THEOREM (SZPILRAJN (MARCZEWSKI) (1934)). *Every strong measure zero set has universal measure zero.*

PROOF. Let μ be any measure on the Borel sets. For any $\varepsilon > 0$ there is a $\delta > 0$ such that if I is any closed subinterval of $[0, 1]$ of diameter less than δ, then $\mu(I) < \varepsilon$. To prove this note that every point has neighborhoods of arbitrarily small μ-measure, since the point has measure zero. So for each $x \in [0, 1]$ let

$x \in I_x$, $\mu(I_x) < \varepsilon$ and let I_x^* be the middle one third of I_x. By compactness there is a finite $F \subseteq [0, 1]$ such that $\{I_x^* : x \in F\}$ covers $[0, 1]$. Let δ be the minimum diameter of I_x^* for $x \in F$. If J is any interval of diameter less than δ there is an I_x for some $x \in F$ with $J \subseteq I_x$ and so $\mu(J) < \varepsilon$.

From this it is easy to see that any strong measure zero set has μ-measure zero. □

Somewhat analogously (to the notion of universal measure zero set) we say that a set of reals X is perfectly meager iff for all perfect sets P the set $X \cap P$ is meager relative to the topology of P. This is also sometimes called 'always of first category' (LUZIN (1933)). A slight weakening of the notion of σ-set is that of a λ-set (or rarefied set, introduced by KURATOWSKI (1933)). A set X is a λ-set iff every countable subset of X is a relative G_δ in X. λ-sets were used by F.B. JONES (1937) to construct a pseudonormal Moore space. This served as the inspiration for nonmetrizable normal Moore spaces based on Q-sets (see also MCAULEY (1956)).

5.2. THEOREM. *If X is a λ-set, then X is perfectly meager.*

PROOF. Suppose P is any perfect set and let $F \subseteq P \cap X$ be a countable set dense in $P \cap X$. Since X is a λ-set it is easy to get a G_δ set G comeager in P with $G \cap X = F$. And hence Y is meager in P. □

The existence of uncountable sets of universal measure zero and uncountable perfectly meager sets does not require any axioms beyond the usual Zermelo–Fraenkel with the axiom of choice.

5.3. THEOREM. *There exists a set of reals X of cardinality ω_1 which has universal measure zero and is perfectly meager.*

PROOF. Select one element from each of the constituents of a nontrivial coanalytic set, then this set will have universal measure zero and will be perfectly meager. More explicitly, let WO be the set of elements of $2^{\omega \times \omega}$ which are the characteristic functions of well-orderings of ω. Then WO is coanalytic (Π_1^1) (see MOSCHOVAKIS (1980), p. 192) and hence universally measurable and has the property of Baire everywhere. For each countable ordinal α choose $x_\alpha \in \mathrm{WO}$ of order type α. Then $X = \{x_\alpha : \alpha < \omega_1\}$ is perfectly meager and has universal measure zero. To see this suppose μ is any Borel measure. Since WO is μ-measurable there exists a Borel set B and a μ-measure zero set M such that $\mathrm{WO} = B \cap M$. But by the boundedness theorem (see MOSCHOVAKIS (1980) p. 196) there is an $\alpha < \omega_1$ such that:

$$B \subseteq \{x \in \mathrm{WO} : \text{order type of } x \text{ is less than } \alpha\}.$$

Thus X has μ-measure zero.

A similar argument shows X is perfectly meager. □

LUZIN (1921) was the first to construct an uncountable perfectly meager set. The idea of using analytic sets seems to be a joint result of Sierpiński and Luzin (see SIERPIŃSKI (1934)). HAUSDORFF (1934) also gave a proof of this theorem using his famous (ω_1, ω_1^*)-gap (see Theorem 5.5). An interesting proof due to Todorčević uses an Aronszajn tree of perfect sets (see Todorčević's Chapter).

One cannot in general do better than ω_1 since it is a theorem of Baumgartner and Laver that in the random real model every universal measure zero set has cardinality less than or equal to ω_1 (see MILLER (19··a) for a proof). The continuum can be made as large as desired in this model. Similarly in the iterated perfect set model (see BAUMGARTNER and LAVER (1979)), every perfectly meager set has cardinality less than or equal to ω_1. In this model $c = 2^{\omega_1}$ so there are only continuum many universal measure zero sets and continuum many perfectly meager sets.

Question (Mauldin). Are there always more than c absolutely measurable sets and more than c sets with the restricted Baire property?

See GRZEGOREK and RYLL–NARDZEWSKI (19··) and FENSTAD and NORMANN [1974] for some related results.

The following theorem of Grzegorek was known assuming the continuum hypothesis or Martin's axiom. The point here is that he uses nothing beyond the usual axioms of set theory (ZFC).

5.4. THEOREM (GRZEGOREK 1980) (1981) (19··). (i) *If κ is the cardinality of the smallest nonmeasurable set of reals, then there is a universal measure zero set of cardinality κ.*

(ii) *If κ is the cardinality of the smallest nonmeager set, then there is a perfectly meager set of cardinality κ.*

(iii) *There is a set of reals X which has universal measure zero but does not have strong measure zero.*

We say that X is a λ'-set iff for every countable set of reals F, contained in X or not, F is a relative G_δ in $X \cup F$. It is easy to show that X is a λ'-set iff for every countable set F, $X \cup F$ is a λ-set (i.e. if $H \subseteq X \cup F$ is countable and G is a G_δ set such that

$$G \cap (X \cup H) = H,$$

then

$$G - (F - H) \cap (X \cup F) = H.)$$

Hausdorff actually proved that there is a set of reals of cardinality ω_1 which has universal measure zero and is a λ'-set (hence perfectly meager). The argument for λ'-set was pointed out by SIERPIŃSKI (1945). Define for X and Y sets of natural numbers, $X \subseteq^* Y$ iff $Y - X$ is finite.

5.5. THEOREM (HAUSDORFF (1934)). *There exist* $\langle X_\alpha : \alpha < \omega_1 \rangle$ *and* $\langle Y_\alpha : \alpha < \omega_1 \rangle$ *such that*:

(i) *for* $\alpha < \beta < \omega_1$, $X_\alpha \subseteq^* X_\beta \subseteq^* Y_\beta \subseteq^* Y_\alpha$;

(ii) *there does not exist* Z *with* $X_\alpha \subseteq^* Z \subseteq^* Y_\alpha$ *for all* $\alpha < \omega_1$.

The proof is also given in LAVER (1976). Laver also gives the proof that the Hausdorff gap has universal measure zero. Let us now see that the gap is a λ'-set. Recall that we identify subsets of ω with their characteristic functions. Let:

$$X = \{X_\alpha : \alpha < \omega_1\} \cup \{Y_\alpha : \alpha < \omega_1\}$$

be the Hausdorff gap from Theorem 5.5. Since the Cantor set is a closed subset of the real line we only have to worry about countable F contained in 2^ω. Define for each $\alpha < \omega_1$:

$$F_\alpha = \{A \subseteq \omega : X_\alpha \subseteq^* A \subseteq^* Y_\alpha\}.$$

Note that for any $B \subseteq \omega$, $\{A \subseteq \omega : A \subseteq B\}$ and $\{A \subseteq \omega : B \subseteq A\}$ are closed sets. Hence each F_α is an F_σ-set.

Let G_α be the complement in 2^ω of F_α. The G_α are strictly increasing G_δ sets whose union is all of 2^ω and for any $\alpha < \omega_1$, $X \cap G_\alpha$ is countable. For any countable $F \subseteq 2^\omega$ there is an $\alpha < \omega_1$ with $F \subseteq G_\alpha$. Now:

$$K = (X \cup F) \cap G_\alpha$$

is countable and hence $K - F$ is F_σ and

$$F = (X \cup F) \cap (G_\alpha - (K - F)).$$

Since F was arbitrary we see that X is a λ'-set. For some other uses of Hausdorff gaps see VAN DOUWEN (1976) and NYIKOS and VAUGHAN (19··).

Rothberger showed that not every λ-set is a λ'-set. The following is a key observation: From the introduction we know that every closed subset of the unit interval disjoint from the rationals corresponds to a compact subset of ω^ω. It is easy to show that for every compact subset C of ω^ω there is an $f \in \omega^\omega$ such that:

$$C \subseteq \{g \in \omega^\omega \mid \text{for all } n < \omega,\ g(n) < f(n)\}.$$

Define for $f, g \in \omega^\omega$, $f <^* g$ iff for all but finitely many $n < \omega$, $f(n) < g(n)$. Note that for any countable set $F \subseteq \omega^\omega$ there exists $f \in \omega^\omega$ such that for all $g \in F$, $g <^* f$. For any $f \in \omega^\omega$ let

$$C_f = \{g \in \omega^\omega \mid g <^* f\}.$$

Then C_f is an F_σ (in fact the countable union of compact sets). And for any F_σ set F disjoint from the rationals we have that for some $f \in \omega^\omega$, $F \subseteq C_f$.

5.6. (THEOREM) (ROTHBERGER (1939)). (i) *Not every λ-set is a λ'-set.*

(ii) *Assuming the continuum hypothesis there is an uncountable λ-set concentrated on the rationals.*

PROOF. Assuming the continuum hypothesis we can find a set:

$$X = \{f_\alpha : \alpha < \omega_1\} \subseteq \omega^\omega$$

such that

(a) $\alpha < \beta$ implies $f_\alpha <^* f_\beta$;

(b) for all $f \in \omega^\omega$ there exists $\alpha < \omega_1$ with $f <^* f_\alpha$.

To see that X is a λ-set consider the G_δ sets, $G_\alpha = \{f \in \omega^\omega$: for infinitely many n, $f(n) < f_\alpha(n)\}$ (just use (a)). However by (b) and earlier remarks we see that for any G_δ set G containing the rationals, $X - G$ is countable. So X is concentrated on the rationals and thus not a λ'-set. This proves (ii).

Part (i) is proved using similar ideas, but no hypothesis beyond ZFC is used. Rothberger shows that the least cardinality of an unbounded subset of ω^ω is also the cardinality of a λ-set which is not a λ'-set. See VAN DOUWEN'S Chapter (10.2) for some similar arguments. □

It is worth noting that the set X we have constructed while being concentrated on the rationals is a λ'-set with respect to the irrationals. The construction of an order type ω_1 subset of $(\omega^\omega, <^*)$ is in fact LUZIN'S (1921) original construction of an uncountable perfectly meager set.

By a slight modification of Rothberger's argument we can show:

5.7. THEOREM. *Assuming the continuum hypothesis, there exists an uncountable σ-set which is concentrated on a countable set.*

PROOF. Construct $X_\alpha \subseteq \omega$ infinite for $\alpha < \omega_1$ such that $\alpha < \beta$ implies $X_\alpha \supseteq^* X_\beta$. By an argument similar to the one used in Theorem 5.6 if the X_α grow fast enough, then

$$X = \{X_\alpha : \alpha < \omega_1\}$$

will be concentrated on the set

$$[\omega]^{<\omega} = \{A : A \subseteq \omega \text{ is finite}\}.$$

By fast enough we mean, look at the natural map showing ω^ω and $[\omega]^\omega$ are homeomorphic (see introduction), then the image of X should satisfy (a) and (b) in the proof of Theorem 5.6. For any set A, $[A]^\omega$ is the set of (countably) infinite subsets of A. The GALVIN–PRIKRY Theorem (1973) says that for every Borel set $B \subseteq [\omega]^\omega$ and $X \in [\omega]^\omega$ there exists $Y \in [X]^\omega$ such that

$$[Y]^\omega \subseteq B \quad \text{or} \quad [Y]^\omega \cap B = \emptyset .$$

(This result has been extended from BOREL sets to $\boldsymbol{\Sigma}^1_1$ (analytic) sets by SILVER (1970) see also ELLENTUCK (1974)). The following lemma easily gives us Theorem 5.7.

5.7.1. LEMMA. *For any Borel set $B \subseteq [\omega]^\omega$ and $X \in [\omega]^\omega$ there exists $Y \in [X]^\omega$ such that B is a relative F_σ set in $\{Z \in [\omega]^\omega : Z \subseteq^* Y\}$.*

PROOF. Construct a sequence $Y_{n+1} \in [Y_n]^\omega$ with $Y_0 = X$ as follows. Let a_n be the least element of Y_n. Repeatedly apply the Galvin–Prikry Theorem to obtain $H_n \subseteq \{A: A \subseteq a_n\}$ and X_{n+1} such that for all $Z \in [\omega]^\omega$ if $Z - a_n \subseteq X_{n+1}$, then $Z \in B$ iff $Z \cap a_n \in H_n$. Now let $Y = \{a_n : n < \omega\}$. Then for any $Z \subseteq^* Y$, $Z \in B$ iff there exists n such that $Z \cap a_n \in H_n$ and $Z - a_n \subseteq Y$. □

Since we are assuming the continuum hypothesis the theorem follows easily from the lemma. □

Next we consider the Baire order of λ-sets.

5.8. THEOREM (MAULDIN (1977)). *Assuming the continuum hypothesis there is a λ-set of Baire order ω_1.*

The proof uses the σ-algebra of abstract rectangles in the plane and also a key lemma on universal sets proved in BING, BLEDSOE, and MAULDIN (1974).

Note that if X is a λ-set in a model M of set theory and N is a countable chain condition forcing extension of M, then X remains a λ-set in N. Thus as a corollary to Theorem 3.5 (MILLER [1979a]) we see that it is consistent to have λ-sets of all possible Baire orders.

I conclude this section with a notion which is weaker than both universal measure zero and perfectly meager. We say that X is an s_0-set iff for every perfect set P there is a perfect set $Q \subseteq P$ such that Q is disjoint from X.

5.9. THEOREM. (SZPILRAJN (MARCZEWSKI) (1935a)). *If X has universal measure zero or X is perfectly meager, then X is an s_0-set.*

PROOF. Suppose x has universal measure zero. If P is any perfect set, then transfer the product measure on 2^ω via any homeomorphism with P to P and call it μ. Then there is a Borel set $B \subseteq P$ of ω measure zero such that $X \cap P \subseteq B$. But since $P - B$ is an uncountable Borel set it contains a perfect subset. A similarly argument works if X is perfectly meager since any comeager set contains an uncountable Borel set. □

In contrast to the case of perfectly meager and universal measure zero there are always large s_0-sets.

5.10. THEOREM. *There exists a s_0-set of cardinality the continuum.*

PROOF. Let P_α for $\alpha < c$ be all perfect subsets of the plane. For any set P in the plane and real x let:

$$P^x = \{y : \langle x, y \rangle \in P\}.$$

We construct an (s_0)-set x as follows. Let $\{x_\alpha : \alpha < c\}$ be the set of all real numbers. For each α choose y_α not an element of:

$$\bigcup\{P_\beta^{x_\alpha} : \beta < \alpha \text{ and } P_\beta^{x_\alpha} \text{ is countable}\}.$$

Now let $X = \{(x_\alpha, y_\alpha) : \alpha < c\}$. Suppose P is any perfect subset of the plane. If for some x_α, P^{x_α} is uncountable, then since $\{x_\alpha\} \times P^{x_\alpha} \cap X \subseteq \{(x_\alpha, y_\alpha)\}$ it is easy to get a perfect $Q \subseteq \{x_\alpha\} \times P^{x_\alpha}$ disjoint from X. On the other hand if P^{x_α} is countable for all $\alpha < c$, then by our construction $P \cap X$ has cardinality less than c. Partition P into continuum many disjoint perfect sets; then one of them misses X. □

For more on (s_0)-sets see MORGAN (1978) Example 3B.

6. Order type of the real line

Baumgartner generalized Cantor's theorem that any two countable dense linear orders are isomorphic. A set of reals X is ω_1-dense iff between any two reals there are ω_1 elements of X.

6.1. THEOREM (BAUMGARTNER (1973)). *It is consistent with Martin's axiom that any two ω_1 dense sets of reals are order isomorphic.*

His forcing argument was rather unusual in that it requires that the continuum hypothesis be true in intermediate models but in the final model it must fail. A similar argument occurs in BAUMGARTNER (1980).

Abraham and Shelah showed that Martin's axiom is not sufficient for this result.

6.2. THEOREM (ABRAHAM–SHELAH (1981)). *It is consistent with Martin's Axiom and the failure of the continuum hypothesis that not every two ω_1-dense sets are order isomorphic.*

Most of the classical work on order types contained in the real line is in SIERPIŃSKI (1950).

7. Unions

It is easy to show that the families of universal measure zero sets, perfectly meager sets, strong measure zero sets, Luzin sets, Sierpiński sets, and sets concentrated on the rationals are closed under countable union. Assuming Martin's Axiom in the case of universal measure zero and perfectly meager, countable can be replaced by less than continuum. This is also true for strong measure zero, but is not so obvious.

7.1. THEOREM (CARLSON 19··a)). *Assuming Martin's Axiom the union of less than continuum many strong measure zero sets has strong measure zero.*

We saw in Section 5 that there is a λ-set whose union with the rationals is not a λ-set. However, it is true that the family of λ' sets is closed under countable union.

7.2. THEOREM (SIERPIŃSKI (1937a)). *The countable union of λ' sets is a λ' set.*

PROOF. Suppose X_n for $n < \omega$ are λ'-sets and F is a countable set. Since X_n is a λ'-set there is a G_δ set G_n such that:

$$F = G_n \cap (X_n \cup F) .$$

But then

$$F = \bigcap_{n<\omega} G_n \cap \Big(\bigcup_{n<\omega} X_n \cup F \Big) . \quad \square$$

7.3. THEOREM (FLEISSNER–MILLER (1980)). *It is consistent that there is an uncountable Q-set which is concentrated on the rationals. So neither the family of Q-sets nor the family of σ-sets need be closed under finite union.*

To prove this theorem instead of constructing the set of reals we construct the model of set theory over the top of a set of Cohen reals.

8. Products

8.1. THEOREM (SZPILRAJN (MARCZEWSKI) (1937)). *The product of two universal measure zero sets has universal measure zero.*

PROOF. Suppose X and Y have universal measure zero and let μ be any measure on $X \times Y$. Define a measure ν on Y by:

$$\nu(B) = \mu(X \times B).$$

Since X has universal measure zero so does $X \times \{y\}$ for any $y \in Y$ and so ν is atomless. Since Y has universal measure zero

$$\mu(X \times Y) = \nu(Y) = 0. \quad \square$$

8.2. THEOREM. *If X and Y are λ-sets (λ'-sets), then $X \times Y$ is a λ-set (λ'-set).*

PROOF. Suppose F is a countable subset of $X \times Y$. Then let $F_x \subseteq X$ and $F_y \subseteq Y$ be countable with $F \subseteq F_x \times F_y$. Since $F_x \times F_y$ is a relative G_δ so is $F = F_x \times F_y - (F_x \times F_y - F)$. A similar argument works for λ'-sets. $\square$

This contrasts sharply with the following theorem.

8.3. THEOREM (FLEISSNER (19··)). *It is consistent that there is a Q-set whose square is not a Q-set.*

However, the following result is true.

8.4. THEOREM (PRZYMUSIŃSKI (19··)). *If there is an uncountable Q-set, then there is one whose square is also a Q-set.*

PROOF. Let $R = \{A \times B\colon A, B \subseteq \omega_1\}$. The next two lemmas are needed to prove Theorem 8.4.

8.4.1. LEMMA. *The graph of any function from ω_1 to ω_1 is a countable intersection of finite unions of elements of R.*

PROOF. Let $f\colon \omega_1 \to \omega_1$ be an arbitrary function. Let $\{x_\alpha\colon \alpha < \omega_1\}$ be any set of distinct real numbers. Let q be the set of rational numbers. Then $f(\alpha) = \beta$ iff

$$\forall r \in Q \quad (r < x_{f(\alpha)} \leftrightarrow r < x_\beta).$$

Hence the graph of f is

$$\bigcap_{r \in Q} ([\{\alpha\colon r < x_{f(\alpha)}\} \times \{\beta\colon r < x_\beta\}] \cup [\{\alpha\colon x_{f(\alpha)} \leq r\} \times \{\beta\colon x_\beta \leq r\}]). \quad \square$$

8.4.2. LEMMA. *There exists a Q-set of cardinality ω_1 iff there exists a countable family $\mathcal{A}$ of subsets of ω_1 such that every subset of ω_1 is a countable union of countable intersections of elements of $\mathcal{A}$.*

PROOF. Let $Y = \{x_\alpha : \alpha < \omega_1\}$ be a Q-set and $\{C_n : n < \omega\}$ a clopen basis for Y closed under finite union and complementation. Then $\{C_n^* : n < \omega\}$ defined by

$$C_n^* = \{\alpha : x_\alpha \in C_n\}$$

has the required property. □

Now we prove Theorem 8.4.

Choose $f_n : \omega_1 \in \omega_1$ so that for each $\alpha < \omega_1$

$$\{f_n(\alpha) : n < \omega\} = \{B : \beta \leqslant \alpha\} .$$

For each $n < \omega$ let

$$D_n = \{(\alpha, f_n(\alpha)) : \alpha < \omega_1\} \quad \text{and} \quad E_n = \{(f_n(\alpha), \alpha) : \alpha < \omega_1\} .$$

From the lemmas we can find $\{A_n : n < \omega\}$ a family of subsets of ω_1 such that for each $n < \omega$, D_n and E_n are the countable intersections of finite unions of $\{A_n \times A_m : n, m < \omega\}$ and every subset of ω_1 is the countable union of countable intersections of elements of $\{A_n : n < \omega\}$. Now let us see that every subset of $\omega_1 \times \omega_1$ is the countable union of countable intersections of finite unions of $\{A_n \times A_m : n, m < \omega\}$ (i.e. F_σ). Suppose $A \subseteq \omega_1 \times \omega_1$. For any n let

$$X_n = \{\alpha : \exists \beta \, (\alpha, \beta) \in D_n \cap A\} .$$

Then since

$$D_n \cap A = (X_n \times \omega_1) \cap D_n$$

we see that $D_n \cap A$ is the countable union of countable intersections of finite unions of elements of $\{A_n \times A_m : n, n < \omega\}$. Similarly for $E_n \cap A$. But by the choice of the f_n,

$$\omega_1 \times \omega_1 = \bigcup_{n<\omega} D_n \cup \bigcup_{n<\omega} E_n$$

and so

$$A = \bigcup_{n<\omega} (D_n \cap A) \cup \bigcup_{n<\omega} (E_n \cap A) .$$

The mapping $\sigma : \omega_1 \to 2^\omega$ defined by

$$\sigma(x)(n) = 0 \quad \text{iff} \quad x \in A_n$$

takes ω_1 onto a Q-set whose square is also a Q-set. □

8.5. THEOREM (SIERPIŃSKI (1935)). *Assuming the continuum hypothesis there exists a Luzin set whose square does not have strong measure zero.*

PROOF. We need the following lemma.

8.5.1. LEMMA. *For any real z and comeager set G there are points x and y in G with $z = x + y$.*

PROOF. Since G is comeager so is $z - G = \{z - g : g \in G\}$, let Y be an element of their intersection. Then $y = z - x$ for some $x \in G$. □

Using this lemma (and the continuum hypothesis) it is now easy to construct a Luzin set X such that for any real z there is an x and y in X such that $z = x + y$. I claim that X^2 is not of strong measure zero. To see this define π from the plane onto the reals by $\pi(x, y) = x + y$. Geometrically this is just a 45° projection onto the real axis. Thus a disk of diameter ε is taken to an interval of diameter $\sqrt{2} \cdot \varepsilon$. Hence the image under π of any strong measure zero set must have strong measure zero. But π takes X^2 onto the real line. □

Clearly the product of two uncountable sets of reals cannot be a Luzin set, a Sierpiński set, or a concentrated set, since horizontal lines are closed measure zero sets.

Question (SZPILRAJN (MARCZEWSKI) (1935b)). Is the product of two perfectly meager sets perfectly meager?

9. Continuous and homeomorphic images and C'' and C'-sets

It is not hard to see that every set homeomorphic to a perfectly meager set is perfectly meager. This is also true for universal measure sets and in fact it characterizes them.

9.1. THEOREM (SZPILRAJN–MARCZEWSKI (1937)). *A set of reals X has universal measure zero iff every set homeomorphic to X has Lebesgue measure zero.*

PROOF. Suppose X has universal measure zero and $\phi: X \to Y$ is a homeomorphism. For any μ a measure on Y define ν a measure on X by letting $\nu(B) = \mu(\phi(B))$. If ν vanishes so does μ. It follows that Y has Lebesgue measure zero. Now to prove the converse suppose X is any set of reals (which for simplicity we will assume is in the unit interval) and suppose every homeomorphic image of X has Lebesgue measure zero. Suppose μ is any measure on the reals such that X does not have μ-measure zero. We may assume that μ does not vanish on any interval since we could replace μ by $\frac{1}{2}(\mu + \lambda)$ where λ is Lebesgue measure. Now define f by

$$f(x) = \mu([0, x]) .$$

Since μ vanishes on no intervals and is atomless we see that f is strictly increasing and continuous and thus a homeomorphism. Define

$$\nu(B) = \mu(f^{-1}(B))$$

for any Borel set B. We are done if we show that ν is Lebesgue measure. Suppose $I = [a, b]$ is any interval. Then

$$f^{-1}(I) = \{y: \mu([0, y]) \in I\} = [c, d]$$

where $\mu([0, c]) = a$ and $\mu([0, d]) = b$. But then

$$\nu(I) = \mu(f^{-1}(I)) = \mu([c, d]) = b - a .$$

Since ν agrees with Lebesgue measure on the intervals and the intervals generate the Borel sets we have that ν is Lebesgue measure. □

This characterization is not true for perfectly meager sets.

9.2. THEOREM (MORGAN (1979)). *There is a set of reals which is not perfectly meager but every set homeomorphic to it is meager.*

PROOF. Let K be the Cantor set and Q the rationals. Let $A \subseteq K$ be a set such that A and $K - A$ meet every perfect subset of K. Let $S = Q \cup A$. S is not perfectly meager since it cannot be meager relative to K. On the other hand S has the property that every nonempty open set U contains a nonempty open set V such that V is countable. But this is true of any homeomorphic image of S and is easily seen to imply first categoricity. □

Clearly any property which is purely topological is preserved by homeomorphisms. The homeomorphic image of a Luzin set is a ν-set—i.e. a set in which every (relatively) meager set is countable. The homeomorphic image of a Sierpiński set is still a Sierpiński set with respect to a different Borel measure. Since λ-sets, σ-sets, and Q-sets are all defined topologically these properties are all preserved by homeomorphisms (but not necessarily one-to-one continuous mappings). Before taking up the homeomorphism problem for λ'-sets, concentrated sets, and strong measure zero sets, we consider one-to-one continuous images.

9.3. THEOREM (KURATOWSKI (1933), SZPILRAJN (MARCZEWSKI) (1937)). *Every set of reals of cardinality ω_1 is the one-to-one continuous image of a set which is both perfectly meager (in fact, a λ'-set) and of universal measure zero. Thus assuming the continuum hypothesis the real line is the continuous image of a set which is both perfectly meager and of universal measure zero.*

PROOF. Let $X = \{x_\alpha : \alpha < \omega_1\}$ be an arbitrary set of reals, $Y = \{y_\alpha : \alpha < \omega_1\}$ be any λ'-set, and let $Z = \{z_\alpha : \alpha < \omega_1\}$ be any set of universal measure zero. Define

$$Q = \{(x_\alpha, y_\alpha, z_\alpha) : \alpha < \omega_1\} .$$

X is the projection of Q onto the first coordinate. We need the following lemma to prove Theorem 9.3.

9.3.1. LEMMA. *Suppose P and T are sets of reals, $f: P \to T$ is a one-to-one continuous map, and T is a* (a) *set of universal measure zero*; (b) *λ-set*; (c) *λ'-set*; *or* (d) *Q-set. Then P is also.*

PROOF. For universal measure zero note that if μ is any measure on P, then

$$\nu(B) = \mu(f^{-1}(B))$$

defines a measure on T. For a λ-set suppose $D \subseteq P$ is countable. Then $f(D)$ is a countable subset of R and so there is a G_δ set G with $G \cap R = f(D)$. Since f is one-to-one $f^{-1}(G) \cap P = D$. This same argument works for Q-sets. For λ'-sets, the following is what we mean. Suppose $f: X \to Y$ is continuous, with $P \subseteq X$ and $T \subseteq Y$, and f takes P one-to-one onto T. If T is a λ'-set with respect to Y, then P is a λ'-set with respect to X. To show that P is a λ'-set with respect to X we must show that for every countable $D \subseteq X$ there is a G_δ set G such that

$$G \cap (P \cup D) = D .$$

Suppose $D \subseteq X$ is countable and let G be a G_δ set in Y such that

$$G \cap (T \cup f(D)) = f(D) .$$

But then $f(D) \subseteq G$ implies $D \subseteq f^{-1}(G)$ and $f^{-1}(G) \cap P = \emptyset$ (since $f^{-1}(G) \cap P \subseteq f^{-1}(G) \cap f^{-1}(T) = f^{-1}(G \cap T) = \emptyset$. Thus

$$f^{-1}(G) \cap (P \cup D) = D .$$

This proves the lemma and the theorem immediately follows. □

This same quasi-diagonal type argument as in 9.3 shows that if we have a Q set of cardinality κ, then every set of reals of cardinality κ is the continuous image of a Q-set. Thus by FLEISSNER–MILLER (1980) it is consistent to have a Q-set whose continuous image is not a Q-set.

Lemma 9.3.1 is not true for perfectly meager sets since we will see (Theorem 9.7) a Luzin set can be mapped one-to-one to a perfectly meager set.

This lemma is also false for σ-sets, i.e. assuming X is any σ-set of cardinality the continuum there exists Y a non σ-set which can be continuously mapped one-to-one onto X. Suppose $X = \{x_\alpha : \alpha < c\}$. To construct Y, let H be any countable dense subset of 2^ω (i.e. any F_σ which is not G_δ) and let $\{G_\alpha : \alpha < c\}$ be all G_δ subsets of $X \times 2^\omega$. For each α choose y_α such that

$$y_\alpha \in (G_\alpha^{x_\alpha} - H) \cup (H - G_\alpha^{x_\alpha}) \quad \text{where } G_\alpha^{x_\alpha} = \{y : (x_\alpha, y) \in G_\alpha\}.$$

This is always possible since H cannot be G_δ in 2^ω. Now let $Y = \{(x_\alpha, y_\alpha) : \alpha < c\}$. The projection map takes Y onto X. The set Y is not a σ-set since $(X \times H) \cap Y$ is not G_δ in Y.

Question. Is it consistent to have a σ-set X which can be mapped continuously onto the reals?

9.4. THEOREM (ROTHBERGER (1941)). *Assuming the continuum hypothesis there exists a set concentrated on the rationals which can be mapped continuously onto 2^ω.*

PROOF. Let $X = \{f_\alpha : \alpha < \omega_1\} \subseteq \omega^\omega$ be a set of order type ω_1 under $<^*$ and for every $g \in \omega^\omega$ there exists $\alpha < \omega_1$ with $g <^* f_\alpha$. We saw in Theorem 5.6(ii) that any such set is concentrated on the rationals. Let $\{x_\alpha : \alpha < \omega_1\} = 2^\omega$. Define $f_\alpha^* \in \omega^\omega$ by

$$f_\alpha^*(n) = 2 f_\alpha(n) + x_\alpha(n)$$

for all $n < \omega$. Let $X^* = \{f_\alpha^* : \alpha < \omega_1\}$ and define $\pi : \omega^\omega \to 2^\omega$ by $\pi(f) = x$ iff for all $n < \omega$ $\pi(x)(n) = f(n) \pmod 2$. Then π maps X^* onto 2^ω. $\square$

Since a concentrated set has strong measure zero it follows that the continuous image of a strong measure zero set need not have strong measure zero.

9.5. THEOREM (SIERPIŃSKI (1945)). *Assuming the continuum hypothesis there is a concentrated set X which is homeomorphic to a set Y which is a λ'-set without strong measure zero.*

PROOF. Using Theorem 9.4 find $X \subseteq 2^\omega$ a concentrated set and a continuous $f : X \to S$ which is one-to-one and onto an uncountable Sierpiński set $S \subseteq 2^\omega$. Let $G = \{(x, f(x)) : x \in X\} \subseteq 2^\omega \times 2^\omega$. G is homeomorphic to X via the map $x \mapsto$

$(x, f(x))$. Let $\pi: 2^\omega \times 2^\omega \to 2^\omega$ be projection onto the 2nd coordinate. Since π is continuous and one-to-one on G it follows from Lemma 9.3.1 that G is a λ'-set (since S is). On the other hand since π is uniformly continuous if G had strong measure zero, then so would $\pi(G) = S$. But S does not even have measure zero. Since the map which shows $2^\omega \times 2^\omega$ is homeomorphic to 2^ω is uniformly continuous we get a subset Y of 2^ω which is homeomorphic to G, is a λ'-set and does not have strong measure zero. □

This shows that the property of being strong measure zero is not topological but depends on the metric. In light of this it is perhaps surprising that the following is true.

9.6. THEOREM (CARLSON (19· ·b)). *If every strong measure zero set of reals is countable, then for every metric space X if X has strong measure zero, then X is countable.*

Sierpiński's question of whether or not the family of strong measure zero sets was closed under continuous image lead ROTHBERGER (1938) to consider two other classes of sets. Strong measure zero sets were also called C-sets or sets with property C. A set of reals has property C' iff for every family $\mathcal{G}_n$ of finite open covers there is a diagonal sequence $U_n \in \mathcal{G}_n$ such that

$$X \subseteq \bigcup_{n<\omega} U_n .$$

Rothberger showed that a set X is a C'-set iff every continuous image of X has strong measure zero. If we drop the condition that the covers $\mathcal{G}_n$ be finite we get the notion of a C''-set. It isn't hard to show that the continuous image of any C''-set is a C''-set. Also, any set concentrated on a countable subset of itself is a C''-set.

Question (Rothberger). *Is every C' set a C'' set?*

Recently there has been some work on very singular sets of cardinality the continuum. These sets can be looked at as generalizations of C'' sets. For reference see GALVIN–MILLER (19· ·).

Now let us consider the continuous image of a Luzin or Sierpiński set.

9.7. THEOREM (LUZIN (1933)). *There exists a one-to-one continuous function from ω^ω to ω^ω which takes every Luzin set to a perfectly meager set.*

PROOF. Let $\omega^{<\omega}$ be the set of finite sequences of ω. Let $\{P_s : s \in \omega^{<\omega}\}$ be a family of perfect subsets of ω^ω constructed as follows. Suppose we have P_s. Let $\{P_{s^\frown\langle n\rangle} : n < \omega\}$ be a family of disjoint perfect subsets of P_s each of which is nowhere dense relative to P_s and for any clopen set C if $P \cap C \neq \emptyset$, then there exists n such that

$$P_{s^\frown\langle n\rangle} \subseteq C \cap P.$$

Define f by

$$\{f(x)\} = \bigcap\{P_{x\restriction n} : n < \omega\}.$$

Now suppose Q is any perfect subset of ω^ω and let

$$R = \bigcup\{P_s : P_s \text{ is nowhere dense in } Q\}.$$

For any $s \in \omega^{<\omega}$ if P_s is not contained in Q, then for some n, $P_{s^\frown\langle n\rangle}$ is disjoint from Q. Otherwise, if P_s is contained in Q, then any $P_{s^\frown\langle n\rangle}$ is nowhere dense in Q. Hence $\{x \in \omega^\omega : f(x) \in R\}$ contains an open dense set. The result follows. □

The next theorem is proved in MILLER (19··a).

9.8. THEOREM. (a) *It is consistent that every set of reals of cardinality the continuum contains the one-to-one continuous image of a Luzin set of cardinality the continuum.*

(b) *It is consistent that every set of reals of cardinality the continuum contains the one-to-one continuous image of a Sierpiński set of cardinality the continuum.*

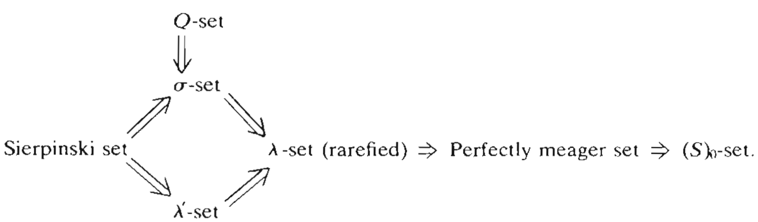
The model for (a) is the Cohen real model and the model for (b) is the random real model. Since the one-to-one preimage of a universal measure zero set has universal measure zero, (b) implies there are no universal measure zero sets of cardinality the continuum. Also, since the one-to-one preimage of a λ-set is a λ-set, (a) implies there are no λ-sets of cardinality the continuum.

10. Implications and definitions

10.1. IMPLICATIONS.

Luzin set ⇒ concentrated set ⇒ c''-set ⇒ c'-set ⇒

⇒ strong measure zero set ⇒ universal measure zero set ⇒ $(S)_0$-set.



10.2. DEFINITIONS. In the following definitions X is a set of reals.

Luzin set (property L). For every meager set M, $X \cap M$ is countable.

Concentrated set. There exist a countable set of reals D such that for every open $G \supseteq D$, $X - G$ is countable.

C″-set (ROTHBERGER property). For every family $\{\mathcal{G}_n : n < \omega\}$ of open covers of X there exists $U_n \in \mathcal{G}_n$ for $n < \omega$ such that $X \subseteq \bigcup_{n<\omega} U_n$.

C′-set. (Same as C'' but each $\mathcal{G}_n$ a finite open cove or X.)

Strong measure zero set (property C). For a sequence of reals $\varepsilon_n > 0$ for $n < \omega$ there exists a set I_n of diameter less than ε_n such that $X \subseteq \bigcup_{n<\omega} I_n$.

Universal measure zero set. For any atomless countably additive measure μ, $\mu(X) = 0$.

$(S)_0$-set. For any perfect set P there exists a perfect set $Q \subseteq P$ disjoint from X.

Sierpiński set (Property S). For every measure zero set M, $X \cap M$ is countable.

Q-set. for every $A \subseteq X$ there exists a G_δ-set G such that $A = X \cap G$.

σ-set. For every F_σ-set F there is a G_δ-set G such that $F \cap X = G \cap X$.

λ-set (rarefied). For every countable set $F \subseteq X$ there is a G_δ-set G such that $F = G \cap X$.

λ′-set. For every countable set F, $X \cup F$ is a λ-set.

Perfectly meager (always of first category). For any perfect set P, $X \cap P$ is meager in P.

References

ABRAHAM, U. and S. SHELAH

[19··] Martin's Axiom does not imply that every two $\aleph_1$-dense sets of reals are isomorphic, *Israel J. Math.*, **38**, 161–176.

ALEXANDROFF, P.S. and P.S. URYSOHN

[1927] Über nulldimensionale Punktmengen, *Mathematische Annalen*, **98**, 89–106.

BALOGH, Z. and H. JUNNILA

[19··] Totally analytic spaces under $V = L$.

BAUMGARTNER, J.

[1973] All $\aleph_1$-dense sets of reals can be isomorphic, *Fund. Math.* **79**, 101–106.

[1980] Chains and antichains in $P(\omega)$, *J. Symbolic Logic*, **45**, 85–92.

[19··] Sacks forcing and the total failure of Martin's Axiom.

BAUMGARTNER, J. and R. LAVER

[1979] Iterated perfect set forcing, *Annals Math. Logic*, **17**, 271–288.

BELL, M. and K. KUNEN

[1981] On the pi-character of ultrafilters, C.R. Math. Rep. Acad. Sci. Canada, **III**, 351–356.

BESICOVITCH, A.S.

[1934] Concentrated and rarified sets of points, *Acta. Math.*, **62**, 289–300.

[1942] Relations between concentrated sets and sets possessing property *C*, *Proc. Cambridge Philos, Soc.*, **38**, 20–23.

BING, R.H., W.W. BLEDSOE and R.D. MAULDIN

[1974] Sets generated by rectangles, *Pacific J. Math.*, **51**, 27–36.

BOREL, E.

[1919] Sur la classification des ensembles de mesure nulle, *Bull. Soc. Math. France*, **47**, 97–125.

BROWN, J.B.

[1977] On the Baire order of concentrated spaces and L_1 spaces, *Fund. Math.*, **94**, 165–172.

CARLSON, T.

[19 · ·a] Strongly meager and strong measure zero sets of reals.

[19 · ·b] Hilbert space and Borel's conjecture.

DARST, R.B.

[1968a] A universal null set which is not concentrated, *Fund. Math.*, **62**, 47–48.

[1968b] Properties of Luzin sets with applications to probability. *Proc. Amer. Math. Soc.* **20**, 348–350.

[1969] Some remarks on Hausdorff measure, *Fund. Math.*, **64**, 325–328.

[1970a] A CBV image of a universal null set need not be a universal null set, *Fund. Math.*, **62**, 219–220.

[1970b] On bimeasurable images of universally measurable sets, *Fund. Math.*, **67**, 381–382.

[1971a] C^∞-function need not be bimeasurable, *Proc. Amer. Math. Soc.*, **27**, 128–132.

[1971b] A characterization of bimeasurable functions in terms of universally measurable sets, *Proc. Amer. Math. Soc.*, **27**, 566–571.

[1973] On the connection between Hausdorff measures and generalized capacity, *Fund. Math.*, **80**, 1–3.

VAN DOUWEN, E.

[1976] Hausdorff gaps and a nice countably paracompact nonnormal space, *Topological Proc.* **1**, 239–242.

VAN DOUWEN, E. and FLEISSNER, W.

[19 · ·] Definable forcing axiom.

VAN DOUWEN, E., TALL, F. and W. WEISS

[1977] Nonmetrizable hereditarily Lindelof spaces with point-countable bases from CH. *Proc. Amer. Math. Soc.*, **64**, 139–145.

ELLENTRUCK, E.

[1974] A new proof that analytic sets are Ramsey, *J. Symbolic Logic*, **39**, 163–165.

ERDÖS, P., K. KUNEN and D. MAULDIN

[1981] Some additive properties of sets of real numbers, *Fund. Math.*, **93**, 187–199.

FENSTAD, J.F. and D. NORMANN

[1974] On absolutely measurable sets, *Fund. Math.*, **81**, 91–98.

FLEISSNER, W.G.

[1978] Current research on *Q*-sets, *Colloq. Math. Soc. Jańos Bolyai* **23**, Topology, Budapest (Hungary), 413–431.

[19 · ·] Squares of *Q*-sets.

FLEISSNER, W.G. and A.W. MILLER

[1980] On *Q*-sets, *Proc. Amer. Math. Soc.*, **78**, 280–284.

FREMLIN, D.H., HANSELL, R.W. and H.J.K. JUNNILA
[19··] Borel functions of bounded class.

FREMLIN, D.H.
[19··] Consequences of Martin's axiom.

GALVIN, F. and A.W. MILLER
[19··] γ-sets and other singular sets of real numbers, *Topology Appl.*

GALVIN, F., MYCIELSKI, J. and R.M. SOLOVAY
[1979] Strong measure zero sets, *AMS Notices*, **26**, A-280.

GALVIN, F. and K. PRIKRY
[1973] Borel sets and Ramsey's theorem, *J. Symbolic Logic*, **38**, 193–198.

GARDNER, R.J.
[1979] On concentrated sets, *Fund. Math.*, **102**, 45–53.

GRZEGOREK, E.
[1978] Remarks on σ-fields without continuous measure, *Coll. Math.*, **39**, 73–75.
[1980] Solution of a problem of Banach on σ-fields without continuous measures, *Bull. L'Acad. Polon. Sciences*, **28**, 7–10.
[1981] On some results of Darst and Sierpiński concerning universal null and universally measureable sets, *Bull. L'Acad. Polon. Sciences*, **29**, 1–5.
[19··] On sets always of the first category.

GRZEGOREK, E. and C. RYLL-NARDZEWSKI
[1981] On universal null sets, *Proc. Amer. Math. Soc.*, **81**, 613–617.
[19··] A remark on absolutely measurable sets.

HANSELL, R.W.
[1980] Some consequences of $(V = L)$ in the theory of analytic sets, *Proc. Amer. Math. Soc.*, **80**, 311–319.

HAUSDORFF, F.
[1934] Summen von $\aleph_1$ Mengen, *Fund. Math.*, **26**, 241–255.

HUREWICZ, W.
[1932] Une remarque sur l'hypothese du continu, *Fund. Math.*, **19**, 8–9.

JECH, T.J.
[1971] Lectures in set theory, *Lecture Notes in Math.*, **217** (Springer-Verlag, Berlin).
[1978] *Set Theory* (Academic Press, New York).

JONES, F.B.
[1937] Concerning normal and completely normal spaces, *Bull. Amer. Math. Soc.*, **43**, 671–676.

KUNEN, K.
[1968] Inaccessibility properties of cardinals, Doctoral Dissertation, Stanford University.
[1976] Luzin spaces, *Topological Proc.*, **1**, 191–199.
[1980] *Set Theory, An Introduction to Independence Proofs* (North-Holland, Amsterdam).

KURATOWSKI, C.
[1933] Sur une famille d'ensembles singuliers, *Fund. Math.*, **21**, 127–128.
[1966] *Topology*, Vol. I (Academic Press, New York).

LAVER, R.
[1976] On the consistency of Borel's conjecture, *Acta. Math.*, **137**, 151–169.

LUZIN, N.
[1914] Sur un problème de M. Baire, *C. R. Hebdomadaires Seances Acad. Sci. Paris*, **158**, 1258–1261.

[1921] Sur l'existence d'un ensemble nondénombrable qui est de première catégorie sur tout ensemble parfait, *Fund. Math.*, **2**, 155–157.

[1933] Sur les ensembles toujours de premiere categorie, *Fund. Math.* **21**, 114–126.

MAHLO, P.

[1913] Über Teilmengen des Kontinuums von dessen Machtigkeit, *Sitzungsberichte der Sachsischen Akademie der Wissenschaften zu Leipzig, Mathematisch-Naturwissenschaftliche Klasse* **65**, 283–315.

MARTIN, D.A. and R.M. SOLOVAY

[1970] Internal Cohen extensions, *Ann. Math. Logic*, (2) **2**, 143–178.

MAULDIN, R.D.

[1970] On the Baire system generated by a linear lattice of functions, *Fund. Math.*, **68**, 51–59.

[1974] Baire function, Borel sets and ordinary function systems, *Advances in Math.*, April.

[1976] Countably generated families, *Proc. A.M.S.*, **54**, 291–297.

[1977] On rectangles and countably generated families, *Fund. Math.*, **95**, 129–139.

MCAULEY, L.F.

[1956] Paracompactness and an example due to F.B. Jones, *Proc. Amer. Math. Soc.* **7**, 1155–1156.

MILLER, A.W.

[1979a] On the length of Borel hierarchies, *Ann. Math. Logic*, **16**, 233–267.

[1979b] On generating the category algebra and the Baire order problem, *Bull. Acad. Polon.*, **27**, 751–755.

[1980] Covering 2^ω with ω_1 disjoint closed sets, *The Kleene Symposium*, (North-Holland, Amsterdam) 415–421.

[1981a] Generic Souslin sets, *Pacific J. Math.*, **97**, 171–181.

[1981b] Some properties of measure and category, *Trans. Amer. Math. Soc.*, **266**, 93–114, Corrections Trans. Amer. Math. Soc.

[1982a] Mapping a set of reals onto the reals, *J. Symbolic Logic*.

[1982b] The Baire category theorem and cardinal of countable cofinality, *J. Symbolic Logic*, **47**, 275–288.

[1982c] A characterization of the least cardinal for which the Baire category theorem fails, *Proc. Amer. Math. Soc.*, **86**, 498–502.

MAZURKIEWICZ, S. and E. SZPILRAJN (MARCZEWSKI)

[1937] Sur la dimension de certains ensembles singuliers, *Fund. Math.*, **28**, 305–308.

MORGAN II, J.C.

[1974] Infinite games and singular sets, *Colloq. Math.*, **29**, 7–17.

[1975] On translation invariant families of sets, *Colloq. Math.*, **34**, 63–68.

[1976] The absolute Baire property, *Pacific J. Math.*, **65**, 421–436.

[1977] Baire category from an abstract viewpoint, *Fund. Math.* **94**, 13–23.

[1978] On the absolute Baire property, *Pacific J. Math.*, **78**, 415–431.

[1979] On sets every homeomorphic image of which has the Baire property, *Proc. Amer. Math. Soc.*, **75**, 351–354.

MOSCHOVAKIS, Y.N.

[1980] *Descriptive Set Theory* (North-Holland, Amsterdam).

NYIKOS, P.J. and J.E. VAUGHAN

[19 · ·] On first countable, countably compact spaces, I: (ω_1, ω_1^*)-gaps.

OXTOBY, J.C.

[1971] *Measure and category* (Springer-Verlag, Amsterdam).

POPROUGÉNKO, G.

[1930] Sur un problème de M. Mazurkiewicz, *Fund. Math.*, **15**, 284–286.

PRZYMUSIŃSKI, T.C.
[1977] Normality and separability of Moore spaces, in: G.M. Reed, ed., *Set Theoretic Topology*, (Academic Press, New York) 325–337.
[19··] On the equivalence of certain set-theoretic and topological conditions.

ROTHBERGER, F.
[1938a] Eine Äquivalenz zwischen der Kontinuumhypothese und der Existenz der Lusinschen und Sierpińskischen Mengen, *Fund. Math.*, **30**, 215–217.
[1938b] Eine verscharfung der eigenschaft *C*, *Fund. Math.*, **30**, 50–55.
[1939] Sur un ensemble de preière catégorie qui est dépourvu de la propriété λ, *Fund. Math.*, **32**, 294–300.
[1941] Sur les families indenombrables des suites de nombres naturels et les problems concernant la propriété *C*, *Proc. Cambridge Philos. Soc.*, **37**, 109–126.
[1944] On families of real functions with a denumerable base, *Ann. of Math.*, **45**, 397–407.
[1948] On some problems of Hausdorff and Sierpiński, *Fund. Math.*, **35**, 29–46.
[1952a] On the property C and a problem of Hausdorff, *Canad. J. Math.*, **4**, 111–116.
[1952b] A remark on the existence of a denumerable base for a family of functions, *Canad. J. Math.*, **4**, 117–119.

ROYDEN, H.L.
[1971] *Real Analysis* (Macmillan, New York).

RUDIN, M.E.
[1977] Martin's axiom, in: J. Barwise, ed., *Handbook of Mathematical Logic* (North-Holland, Amsterdam) 491–501.

SHOENFIELD, J.R.
[1975] Martin's axiom, *Amer. Math. Monthly*, **82**, 610–617.

SIERPIŃSKI, W.
[1923] Sur l'invariance topologique de la propriété de Baire, *Fund. Math.*, **4**, 319–323.
[1924] Sur l'hypothese du continu ($2^{\aleph_0} = \aleph_1$), *Fund. Math.*, **5**, 177–187.
[1928] Sur un ensemble non denombrable, donc toute image continue est de mesure nulle, *Fund. Math.*, **11**, 301–304.
[1934a] *Hypothèse du Continu*, Monografje Matematyczme, Warszawa-Lwow.
[1934b] Sur les ensembles toujours de première catégorie, *Mathematica*, **8**, 191–195.
[1935] Sur le produit combinatoire de deux ensembles jouissant de la propriété *C*, *Fund. Math.*, **24**, 48–50.
[1936] Sur un problème concernant les fonctions de première classe, *Fund. Math.*, **27**, 191–200.
[1937a] Sur une propriété additive d'ensembles, *Computes Rendus Soc. Sci. Varsovie*, **30**, 257–259.
[1937b] Sur le rapport de la propriété (*C*) a la théorie générale des ensembles, *Fund. Math.*, **29**, 91–96.
[1938] Remarque sur le probleme de l'invariance topologique de la propriété (*C*), *Fund. Math.*, **30**, 56–68.
[1939a] Sur les ensembles concentrés, *Fund. Math.*, **32**, 301–305.
[1939b] Sur un ensemble á propriété λ, *Fund. Math.*, **32**, 306–310.
[1945a] Sur la non-invariance topologique de la propriété λ', *Fund. Math.*, **33**, 264–268.
[1945b] Sur deux conséquences d'un théorème de Hausdorff, *Fund. Math.*, **33**, 269–272.
[1950] Sur les d'ordre des ensembles linéaires, *Fund. Math.*, **37**, 253–264.
[1961] *General Topology*, Toronto.

SIERPIŃSKI, W. and E. SZPILRAJN
[1936] Remarque sur le probleme de la mesure, *Fund. Math.* **26**, 256–261.

SILVER, J.
[1970] Every analytic set is Ramsey, *J. Symbolic Logic*, **35**, 60–64.

STEPRĀNS, J.
[1982] Cardinal arithmetic and $\aleph_1$-Borel sets, *Proc. Amer. Math. Soc.*

SZPILRAJN (MARCZEWSKI), E.

[1930] Sur un problème de M. Banach, *Fund. Math.*, **15**, 212–214.

[1933] Sur certains invariants de l'opération (*A*), *Fund. Math.*, **21**, 229–235.

[1934] Remarques sur les fonctions complètement additives d'ensemble et sur les ensembles jouissant de la propriété de Baire, *Fund. Math.*, **22**, 303–311.

[1935a] Sur une classe de fonctions de W. Sierpiński et la classe correspondante d'ensembles, *Fund. Math.*, **24**, 17–34.

[1935b] Problem 68, *Fund. Math.*, **24**, 579.

[1937] O zbiorach i funkcjach bezezglednie mierzalnych (On absolutely measurable sets and functions), Sprawozdania z posiedzen Towarzystwa Naukowego Warszawskiego, Wydzial III, *Comptes Rendus Soc. Sci. Varsovie, Classe III*, **30**, 39–68. An English translation, by John C. Morgan II, is available in manuscript.

[1938a] On the equivalence of some classes of sets, *Fund. Math.* **30**, 235–241.

[1938b] The characteristic function of a sequence of sets and some of its applications, *Fund. Math.*, **31**, 207–223.

[1947] Remarque sur la mesurabilité absolue, *Colloq. Math.* **1**, 42–43.

TALL, F.

[1976] The density topology, *Pacific J. Math.*, **62**, 275–284.

[1978] Normal subspaces of the density topology, *Pacific J. Math.*, **75**, 579–588.

[19··] Some applications of a generalized Martin's Axiom.

WALSH, J.J.

[1979] Infinite dimensional compacta containing no n-dimensional ($n \geqslant 1$) subsets, *Topology*, **18**, 91–95.

ADDED IN PROOF. See the survey paper:

BROWN, J.B. and C.V. COX

[1982] Classical theory of totally imperfect sets, *Real Anal. Exchange*, **7**(2).