The onto mapping property of Sierpinski

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Define

(*) There exists $(\phi_n : \omega_1 \to \omega_1 : n < \omega)$ such that for every $I \in [\omega_1]^{\omega_1}$ there exists n such that $\phi_n(I) = \omega_1$.

This is roughly what Sierpinski [10] refers to as P_3 but I think he brings $\mathbb{R}$ into it. I don't know French so I cannot say for sure what he says but I think he proves that (*) follows from the continuum hypothesis. Here we show that the existence of a Luzin set implies (*) and (*) implies that there exists a nonmeager set of reals of size ω_1. We also show that it is relatively consistent that (*) holds but there is no Luzin set. All the other properties in this paper, (**), (S*), (S**), (B*) are shown to be equivalent to (*).

Proposition 1 *(Sierpinski [10]) CH implies (*).*

proof:

Let $\omega_1^\omega = \bigcup_{\alpha<\omega_1} \mathcal{F}_\alpha$ where the $\mathcal{F}_\alpha$ are countable and increasing. For each α construct $(\phi_n(\alpha) : n < \omega)$ so that for every $g \in \mathcal{F}_\alpha$ there is a some n such that $\phi_n(\alpha) = g(n)$.

Now suppose $I \subseteq \omega_1$. If no ϕ_n maps I onto ω_1, then there exists $g \in \omega_1^\omega$ such that $g(n) \notin \phi_n(I)$ for every n. If $g \in \mathcal{F}_{\alpha_0}$, then $\alpha \notin I$ for every $\alpha \geq \alpha_0$. This is because $g \in \mathcal{F}_\alpha$ and so for some n $g(n) = \phi_n(\alpha)$ and since $g(n) \notin \phi_n(I)$ we have $\alpha \notin I$.
□

Define

(**) There exists $(g_\alpha : \omega \to \omega_1 : \alpha < \omega_1)$ such that for every $g : \omega \to \omega_1$ for all but countably many α there are infinitely many n with $g(n) = g_\alpha(n)$.

Proposition 2 *(**) iff (*).*

proof:

To see (**) implies (*) let $\phi_n(\alpha) = g_\alpha(n)$. Then the proof of the first proposition goes thru.

On the other hand suppose $(\phi_n : \omega_1 \to \omega_1 : n < \omega)$ witnesses (*). First note that for any $I \in [\omega_1]^{\omega_1}$ there are infinitely many n such that $\phi_n(I) = \omega_1$. This is because if there are only finitely many n we could cut down I in finitely many steps so that there were no n with $\phi_n(I) = \omega_1$.

Now define $g_\alpha \in \omega_1^\omega$ by $g_\alpha(n) = \phi_n(\alpha)$. These witness (**). Given any $g : \omega \to \omega_1$ if there is an uncountable $I \subseteq \omega_1$ and $N < \omega$ such that for every $\alpha \in I$ we have $g(n) \neq g_\alpha(n)$ for all $n > N$ then this means that $g(n) \notin \phi_n(I)$ and for all $n > N$ and so (*) fails.

$\Box$

Obviously (**) is false if $\mathfrak{b} > \omega_1$ so (*) is not provable just from ZFC.

Proposition 3 *It is relatively consistent with any cardinal arithmetic that (*) is true and* $\mathfrak{b} = \mathfrak{d} = \omega_1$.

proof:

Start with any M a countable transitive model of ZFC. Our final model is $M[g_\alpha, f_\alpha : \alpha < \omega_1]$ where each $g_\alpha : \omega \to \alpha$ is generic with respect to the poset of finite partial functions from ω to α and $f_\beta \in \omega^\omega$ is Hechler real over $M[g_\alpha, f_\alpha : \alpha < \beta]$. The ω_1-sequence is obtained by finite support ccc forcing. By ccc for any $g \in \omega_1^\omega \cap M[g_\alpha, f_\alpha : \alpha < \omega_1]$ there will be $\alpha_0 < \omega_1$ such that α_0 bounds the range of g and $g \in M[g_\alpha, f_\alpha : \alpha < \alpha_0]$. It follows by product genericity that for every $\alpha \geq \alpha_0$ there are infinitely many n such that $g(n) = g_\alpha(n)$. The Hechler sequence f_α for $\alpha < \omega_1$ shows that $\mathfrak{d} = \omega_1$.

$\Box$

With a little more work we will prove that (*) follows from the existence of a Luzin set (Prop 6). We will also show that (*) implies there is a nonmeager set of reals of size ω_1 (Prop 7) and so in the random real model (*) fails and $\mathfrak{b} = \mathfrak{d} = \omega_1$.

Actually I think Sierpinski considers what appears to be a stronger version:

Define

(S*) There exists $(\phi_n : \omega_1 \to \omega_1 : n < \omega)$ such that for every $I \in [\omega_1]^{\omega_1}$ for all but finitely many n $\quad \phi_n(I) = \omega_1$.

Surprisingly (S*) is equivalent to (*).

Proposition 4 *(S*) iff (*).*

proof:

We show (**) implies (S*).

Let $a_0 = 1$ and $a_{n+1} = 1 + \sum_{i \leq n} a_i$. Let

$$\mathcal{A}_n = \{u \mid \exists D \in [\omega_1]^{a_n} \;\; u : D \to \omega_1\} \text{ and } \prod_{n<\omega} \mathcal{A}_n = \{g \mid \forall n \; g(n) \in \mathcal{A}_n\}$$

Since each $\mathcal{A}_n$ has cardinality ω_1 from (**) we get $(g_\alpha \in \prod_{n<\omega} \mathcal{A}_n : \alpha < \omega_1)$ such that for every $g \in \prod_{n<\omega} \mathcal{A}_n$ for all but countably many α there are infinitely many n such that $g(n) = g_\alpha(n)$. For each $\alpha < \omega_1$ define $h_\alpha : \omega \to \omega_1$ so that if $g_\alpha(n) = u_n : A_n \to \omega_1$ for every n then

$$h_\alpha \restriction (A_n \setminus \cup_{i<n} A_i) = u_n \restriction (A_n \setminus \cup_{i<n} A_i)$$

Since $|A_k| = a_k$ the sets $A_n \setminus \cup_{i<n} A_i$ are nonempty. We claim that the h_α have the following property:

Define

(S**) For any $X \in [\omega]^\omega$ and $h : X \to \omega_1$ for all but countably many α there are infinitely many $n \in X$ with $h(n) = h_\alpha(n)$.

It is enough to see there is at least one $n \in X$ with $h(n) = h_\alpha(n)$. Otherwise if there were only finitely many n for uncountably many α we could throw out from X a fixed finite set for uncountably many α and get a contradiction.

Let $X = \{x_n : n < \omega\}$ listing X in increasing order. Define $g \in \prod_{n<\omega} \mathcal{A}_n$ by $g(n) = h \restriction \{x_i : i < a_n\}$. Now suppose $g_\alpha(n) = g(n)$. This means that if $g_\alpha(n) = u_n : A_n \to \omega_1$, then $A_n = \{x_i : i < a_n\}$ and $u_n = h \restriction A_n$. But since $A_n \setminus \cup_{i<n} A_i$ is nonempty we get that $h_\alpha(x) = h(x)$ for some $x \in X$.

Now define $\phi_n(\alpha) = h_\alpha(n)$. This has the required property (S*). Given I uncountable let X be the $n \in \omega$ with $\phi_n(I) \neq \omega_1$. If X is infinite we would get $h : X \to \omega_1$ such that $h(n) \notin \phi_n(I)$ for all $n \in X$. But this means that for all $\alpha \in I$ and $n \in X$ that $h(n) \neq h_\alpha(n)$ which contradicts (S**).

□

This is related to results in Bartoszynski [2].

Bagemihl-Sprinkle [1] say that Sierpinski states CH implies (S*) but only proves (*). They give a proof from CH of a seemingly stronger version:

Define

(B*) There exists $(\phi_n : \omega_1 \to \omega_1 : n < \omega)$ such that for every $I \in [\omega_1]^{\omega_1}$ for all but finitely many n for all $\beta < \omega_1$ there are uncountably many $\alpha \in I$ with $\phi_n(\alpha) = \beta$, i.e., not only is $\phi_n(I) = \omega_1$ but it is uncountable-to-one.

Proposition 5 *(S*) iff (B*)*

proof:

Let $\pi : \omega_1 \to \omega_1$ be uncountable to one, i.e., for all $\beta < \omega_1$ there are uncountably many $\alpha < \omega_1$ with $\pi(\alpha) = \beta$. If $(\phi_n : \omega_1 \to \omega_1 : n < \omega)$ witness (S*) then $(\pi \circ \phi_n : \omega_1 \to \omega_1 : n < \omega)$ satisfies (B*).
□

Proposition 6 *If there is a Luzin set, then (*) is true.*

proof:

We prove (**). Suppose $\{g_\alpha : \omega \to \omega \ : \ \alpha < \omega_1\}$ is a Luzin set, then it satisfies that for every $k : \omega \to \omega$ for all but countably many $\alpha < \omega_1$ there are infinitely many n such that $k(n) = g_\alpha(n)$.

There is a sequence $(f_\alpha : \alpha \to \omega : \omega \leq \alpha < \omega_1)$ of one-to-one functions which is coherent: for $\alpha < \beta \quad f_\beta \restriction \alpha =^* f_\alpha$, i.e., $f_\beta(\gamma) = f_\alpha(\gamma)$ for all but finitely many $\gamma < \alpha$. This is the construction of an Aronszajn tree which appears in the first edition of Kunen's set theory book [6].

Let $\hat{g}_\alpha : \omega \to \alpha$ be any map which extends $f_\alpha^{-1} \circ g_\alpha$. We claim that for any $k : \omega \to \omega_1$ which is one-to-one that for all but countably many α there are infinitely many n with $\hat{g}_\alpha(n) = k(n)$. To see this suppose $k : \omega \to \beta$ is one-to-one and let $\hat{k} = f_\beta \circ k$ which maps ω to ω. Then for some $\alpha_0 > \beta$ for all $\alpha \geq \alpha_0$ there will be infinitely many n with $g_\alpha(n) = \hat{k}(n)$. This means that $g_\alpha(n) = f_\beta(k(n))$. Since k is one-to-one, there will be infinitely many such n where $f_\beta(k(n)) = f_\alpha(k(n))$. But $g_\alpha(n) = f_\alpha(k(n))$ implies $\hat{g}_\alpha(n) = k(n)$.

To get rid of the requirement that k be one-to-one, let $j : \omega_1 \times \omega \to \omega_1$ be a bijection and $\pi : \omega_1 \to \omega_1$ be projection onto first coordinate, i.e., $\pi(j(\alpha, n)) = \alpha$. Define $h_\alpha(n) = \pi(\hat{g}_\alpha(n))$. Given any $k : \omega \to \omega_1$ define $\hat{k}(n) = j(k(n), n)$. Then since $\hat{k}$ is one-to-one for all but countably many α there will be infinitely many n with $\hat{g}_\alpha(n) = \hat{k}(n)$. But this implies

$$h_\alpha(n) = \pi(\hat{g}_\alpha(n)) = \pi(\hat{k}(n)) = k(n)$$

Hence $(h_\alpha : \alpha < \omega_1)$ satisfies (**).
□

Proposition 7 *Suppose (*), then there exists* $(x_{\alpha,\beta} \in 2^\omega : \alpha, \beta < \omega_1)$ *such that for every dense open* $D \subseteq 2^\omega$ *there exists* $\alpha_0 < \omega_1$ *such that for every* $\alpha \geq \alpha_0$ *there is a* $\beta_\alpha < \omega_1$ *such that* $x_{\alpha,\beta} \in D$ *for every* $\beta \geq \beta_\alpha$*.*

proof:

We use that there are $\{h_\alpha : \omega \to \omega \ : \ \alpha < \omega_1\}$ with the property that for every $X \in [\omega]^\omega$ and $h : \omega \to \omega$ for all but countably many α there are infinitely many $n \in X$ with $h(n) = h_\alpha(n)$ (see (S**) in the proof of Prop 4). This implies that there exists $(X_\alpha \in [\omega]^\omega : \alpha < \omega_1)$ such that for every $Y \in [\omega]^\omega$ for all but countably many α there are infinitely many $x \in X_\alpha$ such that $|Y \cap [x, x^+)| \geq 2$ where x^+ is the least element of X_α greater than x. Fix α and enumerate $X_\alpha = \{k_n : n < \omega\}$ in strict increasing order. Define

$$P_\alpha = \{g : \omega \to FIN(\omega, 2) \ : \ \forall n \ \ g(n) \in 2^{[k_n, k_{n+1})}\}$$

By (S**) there exists $g_{\alpha,\beta} \in P_\alpha$ for $\beta < \omega_1$ with the property that for any h in P_α and infinite $Y \subseteq \omega$ for all but countably many β there are infinitely many $n \in Y$ with $h(n) = g_{\alpha,\beta}(n)$. Define $x_{\alpha,\beta} \in 2^\omega$ by $x_{\alpha,\beta}(m) = g_{\alpha,\beta}(n)(m)$ where n is the unique integer with $k_n \leq m < k_{n+1}$. Equivalently $x_{\alpha,\beta} = \bigcup_n g_{\alpha,\beta}(n)$. (Without loss we may assume $k_0 = 0 \in X_\alpha$.)

Given $D \subseteq 2^\omega$ dense open let $\hat{D} \subseteq 2^{<\omega}$ be the set of all s with $[s] \subseteq D$. Construct an infinite $Z \subseteq \omega$ so that for every $z \in Z$ there exists $t \in 2^{<\omega}$ with $|t| \leq z^+ - z$ such that for every $s \in 2^{<\omega}$ with $|s| \leq z$ we have $s\,t \in \hat{D}$ where $s\,t$ is the concatenation of s with t. By construction there exists α_0 so that for every $\alpha \geq \alpha_0$ the there are infinitely many $x \in X_\alpha$ with $|[x, x^+) \cap Z| \geq 2$.

Fix $\alpha \geq \alpha_0$ and as above $X_\alpha = \{k_n : n < \omega\}$. Let

$$Y = \{n \ : \ |[k_n, k_{n+1}) \cap Z| \geq 2.$$

Note that by the definition of Y there is a $h \in P_\alpha$ with the property that for every $n \in Y$ for every $s \in 2^{k_n}$ we have $s \cup h(n) \in \hat{D}$. For some β_α for every $\beta \geq \beta_\alpha$ there are infinitely many $n \in Y$ with $h(n) = g_{\alpha,\beta}(n)$ and so $x_{\alpha,\beta} \in D$.

□

This is similar to the argument of Miller [9]. Obviously the set of $x_{\alpha,\beta}$ in Prop 7 is nonmeager. Although it seems a little bit like a Luzin set, it isn't. In the first version of this paper I asked:

Does the existence of a nonmeager set of reals of size ω_1 imply (*)?

To my surprise O. G. González [4] proved that in fact, the existence of a nonmeager set of reals of size ω_1, i.e., non(meager)=ω_1 implies (*), so they are equivalent. It also follows from his result that the Luzin like set of Proposition 7 is equivalent to (*). Judah and Shelah [5] have shown that it is consistent that non(meager)=ω_1 and there are no Luzin sets. This holds in the model obtained by countable support iteration of length ω_2 of superperfect tree forcing[1] over a ground model of CH. González [4] also gives a model in which (*) holds and there are no Luzin sets, in fact, in his model it is true that for any nonmeager subset $X \subseteq \omega^\omega$ there is an $f \in \omega^\omega$ such that for uncountably many $g \in X \quad \forall n\; f(n) \neq g(n)$.

This paper was motivated by a result in an earlier version of A.Medini [7] which showed that (*) implies that there is an uncountable $X \subseteq 2^\omega$ with the Grinzing property: for every uncountable $Y \subseteq X$ there is an uncountable family of uncountable subsets of Y with pairwise disjoint closures in 2^ω. To do this Medini used a result from Miller [8]. This has been superceded by a proof in ZFC of an uncountable $X \subseteq 2^\omega$ with the Grinzing property.

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