Sacks forcing, Laver forcing, and Martin's Axiom [1]

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Abstract

In this paper we study the question assuming MA+¬CH does Sacks forcing or Laver forcing collapse cardinals? We show that this question is equivalant to question of what is the additivity of Marczewski's ideal s^0. We give a proof that it is consistent that Sacks forcing collapses cardinals. On the other hand we show that Laver forcing does not collapse cardinals.

Introduction

Let $\mathbb{S}$ be Sacks perfect set forcing [32]; $p \in \mathbb{S}$ iff $p \subseteq 2^{<\omega}$ is a nonempty subtree and for every $s \in p$ there exists $t \supseteq s$ such that $t\hat{\ }0 \in p$ and $t\hat{\ }1 \in p$. The ordering is defined by $p \le q$ iff $p \subseteq q$. Define $[p] = \{x \in 2^\omega : \forall n\ x \restriction n \in p\}$.

The s^0 ideal of subsets of 2^ω is defined by $X \in s^0$ iff for every $p \in \mathbb{S}$ there exists $q \le p$ with $X \cap [q] = \emptyset$. Define $add(s^0) = min\{|F| : F \subseteq s^0, \bigcup F \notin s^0\}$.

Marczewski's ideal s^0, which first appeared in [24], has been studied by a number of authors, Aniszczyk, Frankiewicz, Plewik [4], Brown [11][12][13], Brown, Cox [14], Brown, Prikry [15], Corazza [16], Morgan [29], and Pawlikowski [31]. Aniszczyk [5] has asked if MA implies that the ideal s^0 is $\mathfrak{c}$-additive, i.e., is it true that the union of fewer than continuum many s^0 sets is an s^0 set, i.e., $add(s^0) = \mathfrak{c}$. It is a folklore result that assuming the proper forcing axiom the ideal s^0 is $\mathfrak{c}$-additive (see Abraham [1]). It is also an easy exercise to show the consistency of $add(s^0) = \omega_1$ plus the continuum is large. This happens in the Cohen real model.

Theorem 1.1. *(MA+¬CH) $add(s^0)$ is the minimum κ such that for some $p \in \mathbb{S}$ we have $p \Vdash_{\mathbb{S}} cof(\mathfrak{c}) = \kappa$.*

This means that the question of the additivity of the s^0 ideal is the same as the question of whether Sacks forcing collapses cardinals. In the proof

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we only use that $\mathfrak{c}$ is regular and MA holds for countable posets. It is well known that Sacks forcing cannot blow up the continuum. In fact Sacks [32] showed that every new element y of 2^ω there is a homeomorphism (with perfect domain and range) coded in the ground model which maps the Sacks real to y.

Question: Is it consistent with ZFC that $\mathfrak{c} = \omega_2$, $add(s^0) = \omega_1$, and $\mathbb{S}$ does not collapse $\mathfrak{c}$?

Question (Laver): Does $\mathfrak{d} = \aleph_1$ imply that $\mathbb{S}$ collapses $\mathfrak{c}$ to $\mathfrak{d}$?

The remainder of the section is concerned with three other cardinals associated with the ideal of s^0-sets. These cardinals have been extensively studied for the ideals of measure and category, see for example Judah and Shelah [19].

$non(s^0) = min\{|X| : X \notin s^0, X \subseteq 2^\omega\}$
$cov(s^0) = min\{|F| : F \subseteq s^0, \bigcup F = 2^\omega\}$
$cof(s^0) = min\{|F| : F \subseteq s^0 \text{ and } \forall A \in s^0 \ \exists B \in F \ A \subseteq B\}$

Note that $non(s^0) = \mathfrak{c}$, since any small set of reals is in s^0.

In Veličković's model ([36]) he gets MA, $\mathfrak{c} = \omega_2$, and $cov(s^0) = \omega_1$. The same is true in the model of Theorem 2.1.

Theorem 1.2. *It is relatively consistent with ZFC that $c = \omega_2 = cov(s^0)$ and $add(s^0) = \omega_1$.*

Fremlin noted that $cof(s^0) > \mathfrak{c}$. Theorem 1.3 slightly improves this.

Theorem 1.3. $cof(cof(s^0)) > \mathfrak{c}$

This argument also produces an s^0 set of cardinality $\mathfrak{c}$ (Gurevich and Shelah [18] see also [28]). Pierre Matet [25] has also proved a similar result for $cof(r^0)$ where r^0 is the ideal of Ramsey null sets.

So we get (in ZFC) that

$$\omega_1 \leq add(s^0) \leq cov(s^0) \leq \mathfrak{c} = non(s^0) < cof(cof(s^0)) \leq cof(s^0) \leq 2^{\mathfrak{c}}.$$

The remaining two theorems in this section are easy consistency results to indicate that the inequality

$$\mathfrak{c} < cof(cof(s^0)) \leq cof(s^0) \leq 2^{\mathfrak{c}}$$

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is best possible. For simplicity of notation we only consider models of CH.

Theorem 1.4. *Let $V \models GCH$, κ any cardinal of cofinality greater than ω_1, and let $\mathbb{P}$ be the partial order of countable functions from κ to ω_1. If G is $\mathbb{P}$-generic over V, then $V[G] \models$ "$\omega_1 = \mathfrak{c}$ and $cof(s^0) = 2^{\omega_1} = \kappa$". So, for example, it is consistent that $\omega_2 = cof(cof(s^0)) < cof(s^0) = \aleph_{\omega_2}$.*

Theorem 1.5. *Let V be a model of CH and κ be any regular cardinal with $\omega_1 < \kappa \leq 2^{\omega_1}$. Then there exists a generic extension $V[G]$ (with same cardinals and function 2^λ) in which $cof(s^0) = \kappa$.*

In section 2 we show that it is consistent with MA+¬CH that the addivity of the s^0 ideal is ω_1, (equivalently Sacks forcing collapses the continuum.)

Theorem 2.1. *It is consistent with MA+¬CH that the ideal of s^0 sets is not ω_1-additive.*

This result was also obtained independently by Velickovic [36] about the same time. Velickovic starts with a model of PFA and forces to kill the additivity of the s^0 ideal and proves that his forcing does not add any new subsets of ω_1.

It is clear from the construction that $\mathfrak{c}$ can be made arbitrarily large, but for simplicity we make it ω_2. It is not clear that $add(s^0)$ can be made anything we want. We assume the reader is familiar with the usual proof of the consistency of MA (see Kunen [21]).

A similar theorem has been proved for Silver forcing by Steprans [35]. For Silver forcing the ideal analagous to s^0 is the Mycielski ideal [30].

The technique we use is based on an unpublished proof of Kunen who showed that MA+¬CH does not settle the existence of $(\omega_1^*, \mathfrak{c})$-gaps or $(\mathfrak{c}^*, \mathfrak{c})$-gaps. Kunen [22] also uses this argument in his proof that it is possible for $\aleph_{\omega_1}$ to be the first cardinal for which MA fails. It has also been used to prove other statements are independent of MA+¬CH, see Abraham, Rubin, Shelah [2], Abraham, Todorcevic [3], Devlin [17], and Steprans [35].

Kunen's idea can be summarized as follows. To get the consistency of MA+¬CH+P where P is some statement, do a ccc iteration of of small ccc posets. At each step you either force with a ccc poset Q which preserves the statement P or you force with a ccc poset R which destroys the ccc-ness of

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Q but preserves statement P. In either case you have taken care of Q and preserved statement P.

Section 3 is about Laver forcing $\mathbb{L}$. Conditions in $\mathbb{L}$ are subtrees $p \subseteq \omega^{<\omega}$ with the property that there exists a node $s \in p$ called its root such that:

1. for every $t \in p$ either $t \subseteq s$ or $s \subseteq t$,

2. for every $t \in p$ if $s \subseteq t$, then the set $split(p,t) =^{def} \{n \in \omega : t \hat{\ } n \in p\}$ is infinite.

The order relation is given by subset. This forcing was used by Laver [23] to prove the consistency of the Borel conjecture.

Theorem 3.1. *Assume MA. Then $\mathbb{L}$ does not collapse cardinals.*

Analogously to the s^0 sets we can define the Laver null sets l^0 to be all $X \subseteq \omega^\omega$ such that for every $p \in \mathbb{L}$ there exists $q \leq p$ with $q \in \mathbb{L}$ such that $[q] \cap X = \emptyset$. It is easy to see that Theorem 3.1 also shows that MA implies that $add(l^0) = \mathfrak{c}$.

The corresponding theorem for Mathias forcing ([26]) is easy to prove. The fact that the additivity of the ideal of Ramsey null sets is greater than ω_1 under MA+¬CH plays a key role in Silver's proof [34] that analytic sets are completely Ramsey.

In section 4 we consider a notion of forcing which is half-way in between Sacks forcing $\mathbb{S}$ and Laver forcing $\mathbb{L}$. This forcing is often called superperfect tree forcing or rational perfect set forcing, see Kechris [20], Miller [27], Blass and Shelah [9], Blass [10]

Let $\mathbb{F}$ be superperfect tree forcing which we define as follows. For p a subtree of $\omega^{<\omega}$ define the splitting nodes of p:

$$\text{splitnodes}(p) = \{s \in p : \exists^\infty n \in \omega \ s \hat{\ } n \in p\}.$$

Define $p \in \mathbb{F}$ iff

1. p is a nonempty subtree of $\omega^{<\omega}$,

2. splitnodes(p) is dense in p, i.e., $\forall s \in p \ \exists t \in$ splitnodes(p) with $t \supseteq s$, and

3. if any node in p splits it is a splitting node, i.e., if there exists more than one $n \in \omega$ such that $s \hat{\ } n \in p$, then there are infinitely many $n \in \omega$ such that $s \hat{\ } n \in p$.

Theorem 4.1. *Assume MA. Then $\mathbb{F}$ does not collapse cardinals.*

As is the case with Ramsey null and Laver null sets, MA implies that $add(f^0) = \mathfrak{c}$, where f^0 is the ideal of all subsets $X \subseteq \omega^\omega$ such that for every $p \in \mathbb{F}$ there exists $q \leq p$ with $q \in \mathbb{F}$ such that $[q] \cap X = \emptyset$.

Its seems in general that MA can only handle those forcing which have some kind of infinite splitting going on, e.g. Mathias, Laver, superperfect, whereas for those whose conditions are compact, e.g. Sacks, Silver, it is consistent with MA that they collapse the continuum.

In the appendix we solve a problem posed by M. Foreman.

1 Sacks forcing and Marczewski's ideal

Let $\mathbb{S}$ be Sacks perfect set forcing, $p \in \mathbb{S}$ iff $p \subseteq 2^{<\omega}$ is a nonempty subtree and for every $s \in p$ there exists $t \supseteq s$ such that $t\hat{\ }0 \in p$ and $t\hat{\ }1 \in p$. The ordering is defined by $p \leq q$ iff $p \subseteq q$. Define $[p] = \{x \in 2^\omega : \forall n \ x \upharpoonright n \in p\}$.

The s^0 ideal of subsets of 2^ω is defined by $X \in s^0$ iff for every $p \in \mathbb{S}$ there exists $q \leq p$ with $X \cap [q] = \emptyset$.

$$add(s^0) = min\{|F| : F \subseteq s^0, \bigcup F \notin s^0\}$$

Theorem 1.1 *(MA+¬CH) $add(s^0)$ is the minimum κ such that there exists $p \in \mathbb{S}$ such that $p \Vdash_{\mathbb{S}} cof(\mathfrak{c}) = \kappa$.*

proof:

We will need the following lemma.

Lemma 1.1 *Suppose that $D \subseteq \mathbb{S}$ is open and dense, then there exists $A \subseteq D$ a maximal antichain with the property that for every $p \in \mathbb{S}$ if*
$[p] \subseteq \bigcup_{q \in A}[q]$, *then there exists $A' \in [A]^{<\mathfrak{c}}$ such that $[p] \subseteq \bigcup_{q \in A'}[q]$.*

proof:

Let $\mathbb{S} = \{q_\alpha : \alpha < \mathfrak{c}\}$. Build $A = \{p_\alpha : \alpha < \mathfrak{c}\}$ and $\{x_\alpha : \alpha < \mathfrak{c}\} \subseteq 2^\omega$ by induction on α. At stage α if $[q_\alpha]$ is not covered by $\bigcup_{\beta < \alpha}[p_\beta]$, then choose $x_\alpha \in [q_\alpha] \setminus \bigcup_{\beta < \alpha}[p_\beta]$ otherwise let $x_\alpha = x_0$. If q_α is compatible with some p_β then let $p_\alpha = p_0$. Otherwise since a perfect set can be divided into perfectly

many disjoint perfect sets it is possible to find $p_\alpha \leq q_\alpha$ such that $[p_\alpha]$ is disjoint from $\{x_\beta : \beta \leq \alpha\}$.

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Now suppose $\Vdash cof(\mathfrak{c}) = \kappa$. We show $add(s^0) \leq \kappa$. Let

$$\Vdash_{\mathbb{S}} \text{``}\tau : \kappa \to \mathfrak{c} \text{ is cofinal''}.$$

For each $\alpha < \kappa$ let

$$D_\alpha = \{p \in \mathbb{S} : \exists \beta \; p \Vdash \tau(\alpha) = \beta\}.$$

Let $A_\alpha \subseteq D_\alpha$ be the antichains obtained from the lemma. And let

$$X_\alpha = 2^\omega \setminus \bigcup_{p \in A_\alpha} [p].$$

Since A_α is a maximal antichain it is easy to see that X_α is an s^0 set. Now suppose that $\mathfrak{c}$ is regular and $\kappa < \mathfrak{c}$. We claim that $X = \bigcup_{\alpha < \kappa} X_\alpha$ is not an s^0 set. If it is we must have $[p] \cap X = \emptyset$ for some $p \in \mathbb{S}$. But this implies that $[p] \subseteq \bigcup_{q \in A_\alpha} [q]$ for each α so there exists $A'_\alpha \subseteq A_\alpha$ of cardinality less than $\mathfrak{c}$ such that $[p] \subseteq \bigcup_{q \in A'_\alpha} [q]$. By the definition of D_α this means there exists Y_α of cardinality less than $\mathfrak{c}$ such that

$$p \Vdash \forall \alpha < \kappa \; \tau(\alpha) \in \check{Y}_\alpha.$$

This contradicts the fact that τ is a cofinal map.

For the other direction suppose that $add(s^0) = \kappa$ and suppose for contradiction that $\Vdash_{\mathbb{S}} cof(\mathfrak{c}) > \kappa$. We can assume $\kappa < \mathfrak{c}$. Let X_α be s^0 sets such that

$$X = \bigcup_{\alpha < \kappa} X_\alpha$$

is not s^0. Working below the appropriate p^* (namely, some witness to the fact $X \notin s^0$), we have that every p satisfies $[p] \cap X \neq \emptyset$. Let $D_\alpha = \{p \in \mathbb{S} : [p] \cap X_\alpha = \emptyset\}$. It is easy to see that D_α is open dense. Let $A_\alpha \subseteq D_\alpha$ be any maximal antichain. Let $\mathbb{S} = \{p_\alpha : \alpha < \mathfrak{c}\}$ be listed without repetitions and define an $\mathbb{S}$ name $\tau : \kappa \to \mathfrak{c}$ by $\tau(\alpha)$ is the unique β such that $p_\beta \in G \cap A_\alpha$ where G is the $\mathbb{S}$-generic filter. Since the cofinality of $\mathfrak{c}$ is greater than κ in the extension, we can find $\beta < \mathfrak{c}$ and $p \in \mathbb{S}$ such that

$$p \Vdash \forall \alpha < \kappa \; \tau(\alpha) < \beta.$$

Let $H = \{p_\alpha : \alpha < \beta\}$. Clearly for every $r \leq p$ and $\alpha < \kappa$ there exists $q \in A_\alpha \cap H$ which is compatible with r. Define for $s \in p$, $p_s = \{t \in p : s \subseteq t$ or $t \subseteq s\}$. For a fixed α, there are two possibilities:

1. for every $q \in A_\alpha \cap H$ the set $[q] \cap [p]$ is nowhere dense in $[p]$, or

2. there exists $q \in A_\alpha \cap H$ such that for some $s \in p$ we have $p_s \leq q$.

But (1) is impossible, since by MA we could find a perfect $r \leq p$ such that for every $q \in A_\alpha \cap H$ we have $[r] \cap [q] = \emptyset$. Also (1) is impossible for any p_s in place of p. Hence we can find $E_\alpha \subseteq p$ such that for every $s \in E_\alpha$ there exists $q \in A_\alpha$ such that $p_s \leq q$ and E_α is a dense set of nodes in p, ie. for every $t \in p$ there exists $s \supseteq t$ with $s \in E_\alpha$.

Consider the forcing notion:

$$\mathbb{P} = \{(F, n) : \emptyset \neq F \subseteq p \cap 2^n\}$$

and ordered by $(F, n) \leq (F', n')$ iff $n \geq n'$ and $F' = \{s \restriction n' : s \in F\}$. Since $\mathbb{P}$ is countable, forcing with it is the same as Cohen real forcing. Given G a $\mathbb{P}$-filter let $r = \bigcup\{F : \exists n \ (F, n) \in G\}$. It is easy to write down countably many dense sets which will guarantee that $r \in \mathbb{S}$. Also κ many dense sets which will make it true that for every $\alpha < \kappa$ there exists $n < \omega$ for every $s \in r \cap 2^n$ there exists $q \in A_\alpha$ with $r_s \leq q$. But this implies that for every α

$$[r] \subseteq \bigcup_{q \in A_\alpha} [q]$$

and so $[r] \cap \bigcup_{\alpha < \kappa} X_\alpha = \emptyset$. This contradiction finishes the proof of Theorem 1.1.

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Three other cardinal functions of an ideal I are *non*, *cov*, and *cof*:

$non(I) = min\{|X| : X \notin I, X \subseteq 2^\omega\}$
$cov(I) = min\{|F| : F \subseteq I, \bigcup F = 2^\omega\}$
$cof(I) = min\{|F| : F \subseteq I$ and $\forall A \in I \ \exists B \in F \ A \subseteq B\}$

Theorem 1.2 *It is relatively consistent with ZFC that $c = \omega_2 = cov(s^0)$ and $add(s^0) = \omega_1$.*

proof:

The iterated Sacks forcing model is described in Baumgartner-Laver [8]. It is obtained by starting with a ground model which satisfies CH and then

iterating $\mathbb{S}$ ω_2 times with countable supports. The continuum ends up being ω_2 and no cardinals are collapsed.

The fact that $add(s^0) = \omega_1$ follows from half of the proof of Theorem 1.1 and the theorem (proved in [8]) that Sacks forcing over V_{ω_2} collapses ω_2.

To see that $cov(s^0) = \omega_2$ let V_α for $\alpha \leq \omega_2$ be the iteration sequence. Let $\langle X_\alpha : \alpha < \omega_1 \rangle \in V_{\omega_2}$ be s^0 sets. In V_{ω_2} let $f_\alpha : \mathbb{S} \to \mathbb{S}$ be such that for every $p \in \mathbb{S}$ $f_\alpha(p) \leq p$ and $[f_\alpha(p)] \cap X_\alpha = \emptyset$. Since the iteration has the ω_2 chain condition by a Lowenheim-Skolem argument it is possible to find $\gamma < \omega_2$ so that

$$\langle f_\alpha \upharpoonright \mathbb{S}^{V_\gamma} : \alpha < \omega_1 \rangle \in V_\gamma.$$

But, we claim the γth Sacks real x_γ is not in $\bigcup_{\alpha < \omega_1} X_\alpha$. If it was, then for some condition $p \in \mathbb{P}_{[\gamma, \omega_2)}$ and some $\alpha < \omega_1$ we would have:

$$p \Vdash x_\gamma \in X_\alpha.$$

But letting $q = p(\gamma) \in \mathbb{S}$ and letting $r(\gamma) = f_\alpha(q)$ and $r(\beta) = p(\beta)$ for $\beta > \gamma$ we see that

$$r \Vdash x_\gamma \notin X_\alpha.$$

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Theorem 1.3 $cof(cof(s^0)) > \mathfrak{c}$

proof:

Let $\kappa = cof(s^0)$ and $\{Y_\alpha : \alpha < \kappa\} \subseteq s^0$ be a cofinal family. Let

$$\{p_\alpha : \alpha < \mathfrak{c}\}$$

be all perfect subsets of $2^\omega \times 2^\omega$,

$$\{x_\alpha : \alpha < \mathfrak{c}\} = 2^\omega, \text{ and } L_\alpha = \{(x_\alpha, y) : y \in 2^\omega\}.$$

Build $q_\alpha \subseteq L_\alpha$ perfect such that for every $\beta < \alpha$ if $p_\beta \cap L_\alpha$ is countable, then q_α is disjoint from p_β. This is easily done since any perfect set splits into $\mathfrak{c}$ many disjoint perfect sets. Now we assume for contradiction that $cof(\kappa) \leq \mathfrak{c}$. Let $F : \mathfrak{c} \to \kappa$ have unbounded range. Since $\kappa = cof(s^0)$ for each $\beta < \mathfrak{c}$ the set

$$\{Y_\alpha \cap q_\beta : \alpha < F(\beta)\}$$

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is not cofinal in the s^0 subsets of q_β. So there exists an s^0 set $Z_\beta \subseteq q_\beta$ such that Z_β is not covered by any Y_α with $\alpha < F(\beta)$. Then $Z = \bigcup_{\beta < \mathfrak{c}} Z_\beta$ is not covered by any Y_α, so it suffices to see that Z is an s^0 set. Let $p \subseteq 2^\omega \times 2^\omega$ be any perfect set. If $p \cap L_\beta$ is uncountable for any β, then let $q \subseteq p \cap L_\beta$ be perfect such that $q \cap Z_\beta = \emptyset$, then $q \cap Z = \emptyset$. Otherwise $p \cap L_\beta$ is countable for every β. If $p = p_\alpha$, then by construction $q_\beta \cap p_\alpha = \emptyset$ for every $\beta \geq \alpha$. It follows that

$$(p \cap Z) \subseteq (p \cap \bigcup_{\beta < \alpha} L_\beta).$$

But $p \cap (\bigcup_{\beta < \alpha} L_\beta)$ is a set of cardinality less than $\mathfrak{c}$ and so there exists $q \subseteq p$ perfect such that $q \cap (\bigcup_{\beta < \alpha} L_\beta) = \emptyset$, and hence $q \cap Z = \emptyset$.
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Theorem 1.4 *Let $V \models GCH$, κ any cardinal of cofinality greater than ω_1, and let $\mathbb{P}$ be the partial order of countable functions from κ to ω_1. If G is $\mathbb{P}$-generic over V, then $V[G] \models$ "$\omega_1 = \mathfrak{c}$ and $cof(s^0) = 2^{\omega_1} = \kappa$". So, for example, it is consistent that $\omega_2 = cof(cof(s^0)) < cof(s^0) = \aleph_{\omega_2}$.*

proof:
Countably closed forcing does not add any reals. By the usual chain condition argument and decomposition as a product forcing it is enough to see:

If $G : \omega_1 \rightarrow \omega_1$ is $\omega_1^{<\omega_1}$-generic over V then in $V[G]$ there exists an s^0 set X which is not covered by any s^0 set in V.

By a similar construction as in the proof of Theorem 1.3 we can find in V disjoint perfect sets q_α for $\alpha < \omega_1 = \mathfrak{c}$ such that any $X \subseteq 2^\omega$ which meets each q_α in a singleton is an s^0-set. Now just use the generic function $G : \omega_1 \rightarrow \omega_1$ to pick out a single element of each q_α.
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Theorem 1.5 *Let V be a model of CH and κ be any regular cardinal with $\omega_1 < \kappa \leq 2^{\omega_1}$, then there exists a generic extension $V[G]$ (with same cardinals and function 2^λ) in which $cof(s^0) = \kappa$.*

proof:
To bring down $cof(s^0)$ but leave $2^{\mathfrak{c}}$ large, force with the following partial order $\mathbb{P}$:
$(X, f) \in \mathbb{P}$ iff

1. $X \in s^0$,

2. $f : \mathbb{S} \to \mathbb{S}$ is a countable partial function, and

3. $\forall p \in domain(f) \ \ f(p) \leq p$ and $[f(p)] \cap X = \emptyset$.

The ordering on $\mathbb{P}$ is defined by $(X', f') \leq (X, f)$ iff $X' \supseteq X$ and $f' \supseteq f$. Clearly $\mathbb{P}$ is countably closed and two elements of $\mathbb{P}$ with the same f are compatible, so it is ω_1-centered. It is also true that it is well met, ie. infimums exist.

Now define $\mathbb{P}^{\omega_1}_{\omega}$ to be those elements of $p \in \mathbb{P}^{\omega_1}$ with countable support, ie. there are at most countably many $\alpha < \omega_1$ such that $p(\alpha) \neq (\emptyset, \emptyset)$. It is easy to see that forcing with $\mathbb{P}^{\omega_1}_{\omega}$ adds an $\langle f_\alpha : \alpha < \omega_1 \rangle$ such that if

$$X_\alpha = 2^\omega \setminus \bigcup_{p \in \mathbb{S}} [f_\alpha(p)]$$

then each $X_\alpha \in s^0$ and for every $Y \in s^0 \cap V$ there exists $\alpha < \omega_1$ such that $Y \subseteq X_\alpha$. Now just like in the usual proofs of Generalized Martin's Axiom, see [33], [37], or [7], we iterate forcing with $\mathbb{P}^{\omega_1}_{\omega}$ κ many times with countable supports. Since cofinally often we add a function $f : \omega_1 \to \omega_1$ which is $\omega_1^{<\omega_1}$ generic, by the argument of Theorem 1.4 we see that $cof(s^0) \geq \kappa$. Since the iteration satisfies the ω_2 chain condition every subset of 2^ω appears at some intermediate stage, and is covered by one of the generic s^0 sets. Hence $cof(s^0) \leq \kappa$.

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2   Martin's Axiom and Marczewski's ideal

In this section we start with a model satisfying the continuum hypothesis and by an inductive construction, we will get a model for Martin's Axiom where the Marczewski's ideal, s^0, is not ω_1 additive.

Theorem 2.1 *It is consistent with $MA + \neg CH$ that the ideal of s^0 sets is not ω_1-additive.*

proof:

Recall that $[T] = \{x \in 2^\omega : \forall n < \omega \ x \restriction n \in T\}$.

Define $(*)\langle T_j : j < \zeta \rangle$ where $\zeta \leq \omega_2$ and each $T_j \in \mathbb{S}$ as follows:

$(*)\langle T_j : j < \zeta \rangle$: Given $\langle F_\alpha : \alpha < \omega_1 \rangle$ pairwise disjoint finite subsets of ζ there exists $\alpha \neq \beta$ such that $\forall i \in F_\alpha \forall j \in F_\beta\ [T_i] \cap [T_j] = \emptyset$.

The Construction: By induction on $\zeta \leq \omega_2$ we shall

1. define $\langle P_i, Q_j : j < \zeta, i \leq \zeta \rangle$ a finite support iteration of ccc forcing notions with as usual $P_{\zeta+1} = P_\zeta * Q_\zeta$,

2. define $\langle T_j : j < \zeta \rangle \in V^{P_\zeta}$ such that $V^{P_\zeta} \models (*)\langle T_j : j < \zeta \rangle$,

3. make sure that $\{T_j : j < \omega_2\}$ is dense in $\mathbb{S} \cap V^{P_{\omega_2}}$, and

4. make sure that MA holds in $V^{P_{\omega_2}}$.

In order to make MA true we list all possible P_{ω_2} names for posets of cardinality ω_1 say R_ζ for $\zeta < \omega_2$. We then apply Lemma 2.2 to the ground model V^{P_ζ} to get either that forcing with $\mathbb{P}_\zeta * R_\zeta$ satisfies $(*)\langle T_j : j < \zeta \rangle$ or there exists Q such that forcing with $P_\zeta * Q$ satisfies $(*)\langle T_j : j < \zeta \rangle$ and R_ζ does not have ccc. Let T'_ζ be any element of $\mathbb{S} \cap V^{P_\zeta}$. However make sure that at the end we have $\{T'_\zeta : \zeta < \omega_2\}$ is dense in $\mathbb{S} \cap V^{P_{\omega_2}}$; because $T_\zeta \leq T'_\zeta$ (see $(**)$ below) this ensures that $\{T_\zeta : \zeta < \omega_2\}$ is dense. Note that since $Q_{T'_\zeta}$ (defined below) is a countable poset it cannot destroy $(*)$ nor can adding one tree destroy $(*)$.

We then let $Q_\zeta = R * Q_{T'_\zeta}$ or $Q_\zeta = Q * Q_{T'_\zeta}$ which ever preserves $(*)$. This concludes the proof of Theorem 2.1.
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Lemma 2.1 *In $V^{P_{\omega_2}}$ the additivity of s^0 ideal is ω_1, in fact $cov(s^0) = \omega_1$.*

proof:
Choose $\langle T_{i,j} \in \mathbb{S} : i < \omega_1, j < \omega_2 \rangle$ such that for each i, j there is a unique $\zeta(i,j)$ such that $T_{i,j} \subseteq T_{\zeta(i,j)}$ and also for each $i < \omega_1$

$$A_i = \{T_{i,j} : j < \omega_2\}$$

is a maximal antichain in $\mathbb{S}$. This is easy to do since the family $\{T_\zeta : \zeta < \omega_2\}$ is dense in $\mathbb{S}$ and every condition in $\mathbb{S}$ has a perfect set of incompatible extensions. If

$$X_i = 2^\omega \setminus \bigcup_{T \in A_i} [T]$$

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then each X_i is an s^0-set since A_i is a maximal antichain. But $2^\omega = \bigcup_{i < \omega_1} X_i$, since otherwise if $x \in 2^\omega \setminus \bigcup_{i < \omega_1} X_i$, then choosing $F_\alpha = \{\zeta(\alpha, \beta)\}$ where $x \in [T_{\alpha,\beta}] \subseteq [T_{\zeta(\alpha,\beta)}]$ witnesses the failure of $(*)$.
∎

The following lemma is the key to preserving $(*)$ while at the same time making MA true.

Lemma 2.2 *Suppose* $V \models "(*)\langle T_j : j < \zeta \rangle$, *R is ccc", and* $V^R \models "\neg(*)\langle T_j : j < \zeta \rangle"$. *Then there exists a ccc partial order* Q *such that* $V^Q \models "(*)\langle T_j : j < \zeta \rangle$ *and* R *is not ccc".*

proof:
Let $\langle F'_\alpha : \alpha < \omega_1 \rangle$ be R-names and $r \in R$ such that

$$r \Vdash_R "\langle F'_\alpha : \alpha < \omega_1 \rangle \text{ is a counterexample to } (*)".$$

Let $\langle r_\alpha, F_\alpha : \alpha < \omega_1 \rangle \in V$ be such that each $r_\alpha \leq r$ and

$$r_\alpha \Vdash F'_\alpha = \check{F}_\alpha.$$

For some $A \in [\omega_1]^{\omega_1}$ we have that $\langle F_\alpha : \alpha \in A \rangle$ is a Δ-system. Note that root of this Δ-system must be empty because if for some $\alpha \neq \beta$ $F_\alpha \cap F_\beta \neq \emptyset$, then r_α and r_β are incompatible. But R has the ccc.
Define Q to be the set of all $q \in [A]^{<\omega_0}$ such that if $\alpha \neq \beta \in q$, then

$$\forall i \in F_\alpha \forall j \in F_\beta \ [T_i] \cap [T_j] = \emptyset.$$

Order Q by inclusion. Note that if $q \in Q$ and $\alpha, \beta \in q$ we have r_α and r_β are incompatible in R. Hence forcing with Q adds an uncountable antichain to R. So it is enough to prove the following two claims.

Claim 1. Q satisfies ccc.
proof:
If not let $\langle q_\alpha : \alpha < \omega_1 \rangle$ be pairwise incompatible. For some $B \in [\omega_1]^{\omega_1}$ we have that $\langle q_\alpha : \alpha \in B \rangle$ forms a Δ-system with root q^*. Now $q_\alpha \cup q_\beta \in Q$ iff $(q_\alpha \setminus q^*) \cup (q_\beta \setminus q^*) \in Q$. Therefore without loss of generality we may assume $\langle q_\alpha : \alpha \in B \rangle$ are pairwise disjoint. If we let $K_\alpha = \bigcup_{\delta \in q_\alpha} F_\delta$, then $\langle K_\alpha : \alpha \in B \rangle$ are pairwise disjoint. Therefore by applying $(*)$ in V, there are $\alpha \neq \beta \in B$ satisfying

$$\forall i \in K_\alpha \forall j \in K_\beta \ [T_i] \cap [T_j] = \emptyset$$

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and so $q_\alpha \cup q_\beta \in Q$.
∎

Claim 2. Q preserves $(*)$.
proof:

Let $q \in Q$ and $\langle H'_\alpha : \alpha < \omega_1 \rangle$ be a Q-name such that such that

$$q \Vdash_Q \text{ “}\langle H'_\alpha : \alpha < \omega_1 \rangle \text{ are pairwise disjoint finite subsets of } \zeta\text{”}.$$

Let $\langle q_\alpha, H_\alpha : \alpha < \omega_1 \rangle \in V$ be such that $q_\alpha \leq q$ and

$$q_\alpha \Vdash \text{ “}H'_\alpha = \check{H}_\alpha\text{”}.$$

Then for some $B \in [\omega_1]^{\omega_1}$ the following form Δ-systems: $\langle q_\alpha : \alpha \in B \rangle$ and $\langle H_\alpha : \alpha \in B \rangle$. By ccc of Q we have that the root of $\langle H_\alpha : \alpha \in B \rangle$ must be empty. Let q^* be the root of the q's so $q_\alpha \cap q_\beta = q^*$ for $\alpha \neq \beta \in B$. Now we define

$$K_\alpha = \bigcup_{\delta \in q_\alpha \setminus q^*} F_\delta \cup H_\alpha.$$

Since the H's and F's are pairwise disjoint families it is easy to find $C \in [B]^{\omega_1}$ such that K_α for $\alpha \in C$ are pairwise disjoint.

Hence by $(*)$ in V there exists $\alpha \neq \beta \in C$ such that

$$\forall i \in K_\alpha \forall j \in K_\beta \, [T_i] \cap [T_j] = \emptyset.$$

But this means that $q_\alpha \cup q_\beta \in Q$ and

$$q_\alpha \cup q_\beta \Vdash_Q \text{ “}\forall i \in H'_\alpha \forall j \in H'_\beta \, [T_i] \cap [T_j] = \emptyset\text{”}.$$

Since we started with an arbitrary condition and name we have $(*)$ holds in V^Q. Hence Claim 3 is proven and this finishes the proof of Lemma 2.2.
∎

Next we show that property $(*)$ is preserved at limit stages. Note that $(*)$ is trivially preserved at stage ω_2. The preservation of $(*)$ at stages of cofinality ω_1 is more delicate and requires that we specify the details of exactly how we pick the trees T_i.

For $T' \in \mathbb{S}$ let $Q_{T'} = \{(F, n) : \emptyset \neq F \subseteq T' \cap 2^n\}$

Order $Q_{T'}$ by $(F, n) \leq (G, m)$ iff $n \geq m$ and $G = \{s \restriction m : s \in F\}$. Forcing with $Q_{T'}$ naturally determines a perfect subtree T of T' as follows: if G is $Q_{T'}$ generic, then let

$$T = \{s : \exists (F, n) \in G \, s \in F\}.$$

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In our construction T_j is obtained by forcing with $Q_{T'}$ for some T'. Recall that $T_s = \{t \in T : s \subseteq t \text{ or } t \subseteq s\}$. For T which is $Q_{T'}$-generic it is easy to see that

1. if $T_0 \in \mathbb{S}$ is in the ground model, then for some $n < \omega$ for every $s \in 2^n \cap T_0 \cap T$ $T_s \subseteq T_0$, and

2. for any $p \in Q_{T'}$ there exists $n < \omega, t_0, t_1, q_0, q_1 \leq p$ such that t_0 and t_1 are disjoint, $q_0 \Vdash T \cap 2^n = t_0$, and $q_1 \Vdash T \cap 2^n = t_1$.

Our construction will satisfy the following:

(∗∗) For every $\alpha < \omega_2$ T_α is $Q_{T'_\alpha}$-generic over V^{P_α} for some $T'_\alpha \in \mathbb{S} \cap V^{P_\alpha}$.

Lemma 2.3 *Assume (∗∗) and suppose $F = \{\alpha_l : l < m\} \subseteq \zeta$ is finite and enumerated in increasing order, and $p \in P_\zeta$, then there exists $q \leq p$, $n < \omega$, and $\langle Q_l \subseteq 2^n : l < m \rangle$ such that for each $l < m$*

$$q \Vdash \text{“} T_{\alpha_l} \cap 2^n = \check{Q}_l \text{”}$$

and for any $k < l < m$ if $s \in Q_l \cap Q_k$ then

$$q \Vdash \text{“} T_{\alpha_l s} \subseteq T_{\alpha_k} \text{”}.$$

proof:
    Left to the reader.
∎

Lemma 2.4 *Assume (∗∗) and suppose $\gamma \leq \min(H)$ where $H \subseteq \zeta$ is finite and $p \in P_\zeta$, then there exists $q_0, q_1 \leq p$ with $q_0 \restriction \gamma = q_1 \restriction \gamma$ and disjoint $t_0, t_1 \subseteq 2^n$ for some $n < \omega$ such that for each $\alpha \in H$ and $i = 0, 1$*

$$q_i \Vdash \text{“} T_\alpha \cap 2^n \subseteq \check{t}_i \text{”}.$$

proof:
It is easy to show:
    Suppose P_1, P_2 are arbitrary posets and $T'_i \in \mathbb{S} \cap V^{P_i}$. Given $p_1 \in P_1 * Q_{T'_1}$ and $p_2 \in P_2 * Q_{T'_2}$ there exists $n < \omega, q_1 \leq p_1, q_2 \leq p_2, t_0 \subseteq 2^n, t_1 \subseteq 2^n$ such that $t_0 \cap t_1 = \emptyset$ and for $i = 0, 1$ $q_i \Vdash T_i \cap 2^n \subseteq \check{t}_i$.
    The lemma follows by iteratively applying this statement to all pairs in H with V^{P_γ} as the ground model.
∎

Lemma 2.5 *Assume* $(**)$ *and suppose that* ζ *is a limit ordinal and for all* $\alpha < \zeta$

$$V^{P_\alpha} \models (*) \langle T_j : j < \alpha \rangle,$$

then $V^{P_\zeta} \models (*) \langle T_j : j < \zeta \rangle.$

proof:
  Let

$$p \Vdash \text{``} \langle F'_\alpha : \alpha < \omega_1 \rangle \text{ are pairwise disjoint finite subsets of } \zeta\text{''}.$$

We must show there exists $q \leq p$ and $\alpha \neq \beta$ such that

$$q \Vdash \forall i \in F'_\alpha \forall j \in F'_\beta \ [T_i] \cap [T_j] = \emptyset.$$

Let $\langle p_\alpha, F_\alpha : \alpha < \omega_1 \rangle \in V$ be such that $p_\alpha \leq p$ and

$$p_\alpha \Vdash F'_\alpha = \check{F}_\alpha.$$

For some $A \in [\omega_1]^{\omega_1}$ $\{F_\alpha : \alpha \in A\}$ forms a Δ-system. The root of this Δ-system must be empty, since P_ζ satisfies ccc.

Case 1. Cofinality of ζ is ω.
  For some $B \in [A]^{\omega_1}$, and $\gamma < \zeta$ we have that $F_\alpha \subseteq \gamma$ and $p_\alpha \in P_\gamma$ for every $\alpha \in B$. Let G be a P_γ-filter such that $C = \{\alpha \in B : p_\alpha \in G\}$ is uncountable. (Note that such a G must exist, else there would exist a maximal antichain Q such that for every $q \in Q$ there exists $\alpha < \omega_1$ $q \Vdash C \subseteq \alpha$. Since Q would be countable this would imply that A is countable.) Then by applying $(*)$ in $V[G]$ there exists $\alpha \neq \beta \in C$ such that $\forall i \in F_\alpha \forall j \in F_\beta \ [T_i] \cap [T_j] = \emptyset$. Then $q \in G$ with $q \leq p_\alpha$ and $q \leq p_\beta$ is as required.

Case 2. Cofinality of ζ is ω_1.
  Apply Lemma 2.3. By cutting down to uncountable subset of A we can assume that $|F_\beta| = m$ and $\langle Q_l : l < m \rangle$ are the same for each $\beta \in A$.
  By passing to an uncountable subset of A we can assume that there exists $\gamma < \zeta$ such that $F_\alpha = G_\alpha \cup H_\alpha$ where for each $\alpha \in A$ $G_\alpha \subseteq \gamma$ and $\langle min(H_\alpha) : \alpha \in A \rangle$ is unbounded in ζ, and if $\alpha, \beta \in A$ and $\alpha < \beta$ then $max(H_\alpha) < min(H_\beta)$. Apply Lemma 2.4 to each H_α obtaining q^i_α for $i = 0, 1$ with

$$q_\alpha =^{def} q^0_\alpha \restriction min(H_\alpha) = q^1_\alpha \restriction min(H_\alpha)$$

and disjoint t_0^α, t_1^α. Again by passing to an uncountable subset we may assume t_0, t_1 are the same for each $\alpha \in A$. By cutting down A and increasing γ we may suppose that $\{\text{support}(q_\alpha^0) \cup \text{support}(q_\alpha^1) : \alpha \in A\}$ is a Δ-system whose root is a subset of γ. By the same argument as was used in Case 1, we can find distinct $\alpha, \beta \in A$ such that q_α and q_β are compatible and

$$\forall i \in G_\alpha \forall j \in G_\beta \ [T_i] \cap [T_j] = \emptyset.$$

Claim 1. q_α^0 and q_β^1 are compatible.
proof:

By definition $q_\delta = q_\delta^i \upharpoonright min(H_\delta)$. Also q_α and q_β are compatible elements of P_γ. Since the supports form a Δ-system with root contained in γ they are compatible.
∎

Claim 2. $q_\alpha^0 \cup q_\beta^1 \Vdash \forall i \in F_\alpha \forall j \in F_\beta \ [T_i] \cap [T_j] = \emptyset$.
proof:

Case a. $i \in G_\alpha$ and $j \in G_\beta$.
    This is true by the way we picked α and β.

Case b. $i \in H_\alpha$ and $j \in H_\beta$.
    $q_\alpha^0 \Vdash T_i \cap 2^n \subset t_0$ and $q_\alpha^1 \Vdash T_j \cap 2^n \subset t_1$, but $t_0 \cap t_1 = \emptyset$.

Case c. $\alpha_l = i \in H_\alpha$ and $\beta_k = j \in G_\beta$.
    Let $F_\alpha = \{\alpha_l : l < m\}$ and $F_\beta = \{\beta_l : l < m\}$, and so $k < l$. Note that

$$q_\alpha \Vdash T_{\alpha_l} \cap 2^n = Q_l$$

and

$$q_\beta \Vdash T_{\alpha_k} \cap 2^n = Q_k.$$

If $s \in Q_l \cap Q_k$, then

$$q_\alpha \Vdash T_{\alpha_l s} \subseteq T_{\alpha_k}.$$

But by Case (a) we know

$$q_\alpha \cup q_\beta \Vdash [T_{\alpha_k}] \cap [T_{\beta_k}] = \emptyset.$$

Consequently $q_\alpha \cup q_\beta \Vdash [T_{\alpha_l}] \cap [T_{\beta_k}] = \emptyset$.

Case d. $\alpha_l = i \in G_\alpha$ and $\beta_k = j \in H_\beta$.
    Same as Case (c).
∎

3 Laver tree forcing

Let $\mathbb{L}$ be Laver tree forcing, that is conditions are subtrees $p \subseteq \omega^{<\omega}$ with the property that there exists a node $s \in p$ called its root such that:

1. for every $t \in p$ either $t \subseteq s$ or $s \subseteq t$,

2. for every $t \in p$ if $s \subseteq t$, then the set $split(p, t) =^{def} \{n \in \omega : t\widehat{\ }n \in p\}$ is infinite.

Theorem 3.1 *Assume MA. Then $\mathbb{L}$ does not collapse cardinals.*

Given $A = \langle A_s \in [\omega]^\omega : s \in \omega^{<\omega} \rangle$ and $s \in \omega^{<\omega}$ define $p_s(A) = p \in \mathbb{L}$ to be the unique Laver tree such that the root of p is s and for every $t \supseteq s$ with $t \in p$ we have that $split(p, t) = A_t$.

Lemma 3.1 *Suppose $\Vdash \tau \in V$ and $B = \langle B_s : s \in \omega^{<\omega} \rangle$ where each $B_s \in [\omega]^\omega$. Then there exists a countable X and $A = \langle A_s \in [B_s]^\omega : s \in \omega^{<\omega} \rangle$ such that for every $s \in \omega^{<\omega}$*
$$p_s(A) \Vdash \tau \in X.$$

proof:

Laver proved that for any $p \in \mathbb{L}$ there exists $q \leq p$ with the same root and X countable such that
$$q \Vdash \tau \in X.$$

Build $p_n \in \mathbb{L}$ as follows. At stage n, let s be the nth element of $\omega^{<\omega}$. If $s \in p_m$ for some $m < n$, then do nothing. Otherwise, take $p_n \leq p_s(B)$ and X_n countable such that s is the root of p_n and $p_n \Vdash \tau \in X_n$. For every $t \in p_n$ such that $s \subseteq t$, let $A_t = split(p_n, t)$. Finally let $X = \bigcup_{n<\omega} X_n$.
∎

The next lemma proves the theorem.

Lemma 3.2 *Suppose MA, $\kappa < \mathfrak{c}$, and $p \Vdash \tau : \kappa \to V$, then there exist $q \leq p$ and $\langle X_\alpha : \alpha < \kappa \rangle$ such that for every α X_α is countable and $q \Vdash \tau(\alpha) \in \check{X}_\alpha$.*

proof:

To simplify notation let $p = \omega^{<\omega}$.

Let $Q = \{\langle A_s : s \in \omega^{<\omega} \rangle : A_s \in [\omega]^\omega\}$ and for $A, B \in Q$ define $A \subseteq^* B$ iff for all $s \in \omega^{<\omega}$ $A_s \setminus B_s$ is finite.

Build $A_\alpha \in Q$ for $\alpha < \kappa$ such that

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1. $\alpha < \beta$ implies $A_\beta \subseteq^* A_\alpha$.

2. there exists a countable set X_α such that for every $s \in \omega^{<\omega}$

$$p_s(A_\alpha) \Vdash \tau(\alpha) \in X_\alpha.$$

At stage α use the MA to get $A \in Q$ such that for all $\beta < \alpha$ $A \subseteq^* A_\beta$. (This is a well known consequence of Martin's Axiom, apply Solovay's Lemma, Kunen [21] p.57, to each of the families $\{A_{\beta s} : \beta < \alpha\}$ for $s \in \omega^{<\omega}$.) Then use Lemma 3.1 to get $A_\alpha \subseteq^* A$ as desired.

Now consider the following poset:

$$\mathbb{P} = \{(T, W) : T \text{ is a finite subtree of } \omega^{<\omega}, W \in [\kappa]^{<\omega}\}$$

Order $\mathbb{P}$ by $(T, W) \leq (T', W')$ iff

1. $W \supseteq W'$, $T \supseteq T'$, and

2. $\forall n < \omega \ \forall s \in (T \setminus T') \cap \omega^{n+1} \ \forall \alpha \in W' \ s(n) \in A_{\alpha,s}$.

Since any two conditions with the same T part are compatible, $\mathbb{P}$ is σ-centered, so we can apply MA to it. For G a $\mathbb{P}$-filter let

$$q = \cup\{T : \exists W \ (T, W) \in G\}.$$

If G meets the dense subsets of $\mathbb{P}$ of the form

$$D_{s,n} = \{p \in \mathbb{P} : p \Vdash s \notin q \text{ or } \exists m > n \ s\char94 m \in T_p\}$$

then we will have that $q \in \mathbb{L}$ with the empty sequence as its root. For any $\alpha < \kappa$ let

$$D_\alpha = \{p \in \mathbb{P} : \alpha \in W_p\}.$$

Each D_α is dense.

Hence by MA we can get $q \in \mathbb{L}$ with the empty node as root such that for every $\alpha < \kappa$ there is finite subtree $T \subseteq q$ such that for every $t \in q \setminus T$ we have $t \in p_{t\restriction(|t|-1)}(A_\alpha)$. This implies that for every $r \leq q$ there exists s such that $p_s(A_\alpha)$ is compatible with r. It follows that

$$q \Vdash \forall \alpha < \kappa \ \tau(\alpha) \in X_\alpha.$$

∎

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4 Superperfect trees

Superperfect tree forcing $\mathbb{F}$ is defined as follows. For p a subtree of $\omega^{<\omega}$ define the splitting nodes of p:

$$\mathrm{splitnodes}(p) = \{s \in p : \exists^\infty n \in \omega \; s\hat{\ }n \in p\}.$$

Define $p \in \mathbb{F}$ iff

1. p is a nonempty subtree of $\omega^{<\omega}$,

2. $\mathrm{splitnodes}(p)$ is dense in p, ie $\forall s \in p \; \exists t \in \mathrm{splitnodes}(p)$ with $t \supseteq s$, and

3. if any node in p splits it is a splitting node, ie if there exists more than one $n \in \omega$ such that $s\hat{\ }n \in p$, then there are infinitely many $n \in \omega$ such that $s\hat{\ }n \in p$.

Theorem 4.1 *Assume MA. Then $\mathbb{F}$ does not collapse cardinals.*[3]

proof:
Call a sequence $\langle P_s : s \in \omega^{<\omega} \rangle$ good iff

1. each $P_s \subseteq \omega^{<\omega}$ is infinite,

2. $t \in P_s$ implies $s \subsetneq t$, and

3. for $s \in \omega^n$ if $t, t' \in P_s$ and $t \neq t'$, then $t(n) \neq t'(n)$.

Given any good sequence $\langle P_s : s \in \omega^{<\omega} \rangle$ we determine $\langle p_s \in \mathbb{F} : s \in \omega^{<\omega} \rangle$ as follows. For each s let S be is smallest subset of $\omega^{<\omega}$ such that $P_s \subseteq S$ and if $t \in S$ then $P_t \subseteq S$. Then p_s is the unique condition in $\mathbb{F}$ such that $S = \mathrm{splitnodes}(p_s)$. In other words, P_s says that s is a splitting node and the splitting nodes immediately below s are P_s. Define $\langle P_s : s \in \omega^{<\omega} \rangle \leq \langle Q_s : s \in \omega^{<\omega} \rangle$ iff $p_s \subseteq q_s$ for each $s \in \omega^{<\omega}$. An equivalent definition would be for each $s \in \omega^{<\omega}$ and $t \in P_s$ there exists k and $s_0, s_1, \ldots, s_k$ where $s = s_0 \subseteq s_1 \subseteq \ldots \subseteq s_k = t \; s_{i+1} \in Q_{s_i}$ for $i = 0, 1, \ldots, k-1$.

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[3]Added post publication: Goldstern, Roslanowski, and Spinas found a mistake in this proof. The relation $\leq^*$ is not transitive. A correct proof is given in the paper: Goldstern, Martin; Johnson, Mark J.; Spinas, Otmar Towers on trees. Proc. Amer. Math. Soc. 122 (1994), no. 2, 557–564.

Lemma 4.1 *Given τ such that $\Vdash_{\mathbb{F}}$ "$\tau \in V$", and good $\langle P_s : s \in \omega^{<\omega} \rangle$ there exists a good $\langle Q_s : s \in \omega^{<\omega} \rangle \leq \langle P_s : s \in \omega^{<\omega} \rangle$ and a countable set Σ such that for every $s \in \omega^{<\omega}$ $q_s \Vdash$ "$\tau \in \Sigma$".*

proof:

For any $p \in \mathbb{F}$ with smallest splitting node s (ie root) there exists $q \subseteq p$ such that $s \in \text{splitnodes}(q)$ and a countable Σ such that $q \Vdash$ "$\tau \in \Sigma$". Now just apply this fact repeatedly down the $s \in \omega^{<\omega}$.
∎

Define $\langle P_s : s \in \omega^{<\omega} \rangle \leq^* \langle Q_s : s \in \omega^{<\omega} \rangle$ iff there exists $\langle P'_s : s \in \omega^{<\omega} \rangle \leq \langle Q_s : s \in \omega^{<\omega} \rangle$ such that for every $s \in \omega^{<\omega}$ $P_s =^* P'_s$ (equal mod finite). Similarly for $p, q \in \mathbb{F}$ define $p \leq^* q$ iff there exists $f : \text{splitnodes}(p) \to \omega$ such that $p_f \subseteq q$, where

$$p_f = p \setminus \{r \in \omega^{<\omega} : \exists s \in \text{splitnodes}(p) \exists n < f(s)\ s\,\hat{}\,n \subseteq r\}.$$

We think of p_f as being obtained from p by pruning finitely many nodes from beneath each splitting node of p. Note that $p_f \in \mathbb{F}$.

Lemma 4.2 *The following are equivalent:*

1. $\langle P_s : s \in \omega^{<\omega} \rangle \leq^* \langle Q_s : s \in \omega^{<\omega} \rangle$

2. *for every $s \in \omega^{<\omega}$, $p_s \leq^* q_s$.*

proof:

Left to reader.
∎

Lemma 4.3 *(MA) Suppose $\gamma < \mathfrak{c}$ and $\langle P_s^\alpha : s \in \omega^{<\omega} \rangle$ for $\alpha < \gamma$ are good and have the property that $\alpha > \beta$ implies $\langle P_s^\alpha : s \in \omega^{<\omega} \rangle \leq^* \langle P_s^\beta : s \in \omega^{<\omega} \rangle$. Then there exists a good $\langle P_s : s \in \omega^{<\omega} \rangle$ such that for every $\alpha < \gamma$ $\langle P_s : s \in \omega^{<\omega} \rangle \leq^* \langle P_s^\alpha : s \in \omega^{<\omega} \rangle$.*

proof:

Let Q be the following poset, $(A_s : s \in F, H) \in Q$ iff

1. $H \subseteq \gamma$, $F \subseteq \omega^{<\omega}$, $A_s \subseteq \omega^{<\omega}$ for $s \in F$ are all finite,

2. $t \in A_s$ implies $s \subsetneq t$, and

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3. $t, t' \in A_s$ and $t \neq t'$ implies $t(n) \neq t'(n)$, where $s \in \omega^n$.

We define $(\hat{A}_s : s \in \hat{F}, \hat{H}) \leq (A_s : s \in F, H)$ iff

1. $\hat{F} \supseteq F$, $\hat{H} \supseteq H$, $\hat{A}_s \supseteq A_s$ for $s \in F$, and

2. for each $s \in F$ if $t \in \hat{A}_s \setminus A_s$ and $\alpha \in H$, then $t \in \mathrm{splitnodes}(p_s^\alpha)$.

Note that Q is ccc, in fact σ-centered, since $(A_s : s \in F, H \cup \hat{H})$ extends both $(A_s : s \in F, H)$ and $(A_s : s \in F, \hat{H})$. For any $\alpha < \gamma$

$$\{(A_s : s \in F, H) \in Q : \alpha \in H\}$$

is dense in Q, since $(A_s : s \in F, H \cup \{\alpha\}) \leq (A_s : s \in F, H)$. In order to quarantee that A_t grows up into an infinite set, we need only check that the following sets are dense. Fix $t \in \omega^n$ and $m < \omega$ and define

$$E_{t,m} = \{(A_s : s \in F, H) \in Q : t \in F, \exists r \in A_t \ r(n) > m\}$$

To check this let $(A_s : s \in F, H) \in Q$ and put t into F by letting $A_t = \emptyset$ if neccessary. Let $\alpha = max\{H\}$ and let $p = p_s^\alpha$. Then there exists $f : \mathrm{splitnodes}(p) \to \omega$ such that for every $\beta \in H$ we have $p_f \subseteq p_s^\beta$. Consequently any $r \in \mathrm{splitnodes}(p_f)$ with $r \supset s$ and $r(n) > m$ can be added to A_t. Finally if G is sufficiently Q-generic, then $\langle P_s : s \in \omega^{<\omega} \rangle$ defined by

$$P_t = \bigcup \{A_t : \exists (A_s : s \in F, H) \in G, t \in F\}$$

is as required.
∎

Lemma 4.4 *(MA) Suppose $\kappa < \mathfrak{c}$ is an uncountable regular cardinal and $p_\alpha \in \mathbb{F}$ for $\alpha < \kappa$ and $r \in \mathbb{F}$ have the property that for every $\alpha < \kappa$ $r \leq^* p_\alpha$. Then there exists $q \subseteq r$ and $\Gamma \in [\kappa]^\kappa$ such that $q \subseteq p_\alpha$ for every $\alpha \in \Gamma$.*

proof:
For each $\alpha < \kappa$ let $f_\alpha : \mathrm{splitnodes}(r) \to \omega$ be such that $r_{f_\alpha} \subseteq p_\alpha$ where as before,

$$r_f = r \setminus \{t \in \omega^{<\omega} : \exists s \in \mathrm{splitnodes}(f) \exists m < f(s); s^\frown m \subset t\}$$

that is $f_\alpha(s)$ tells what finite set of nodes below the splitting node s of r we should prune from r so as to end up with subtree of p_α. It is well known that

MA gives us $f : \text{splitnodes}(r) \to \omega$ that for all $\alpha < \kappa$ for all but finitely many $s \in \text{splitnodes}(r)$ $f_\alpha(s) < f(s)$. By changing f on a finite set we can find $\Gamma \in [\kappa]^\kappa$ such that for all $\alpha \in \Gamma$ and for all $s \in \text{splitnodes}(r)$ $f_\alpha(s) < f(s)$. It follows from this that $q = r_f \subseteq p_\alpha$ for all $\alpha \in \Gamma$

∎

Proof of Theorem 4.1: Suppose that forcing with $\mathbb{F}$ did collapse cardinals, then there would exist regular cardinals $\kappa < \lambda \leq \mathfrak{c}$, $p \in \mathbb{F}$ and a name τ such that

$$p \Vdash \tau : \kappa \to \lambda \text{ is increasing and cofinal.}$$

To simplyfy notation assume $p = \omega^{<\omega}$. Using Lemma 4.1 and Lemma 4.3 build a sequence of good $\langle P_s^\alpha : s \in \omega^{<\omega} \rangle$ for $\alpha < \kappa$ such that

1. $\alpha > \beta$ implies $\langle P_s^\alpha : s \in \omega^{<\omega} \rangle \leq^* \langle P_s^\beta : s \in \omega^{<\omega} \rangle$ and

2. for any α there exists a countable Σ such that for every $s \in \omega^{<\omega}$ $p_s^\alpha \Vdash \text{``}\tau(\alpha) \in \Sigma\text{''}$.

Use Lemma 4.3 one more time to obtain $\langle R_s : s \in \omega^{<\omega} \rangle$ such that $\langle R_s : s \in \omega^{<\omega} \rangle \leq^* \langle P_s^\alpha : s \in \omega^{<\omega} \rangle$ for each $\alpha < \kappa$. Fix s (say the empty node) and define $r = r_s$ and $p_\alpha = p_s^\alpha$. Since these satisfy the hypothesis of Lemma 4.4 we can obtain $q \subseteq r$ and $\Gamma \in [\kappa]^\kappa$ such that $q \subseteq p_\alpha$ for every $\alpha \in \Gamma$. But now there exists countable sets $\{\Sigma_\alpha : \alpha \in \Gamma\}$ such that

$$q \Vdash \forall \alpha \in \Gamma \ \tau(\alpha) \in \Sigma_\alpha$$

which means that the range of τ cannot be cofinal in λ.

∎

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Appendix: Stationary subsets of ω_2

In this appendix we solve a problem posed by M. Foreman: Does forcing with $\mathbb{S}$ add a stationary subset of ω_2 which does not contain a stationary subset of the ground model?

Note that if ω_2 is collapsed to ω_1, then there exists a club subset of ω_2 which does not contain an unbounded subset of the ground model. If $\mathfrak{c} = \omega_1$, then every new stationary subset S of ω_2 contains an old stationary set. This is because $\mathbb{S}$ has cardinality ω_1 and so some $p \in \mathbb{S}$ must force stationarily

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many $\alpha \in S$. Amoeba Sacks forcing is defined by $\mathbb{P} = \{(p,n) : n < \omega, p \in \mathbb{S}\}$ where $(p,n) \leq (q,m)$ iff $p \subseteq q$, $n \geq m$, and $p \cap 2^m = q \cap 2^m$. It is proper. If $\mathfrak{c} > \omega_2$ and MA(Amoeba Sacks), then every stationary subset of ω_2 contains a ground model stationary set. To see this note that if

$$p \Vdash_{\mathbb{S}} \text{``}S \subseteq \omega_2 \text{ is stationary''}$$

then by using Amoeba Sacks forcing we can find $q \leq p$ such that for every $\alpha < \omega_2$ there exists $n < \omega$ such that for every $s \in 2^n \cap q$ either $q_s \Vdash \alpha \in S$ or $q_s \Vdash \alpha \notin S$. Hence there must be some $s \in q$ such that

$$\{\alpha < \omega_2 : q_s \Vdash \alpha \in S\}$$

is stationary.

Theorem 1 *Suppose* $\mathfrak{c} = \omega_2$ *then forcing with* $\mathbb{S}$ *adds a stationary subset of* ω_2 *that does not contain a stationary set from the ground model.*

proof:
  Let $\mathbb{S} = \{q_\alpha : \alpha < \omega_2\}$. Use Lemma 2 to obtain for each $\beta < \omega_2$ an antichain $\{p_{\beta\alpha} \leq q_\alpha : \alpha < \beta\}$. Let $p^i_{\beta\alpha} \leq p_{\beta\alpha}$ for $i = 0,1$ be two incompatible extensions.
  Now let $S = \{\langle p^1_{\beta\alpha}, \check{\beta} \rangle : \alpha < \beta < \omega_2\}$. Then the following two facts hold:

  1. $\Vdash_{\mathbb{S}} S \subseteq \check{\omega}_2$

  2. for every $p \in \mathbb{S}$ $\{\alpha < \omega_2 : p \Vdash \alpha \in S$ or $p \Vdash \alpha \notin S\}$ is bounded in ω_2.

Thus neither S nor its complement can contain a stationary set (or even an unbounded subset of ω_2) which is in the ground model. Since one of the two must be stationary the theorem is proved.
∎

Lemma 2 *Suppose* $\mathfrak{c} = \omega_2$ *and* $Q = \{q_\alpha : \alpha < \omega_1\} \subseteq \mathbb{S}$, *then there exists an antichain* $\{p_\alpha : \alpha < \omega_1\}$ *such that for each* $\alpha < \omega_1$ $p_\alpha \leq q_\alpha$.

proof:
  Inductively construct p_α so that for every $\beta > \alpha$ $[p_\alpha] \cap [q_\beta]$ is nowhere dense in $[q_\beta]$. This is easy to do. At stage α, find $r \in \mathbb{S}$ such that

$$[r] \subseteq \left([q_\alpha] \setminus \bigcup_{\beta < \alpha} [p_\beta]\right).$$

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Then split $[r]$ into ω_2 perfect disjoint sets. Note that for any $\beta > \alpha$ at most countably many of these perfect sets can fail to be nowhere dense in $[q_\beta]$. Let p_α be any of the remaining ones.
∎

By a theorem of Balcar and Vojtas [6], Lemma 2 is true in much more generality. All that is really needed is: every $p \in \mathbb{P}$ has ω_2 incompatible extensions. Consequently the result holds for all of the standard tree kinds of forcing, eg Laver, Mathias, Silver, etc.