Remark 3.4
A Dedekind Finite Borel Set

Arnold W. Miller
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Asaf Karagila pointed out that Remark 3.4 in my paper, A Dedekind finite Borel set [5], directly contradicts Theorem 3.3 (c) in Does GCH imply AC locally? by A. Kanamori and D. Pincus [3]. Neither of the papers provides a proof.[1] Here we give a proof of Remark 3.4 [5].

The Feferman-Levy model $\mathcal{N}$ is described in Cohen [1] p.143, Jech [2] p.142, and in the on-line only appendix to Miller [4]. We use the notation from [4]. $\mathbb{C}ol$ is the Levy collapse of $\aleph_\omega$, $\mathbb{C}ol_n$ the collapse of $\aleph_k$ for $k < n$, $G_n = \mathbb{C}ol_n \cap G$ where G $\mathbb{C}ol$-generic over M. The model $\mathcal{N}$ is the symmetry model $M \subseteq \mathcal{N} \subseteq M[G]$ determined by the group $\mathcal{H}$ of automorphism of $\mathbb{C}ol$ which finitely permute finitely many of the collapsing maps domains and the filter of subgroups generated by $(H_n \ : \ n < \omega)$ where

$$H_n = \{\rho \in \mathcal{H} \ : \ \forall p \in \mathbb{C}ol_n \ \ \rho(p) = p\}.$$

Definition 1 *Let $\langle , \rangle : \omega \times \omega \to \omega$ be a fixed bijection, i.e., a pairing function. For each $n \in \omega$ define the map $\pi_n : 2^\omega \to 2^\omega$ by:*

$$\pi_n(x) = y \text{ iff } \forall m \in \omega \ \ y(m) = x(\langle n, m\rangle).$$

Definition 2 *In the Feferman-Levy model $\mathcal{N}$, take $F_n = M[G_n] \cap 2^\omega$. Define*

$$B = \{x \in 2^\omega \ : \ \forall n \ \ \pi_n(x) \in F_{n+1} \setminus F_n \text{ or } [\pi_n(x) \in F_n \text{ and } \pi_n(x) = \pi_{n+1}(x)]\}.$$

The set B is uncountable because there is a map h from B onto 2^ω. Define h by $h(x) = \pi_n(x)$ iff $\pi_n(x) = \pi_m(x)$ for all $m > n$. Such an n must exist because for any x there exists n such that $x \in F_n$ and hence $\pi_m(x) \in F_n$ for all m. It is easy to check that h maps B onto 2^ω.

Proposition 3 *In the Feferman-Levy model $\mathcal{N}$ the set B has the property that there is no one-to-one map taking 2^ω into B.*

[1] Aki Kanomori tells me that Theorem 3.3 (c) was incorrectly stated. Instead of $\mathcal{N}$ it should be the Solovay-type inner model $\mathcal{N}_0 \subseteq \mathcal{N}$. This is the model $HOD({}^\omega ON)$ of $\mathcal{N}$. See page 155 of **Consequences of the Axiom of Choice** by Rubin and Howard who call $\mathcal{N}_0$ Truss's model.

Proof
Suppose for contradiction that there is in $\mathcal{N}$ a one-to-one map $f : 2^\omega \to B$. Define

$$C_n = \{x \in B \ : \ \forall m > n \ \ \pi_m(x) = \pi_n(x)\}$$

and note that $B = \bigcup_n C_n$ and each C_n is countable.

Let

$$p \Vdash \overset{\circ}{f}: 2^\omega \to \overset{\circ}{B} \ \text{ is one-to-one}$$

and suppose that $p \in \mathbb{C}ol_n$ and H_n (the subgroup of $\mathcal{H}$ which is the identity on $\mathbb{C}ol_n$) fixes $\overset{\circ}{f}$. Since C_n is countable and f is one-to-one, we may choose $N >> n$ and $q \le p$ such that

$$q \Vdash f^{-1}(\overset{\circ}{C}_n) \subseteq \overset{\circ}{F}_N \ .$$

Now let $\overset{\circ}{x}_N$ be a canonical name for a real which codes the generic collapse of $\aleph_N$. The important property it has is that $x_N \in F_{N+1} \backslash F_N$ and its values are completely determined by N^{th} coordinate of the generic filter and not any earlier ones, e.g., the $\le n$ coordinates. Take $r \le q$ such that

$$r \Vdash \overset{\circ}{f}\ (\overset{\circ}{x}_N) = \overset{\circ}{y}, \pi_n(\overset{\circ}{y}) = \tau, \ \text{ and } \tau \in \overset{\circ}{F}_{n+1} \setminus \overset{\circ}{F}_n$$

where τ is a $\mathbb{C}ol_{n+1}$-name for an element of 2^ω. For G $\mathbb{C}ol$-generic over M and containing r note that $\tau^{G_{n+1}} \in M[G_{n+1}] \backslash M[G_n]$. So we can find k and $r_0, r_1 \le r$ such that

- r_0 and r_1 are the same on every coordinate except n,
- $r_0 \Vdash \tau(k) = 0$, and
- $r_1 \Vdash \tau(k) = 1$.

Now let $\rho \in \mathcal{H}$ be an automorphism of $\mathbb{C}ol$ which fixes every coordinate except n and makes $\rho(r_0)$ and r_1 compatible. To do this move the domain of $r_0 \restriction (\{n\} \times \omega)$ disjoint from the domain of r_1. Since $\rho \in H_n$ it fixes $\overset{\circ}{f}$, i.e., $\rho(\overset{\circ}{f}) = \overset{\circ}{f}$. Since ρ fixes the N^{th} coordinate $\rho(\overset{\circ}{x}_N) = \overset{\circ}{x}_N$. But this means that

$$\rho(r_0) \Vdash \pi_n(\overset{\circ}{f}\ (\overset{\circ}{x}_N))(k) = 0$$

and

$$r_1 \Vdash \pi_n(\overset{\circ}{f}\ (\overset{\circ}{x}_N))(k) = 1$$

which contradicts that $\rho(r_0)$ and r_1 are compatible.
QED

References

[1] Cohen, Paul J.; **Set theory and the continuum hypothesis.** W. A. Benjamin, Inc., New York-Amsterdam 1966 vi+154 pp.

[2] Jech, Thomas J.; **The axiom of choice.** Studies in Logic and the Foundations of Mathematics, Vol. 75. North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973. xi+202 pp.

[3] Kanamori, A.; Pincus, D.; Does GCH imply AC locally?, Paul Erdos and his mathematics, II (Budapest, 1999), 413-426, Bolyai Soc. Math. Stud., 11, Jnos Bolyai Math. Soc., Budapest, 2002.

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[4] Miller, Arnold W.; Long Borel hierarchies, Math Logic Quarterly, 54(2008), 301-316.

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[5] Miller, Arnold W.; A Dedekind Finite Borel Set, Arch. Math. Logic 50 (2011), no. 1-2, 1–17.

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Arnold W. Miller
miller@math.wisc.edu
http://www.math.wisc.edu/∼miller
University of Wisconsin-Madison
Department of Mathematics, Van Vleck Hall
480 Lincoln Drive
Madison, Wisconsin 53706-1388