The cardinal characteristic for relative γ-sets

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Abstract: For X a separable metric space define $\mathfrak{p}(X)$ to be the smallest cardinality of a subset Z of X which is not a relative γ-set in X, i.e., there exists an ω-cover of X with no γ-subcover of Z. We give a characterization of $\mathfrak{p}(2^\omega)$ and $\mathfrak{p}(\omega^\omega)$ in terms of definable free filters on ω which is related to the pseudo-intersection number $\mathfrak{p}$. We show that for every uncountable standard analytic space X that either $\mathfrak{p}(X) = \mathfrak{p}(2^\omega)$ or $\mathfrak{p}(X) = \mathfrak{p}(\omega^\omega)$. We show that the following statements are each relatively consistent with ZFC: (a) $\mathfrak{p} = \mathfrak{p}(\omega^\omega) < \mathfrak{p}(2^\omega)$ and (b) $\mathfrak{p} < \mathfrak{p}(\omega^\omega) = \mathfrak{p}(2^\omega)$

First we remind the reader of the definition of a γ-set. An open cover $\mathcal{U}$ of a topological space X is an ω-cover iff for every finite $F \subseteq X$ there exists $U \in \mathcal{U}$ with $F \subseteq U$. The space X is a γ-set iff for every ω-cover $\mathcal{U}$ of X there exists a sequence $(U_n \in \mathcal{U} : n < \omega)$ such that for every $x \in X$ for all but finitely many n we have $x \in U_n$, equivalently

$$X = \bigcup_{m<\omega} \bigcap_{n>m} U_n \quad \text{or} \quad \forall x \in X\ \forall^\infty n \in \omega\ \ x \in U_n.$$

We refer to the sequence $(U_n : n < \omega)$ as a γ-cover of X, although technically we are supposed to assume that the U_n are distinct. In this paper all our spaces are separable metric spaces, so we may assume that all ω-covers are countable. This is because we can replace an arbitrary ω-cover with a refinement consisting of finite unions of basic open sets.

The γ-sets were first considered by Gerlits and Nagy [5]. One of the things that they showed was the following. The pseudo-intersection number $\mathfrak{p}$ is defined as follows:

$$\mathfrak{p} = \min\{|\mathcal{F}| \ :\ \mathcal{F} \subseteq [\omega]^\omega \text{ has the FIP and } \neg\exists X \in [\omega]^\omega\ \forall Y \in \mathcal{F}\ X \subseteq^* Y\}$$

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where FIP stands for the finite intersection property, i.e., every finite subset of $\mathcal{F}$ has infinite intersection, and $\subseteq^*$ denotes inclusion mod finite. The set X in this definition is called the pseudo-intersection of the family $\mathcal{F}$.

Gerlits and Nagy [5] showed that every γ-set has strong measure zero (in fact, the Rothberger property C'') and that Martin's Axiom implies every set of reals of size smaller than the continuum is a γ-set. Their arguments show that

$$\mathfrak{p} = \text{non}(\gamma\text{-set }) =^{def} \min\{|X| \;:\; X \text{ is not a } \gamma\text{-set }\}$$

where we only consider separable metric spaces X.

The property of being a γ-set is not hereditary. In fact, a γ-set X of size continuum is constructed in Galvin and Miller [4] using MA, which has the property that there exists a countable $F \subseteq X$ such that $X \setminus F$ is not a γ-set. However, any closed subspace of a γ-set is a γ-set.

Babinkostova, Guido and Kocinac [1] have defined the notion of a relative γ-set. This is also studied in Babinkostova, Kocinac, and Scheepers [2]. For $X \subseteq Y$ define X to be a γ-set relative to Y iff for every open ω-cover $\mathcal{U}$ of Y there exists a sequence $(U_n \in \mathcal{U} : n < \omega)$ such that

$$X \subseteq \bigcup_{m<\omega} \bigcap_{n>m} U_n.$$

Note that if $Z \subseteq X \subseteq Y$ and X is a relative γ-set in Y, then Z is also.

Define the following cardinal number:

$$\mathfrak{p}(Y) = \min\{|X| \;:\; X \subseteq Y \text{ is not a } \gamma\text{-set relative to } Y\}.$$

Perhaps it should be written non(γ relative to Y).

In Just, Scheepers, Szeptycki, and Miller [8] many cardinal characteristics for covering properties are shown to be equal to well-known cardinals. Scheepers has noted that the cardinal numbers of the relativized version of the Rothberger property C'' work out to be either cov(meager) (the cardinality of the smallest cover of the real line with meager sets) or non(SMZ) (the cardinality of the smallest non strong measure zero set of reals).

Scheepers has raised the question of what we can say about the relativized versions for the γ-property. We begin with the easy

Proposition 1 $\mathfrak{p} \leq \mathfrak{p}(\omega^\omega) \leq \mathfrak{p}(2^\omega) \leq \mathfrak{c}$

Proof
If X is a γ-set, then it is a γ-set relative to any superspace. Let $|X| = \mathfrak{p}(\omega^\omega)$ be a subset of ω^ω which is not a relative γ-set. Then X is not a γ-set relative to itself, and hence $\mathfrak{p} \leq |X| = \mathfrak{p}(\omega^\omega)$.

For the second inequality, suppose $X \subseteq 2^\omega$ is not a γ-set relative to 2^ω with $|X| = \mathfrak{p}(2^\omega)$. Let $\mathcal{U}$ be an ω-cover of 2^ω witnessing that X is not a relative γ-set in 2^ω. Then

$$\{U \cup (\omega^\omega \setminus 2^\omega) \ : \ U \in \mathcal{U}\}$$

is an ω-cover of ω^ω witnessing that X is not a γ-set relative to ω^ω, and so $\mathfrak{p}(\omega^\omega) \leq |X| = \mathfrak{p}(2^\omega)$.
QED

We give another characterization of $\mathfrak{p}(\omega^\omega)$ and $\mathfrak{p}(2^\omega)$. A filter is free iff it contains the cofinite sets. For $\mathcal{F} \subseteq P(\omega)$ a free filter on ω, define

$$\mathfrak{p}_{\mathcal{F}} = \min\{|X| \ : \ X \subseteq \mathcal{F} \quad \text{and} \quad \neg\exists a \in [\omega]^\omega \ \ \forall b \in X \ \ a \subseteq^* b\}.$$

Note that $\mathfrak{p}$ is the minimum of $\mathfrak{p}_{\mathcal{F}}$ for $\mathcal{F} \subseteq P(\omega)$ a free filter, since every family with the FIP generates a filter. We have the following characterizations:

Theorem 2 *(a) $\mathfrak{p}(\omega^\omega)$ is the minimum of $\mathfrak{p}_{\mathcal{F}}$ such that $\mathcal{F} \subseteq P(\omega)$ is a $\mathbf{\Sigma}^1_1$ free filter.*

(b) $\mathfrak{p}(2^\omega)$ is the minimum of $\mathfrak{p}_{\mathcal{F}}$ such that $\mathcal{F} \subseteq P(\omega)$ is a $\mathbf{\Sigma}^0_2$ free filter.

Proof

Suppose $X \subseteq \omega^\omega$ with $|X| = \mathfrak{p}(\omega^\omega)$ and $\mathcal{U}$ is an open ω-cover of ω^ω witnessing that X is not a relative γ-set. Without loss of generality we may assume that $\mathcal{U}$ is a countable family of clopen sets, say $\mathcal{U} = \{U_n : n \in \omega\}$. Let $f : \omega^\omega \to P(\omega)$ be the Marczewski [12] characteristic function of sequence

$$f(x) = \{n : x \in U_n\}.$$

This is a continuous mapping so its image $\mathcal{G} = f(\omega^\omega)$ is $\mathbf{\Sigma}^1_1$. Since $\mathcal{U}$ was an ω-cover the image $\mathcal{G}$ has the FIP and note that the filter $\mathcal{F}$ generated by a $\mathbf{\Sigma}^1_1$ family $\mathcal{G}$ with the FIP is $\mathbf{\Sigma}^1_1$, i.e.,

$$X \in \mathcal{F} \text{ iff } \exists F \in [\mathcal{G}]^{<\omega} \quad \cap F \subseteq X.$$

Now assume $|X| < \mathfrak{p}_{\mathcal{F}}$ and hence $|f(X)| < \mathfrak{p}_{\mathcal{F}}$. Then there exists $a \in [\omega]^\omega$ such that for each $b \in X$ we have that $a \subseteq^* f(b)$. It follows that $(U_n \ : \ n \in a)$ is a γ-cover of X which is a contradiction. Hence $\mathfrak{p}(\omega^\omega) = |X| \geq \mathfrak{p}_{\mathcal{F}}$ and so

$$\mathfrak{p}(\omega^\omega) \geq \min\{\mathfrak{p}_{\mathcal{F}} : \mathcal{F} \text{ is a } \mathbf{\Sigma}^1_1 \text{ free filter }\}.$$

To see the other inequality, suppose $\mathcal{F} \subseteq P(\omega)$ is a $\mathbf{\Sigma}^1_1$ filter and $X \subseteq \mathcal{F}$ has no pseudo-intersection with $|X| = \mathfrak{p}_{\mathcal{F}}$. Let $f : \omega^\omega \to \mathcal{F}$ be a continuous onto map. For each $n \in \omega$ define $U_n = f^{-1}(\{x \in \mathcal{F} \;:\; n \in x\})$. Define $\mathcal{U} = \{U_n \;:\; n \in \omega\}$. Then $\mathcal{U}$ is an ω-cover of ω^ω. Choose $Y \subseteq \omega^\omega$ with $f(Y) = X$ and $|Y| = |X|$. If Y is relative γ in ω^ω, then there exists $a \in [\omega]^\omega$ such that $(U_n \;:\; n \in a)$ is a γ-cover of Y. For each $b \in X$ we have $c \in Y$ with $f(c) = b$. For each n if $c \in U_n$, then $f(c) \in f(U_n)$ and so $n \in b$. It follows that $a \subseteq^* c$ for all $c \in X$. Since we are assuming that there is no such a, we must have that Y is not a γ-set relative to ω^ω and therefore

$$\mathfrak{p}(\omega^\omega) \leq |Y| = |X| = \mathfrak{p}_{\mathcal{F}}$$

and therefor

$$\mathfrak{p}(\omega^\omega) \leq \min\{\mathfrak{p}_{\mathcal{F}} : \mathcal{F} \text{ is a } \mathbf{\Sigma}^1_1 \text{ free filter }\}.$$

The proof for $\mathfrak{p}(2^\omega)$ is similar. To see that

$$\mathfrak{p}(2^\omega) \geq \min\{\mathfrak{p}_{\mathcal{F}} : \mathcal{F} \text{ is a } \mathbf{\Sigma}^0_2 \text{ free filter }\}$$

choose $X \subseteq 2^\omega$ with $|X| = \mathfrak{p}(2^\omega)$ and $\mathcal{U}$ a countable clopen ω-cover of 2^ω with no γ-subcover of X. Let $f : 2^\omega \to P(\omega)$ be defined by $f(x) = \{n : x \in U_n\}$. Then f is continuous and so its range is a compact set $f(2^\omega) = C \subseteq P(\omega)$ which has the FIP. Note that the filter $\mathcal{F}$ generated by C is $\mathbf{\Sigma}^0_2$ in $P(\omega)$. To see this note that for each $n < \omega$ the map $h : C^n \to P(\omega)$ defined by

$$h(X_1, \ldots, X_n) = X_1 \cap X_2 \cap \cdots \cap X_n$$

is continuous and so its range C_n is compact. For each n let D_n be the compact set

$$D_n = \{(x, y) \;:\; x \in C_n \text{ and } x \subseteq y\}.$$

and let $\pi(x, y) = y$ be the projection onto the second coordinate. Then

$$\mathcal{F} = \cup_{n<\omega} \pi(D_n)$$

and so $\mathcal{F}$ is $\mathbf{\Sigma}^0_2$.

Hence, if $|X| < \mathfrak{p}_{\mathcal{F}}$, then $|f(X)| < \mathfrak{p}_{\mathcal{F}}$ and therefor there exists $a \in [\omega]^\omega$ with $a \subseteq^* f(x)$ for each $x \in X$ and therefor $x \in U_n$ for all but finitely many $n \in a$ and $(U_n : n \in a)$ is a γ-cover of X, which is a contradiction.

To see that

$$\mathfrak{p}(2^\omega) \leq \min\{\mathfrak{p}_{\mathcal{F}} : \mathcal{F} \text{ is a } \mathbf{\Sigma}^0_2 \text{ free filter }\}$$

suppose that $\mathcal{F}$ is a $\mathbf{\Sigma}^0_2$ free filter in $P(\omega)$ and for contradiction $\mathfrak{p}_{\mathcal{F}} < \mathfrak{p}(2^\omega)$. First note that there exists a compact $C \subseteq \mathcal{F}$ such that for every $x \in \mathcal{F}$ there exists a $y \in C$ with $x =^* y$. To see this, suppose that $\mathcal{F} = \cup_{n<\omega} C_n$. For each $n < \omega$ let

$$C^*_n = \{x \subseteq \omega : n \subseteq x \text{ and } \exists y \in C_n \;\; \forall i \geq n(i \in y \text{ iff } i \in x)\}$$

then $C = \cup_{n<\omega} C^*_n$ does the trick. Now suppose X is a subset of $\mathcal{F}$ with no pseudo-intersection and $|X| = \mathfrak{p}_{\mathcal{F}} < \mathfrak{p}(2^\omega)$. Choose a map $f : 2^\omega \to C$ which is continuous and onto and select $Y \subseteq 2^\omega$ with $|Y| = |X|$ such that for each $x \in X$ there exists $y \in Y$ with $f(y) =^* x$. Let

$$U_n = f^{-1}(\{x \in C \;:\; n \in x\}).$$

Then $\mathcal{U} = \{U_n : n < \omega\}$ is an ω-cover of 2^ω and since Y is a relative γ-set there exists $a \in [\omega]^\omega$ such that for every $y \in Y$ we have that $y \in U_n$ for all but finitely many $n \in a$. Hence for each $x \in X$ there is $y \in Y$ with $a \subseteq^* f(y) =^* x$ which means that X does have a pseudo-intersection which is contrary to what we assumed.
QED

For another paper studying the connection between γ-sets and free filters, see LaFlamme and Scheepers [10].

Lemma 3 *(a) Suppose that X is homeomorphic to a closed subspace of Y, then $\mathfrak{p}(Y) \leq \mathfrak{p}(X)$.*

(b) Suppose that $f : X \to Y$ is continuous and onto, then $\mathfrak{p}(X) \leq \mathfrak{p}(Y)$.

Proof

(a) Suppose $Z \subseteq X$ with $|Z| = \mathfrak{p}(X)$ is not relatively γ in X and this is witnessed by a family $\mathcal{U}$ of open sets of Y which is an ω-cover of X. Then

$$\{U \cup (Y \setminus X) \;:\; U \in \mathcal{U}\}$$

is an ω-cover of Y which shows that Z is not relatively γ in Y. Hence $\mathfrak{p}(Y) \leq |Z| = \mathfrak{p}(X)$.

(b) Suppose $Z \subseteq Y$ with $|Z| = \mathfrak{p}(Y)$ is not relatively γ in Y and this is witnessed by an ω-cover $\mathcal{U}$. Choose $W \subseteq X$ with $|W| = |Z|$ and $f(W) = Z$. Let $\mathcal{V} = \{f^{-1}(U) : U \in \mathcal{U}\}$. Since f is onto, $\mathcal{V}$ is an ω-cover of X. We claim that there is no sequence $(U_n \in \mathcal{U} : n < \omega)$ such that $(f^{-1}(U_n) : n < \omega)$ is a γ-cover of W. This is because $x \in f^{-1}(U_n)$ implies $f(x) \in U_n$ and since $f(W) = Z$, then $(U_n : n < \omega)$ would be a γ-cover of Z. It follows that $\mathfrak{p}(X) \leq |W| = |Z| = \mathfrak{p}(Y)$.
QED

Theorem 4 *Suppose X is an uncountable $\mathbf{\Sigma}^1_1$ set in a Polish space, i.e., a nontrivial standard analytic space, then*
(a) if X is not σ-compact, then $\mathfrak{p}(X) = \mathfrak{p}(\omega^\omega)$ and
(b) if X is σ-compact, then $\mathfrak{p}(X) = \mathfrak{p}(2^\omega)$.

Proof

Every $\mathbf{\Sigma}^1_1$ set is the continuous image of ω^ω and every uncountable $\mathbf{\Sigma}^1_1$ set contains a homeomorphic copy of 2^ω. It follows from Lemma 3 that

$$\mathfrak{p}(\omega^\omega) \leq \mathfrak{p}(X) \leq \mathfrak{p}(2^\omega).$$

(a) If X is not σ-compact, then Hurewicz [6] (see Kechris [9] 21.18) proved that there exists a closed subspace of X which is homeomorphic to ω^ω. Hence by Lemma 3 we have $\mathfrak{p}(X) \leq \mathfrak{p}(\omega^\omega)$.

(b) Suppose X is σ-compact. We need to show that $\mathfrak{p}(2^\omega) \leq \mathfrak{p}(X)$. We first consider the special case that $X = \omega \times 2^\omega$. Choose $Y \subseteq \omega \times 2^\omega$ to be non relatively γ in $\omega \times 2^\omega$ with $|Y| = \mathfrak{p}(\omega \times 2^\omega)$. Since $\omega \times 2^\omega$ is zero dimensional we can assume that there exists an ω-cover $\mathcal{U} = \{C_n : n < \omega\}$ of clopen sets in $\omega \times 2^\omega$ with no γ-subcover of Y. As in the proof of Theorem 2 we consider $f : \omega \times 2^\omega \to P(\omega)$ defined by

$$f(x) = \{n < \omega : x \in C_n\}.$$

The function f is continuous since the C_n are clopen. The image

$$f(\omega \times 2^\omega) \subseteq P(\omega)$$

is a σ-compact family of subsets of ω with the finite intersection property. Hence $f(\omega \times 2^\omega)$ generates a σ-compact filter $\mathcal{F}$ as in the proof of Theorem 2. Note that $f(Y)$ is a subset of $\mathcal{F}$ without a pseudo-intersection. Hence

$\mathfrak{p}_{\mathcal{F}} \leq |f(Y)| \leq |Y| = \mathfrak{p}(\omega \times 2^\omega)$ and so we have $\mathfrak{p}(2^\omega) \leq \mathfrak{p}(\omega \times 2^\omega)$ and hence $\mathfrak{p}(2^\omega) = \mathfrak{p}(\omega \times 2^\omega)$.

Now suppose that X is any σ-compact metric space. Note that there is a continuous onto mapping $f : \omega \times 2^\omega \to X$ and so by Lemma 3 we have that

$$\mathfrak{p}(X) \geq \mathfrak{p}(\omega \times 2^\omega) = \mathfrak{p}(2^\omega).$$

QED

The main result of this paper is the following theorem:

Theorem 5 *The following statements are each relatively consistent with ZFC:*
(a) $\mathfrak{p} = \mathfrak{p}(\omega^\omega) < \mathfrak{p}(2^\omega)$ *and*
(b) $\mathfrak{p} < \mathfrak{p}(\omega^\omega) = \mathfrak{p}(2^\omega)$

Proof

Part(a).

Given an ω-cover $\mathcal{U}$ of 2^ω define the poset $\mathbb{P}(\mathcal{U})$ as follows:

1. $p \in \mathbb{P}(\mathcal{U})$ iff $p = (F, (U_n \in \mathcal{U} : n < N))$ where $N < \omega$ and $F \in [2^\omega]^{<\omega}$.
2. $p \leq q$ iff $F^p \supseteq F^q$, $N^p \geq N^q$, $U_n^p = U_n^q$ for each $n < N^q$, and $x \in U_n^p$ for each $x \in F^q$ and n with $N^q \leq n < N^p$.

This poset is the obvious one for generically creating a γ-subcover of $\mathcal{U}$ for the ground model elements of 2^ω.

Lemma 6 *The partial order* $\mathbb{P}(\mathcal{U})$ *is* σ*-centered. Furthermore, suppose* G *is* $\mathbb{P}(\mathcal{U})$*-generic over* V*. Define* $(U_n : n < \omega)$ *by* $U_n = U_n^p$ *for any* $p \in G$ *with* $N^p > n$*. Then* $\forall x \in V \cap 2^\omega \;\; \forall^\infty n \;\; x \in U_n$.

Proof

σ-centered is clear, since if $(N_n^p : n < N^p) = (N_n^q : n < N^q)$ then the condition $(F^p \cup F^q, (N_n^p : n < N^p))$ extends both p and q. The fact that U_n is defined for every $n < \omega$ follows from $\mathcal{U}$ being an ω-cover and a density argument, i.e., given any p with $N_p \leq n$ extend it by adding U_k which cover F_p. To see that $(U_n : n < \omega)$ is a γ-cover of $2^\omega \cap V$ let $x \in 2^\omega$ be in the ground model V. The set

$$D = \{p \in \mathbb{P}(\mathcal{U}) \;: x \in F_p\}$$

is dense in $\mathbb{P}(\mathcal{U})$ and if $x \in F_p$ for some $p \in G$ then $x \in U_n$ for every $n \geq N_p$. QED

The model for $\mathfrak{p} = \mathfrak{p}(\omega^\omega) < \mathfrak{p}(2^\omega)$ is obtained by starting with a model of GCH and doing a finite support iteration of $\mathbb{P}(\mathcal{U}_\alpha)$ for $\alpha < \omega_2$ where at each stage in the iteration

$$V[G_\alpha] \models \mathcal{U}_\alpha \text{ is an } \omega\text{-cover of } 2^\omega$$

and where we have dove-tailed so as to ensure that for any $\mathcal{U}$ such that

$$V[G_{\omega_2}] \models \mathcal{U} \text{ is a countable } \omega\text{-cover of } 2^\omega$$

then for some $\alpha < \omega_2$ we have that $\mathcal{U} = \mathcal{U}_\alpha$. This dovetailing can be done since there are only continuum many countable ω-covers of 2^ω and the intermediate models satisfy the continuum hypothesis. In the model $V[G_{\omega_2}]$ we have that $\mathfrak{p}(2^\omega) = \omega_2$, so we need only show that $\mathfrak{p}(\omega^\omega) = \omega_1$. As usual, define Rothberger's unbounded number:

$$\mathfrak{b} = \min\{|X| \; : \; X \subseteq \omega^\omega \;\; \forall g \in \omega^\omega \;\; \exists f \in X \;\; \exists^\infty n \;\; f(n) > g(n)\}.$$

Lemma 7 $\mathfrak{p}(\omega^\omega) \leq \mathfrak{b}$

Proof
Suppose $X \subseteq \omega^\omega$ and $|X| < \mathfrak{p}(\omega^\omega)$. We need to show that X is eventually dominated. Without loss of generality we may assume that the elements of X are increasing and X is infinite. For each $n < \omega$ let

$$\mathcal{U}_n = \{U^n_m : m < \omega\} \text{ where } U^n_m = \{f \in \omega^\omega : f(n) < m\}.$$

Each $\mathcal{U}_n$ is an ω-cover of ω^ω. There is a standard trick due to Gerlits and Nagy [5] for replacing a sequence of ω-covers by a single ω-cover. Let

$$\{x_n : n < \omega\} \subseteq X$$

be distinct and let

$$\mathcal{U} = \{U \setminus \{x_n\} : n < \omega, \; U \in \mathcal{U}_n\}.$$

Then $\mathcal{U}$ is an ω-cover of ω^ω, since given any finite set F then $x_n \notin F$ for large enough n and so $F \subseteq U \setminus \{x_n\}$ for some $U \in \mathcal{U}_n$.

Since X is a relative γ-set in ω^ω there exists a sequence from $\mathcal{U}$ which is a γ-cover of X. Now since we threw out x_n from each element $\mathcal{U}_n$ at most finitely many of the elements of this sequence can come from the same $\mathcal{U}_n$. By taking an infinite subsequence we may assume that $(U^n_{g(n)} : n \in A)$ is a γ-cover of X for some infinite $A \subseteq \omega$. It follows that for every $f \in X$ that

$$\forall^\infty n \in A \;\; f(n) < g(n).$$

Since the $f \in X$ are increasing if we extend g to all of ω by letting $g(m) = g(n)$ where $n \in A$ is minimal so that $n \geq m$, then g eventually dominates every $f \in X$ on all of ω.

It follows that $|X| < \mathfrak{b}$. Since X was arbitrary we get that $\mathfrak{p}(\omega^\omega) \leq \mathfrak{b}$.
QED

Our goal is to show that $\mathfrak{b} = \omega_1$ holds in this model. For the next two lemmas we assume $\mathcal{U}$ is an ω-cover of 2^ω and the forcing is $\mathbb{P}(\mathcal{U})$.

Lemma 8 *Suppose we are given $(U_n \in \mathcal{U} : n < N)$, $k < \omega$, and a term τ such that $|\vdash \tau \in \omega$. Then there exists $l < \omega$ such that for every $p \in \mathbb{P}(\mathcal{U})$ with $|F^p| \leq k$ and $(U_n \in \mathcal{U} : n < N) = (U^p_n \in \mathcal{U} : n < N^p)$ there exists $q \leq p$ such that $q |\vdash \tau < l$.*

Proof
Call $q \in \mathbb{P}(\mathcal{U})$ good iff

1. $N^q \geq N$
2. $U_n = U^q_n$ for all $n < N$, and
3. q decides τ, i.e. for some m, $\quad q|\vdash \tau = m$.

For good q define:

$$U(q) = \{(x_1, \ldots, x_k) \in (2^\omega)^k \;:\; \forall i \quad (N \leq i < N_q) \rightarrow \{x_1, \ldots, x_k\} \subseteq U^q_i\}.$$

Note that each $U(q)$ is an open subset of $(2^\omega)^k$. The family

$$\{U(q) : q \text{ is good }\}$$

covers $(2^\omega)^k$. This is because given any $(x_1, \ldots, x_k)$ there exists a condition $q \leq (\{x_1, \ldots, x_k\}, (U_n : n < N))$ which decides τ and therefor is good. By compactness there exist finitely many good q, say Γ, such that $\{U(q) : q \in \Gamma\}$ covers $(2^\omega)^k$.

Since each good q decides τ and Γ is finite, we can find l so that for each $q \in \Gamma$

$$q \Vdash \tau < l.$$

Note that for any p as in the Lemma, if $F^p \subseteq \{x_1, \ldots, x_k\}$ where

$$(x_1, \ldots, x_k) \in U(q),$$

then q and p are compatible since $(F^p \cup F^q, (N^q_n : n < N^q))$ extends both of them.
QED

It is not hard to see from this lemma that our forcing does not add a dominating sequence. In order to prove the full result we need to show this for the iteration. To do this we prove the following stronger, but more technical, property (see Bartoszynski and Judah [3] definition 6.4.4).

Lemma 9 *The poset $\mathbb{P}(\mathcal{U})$ is really $\sqsubseteq^{\text{bounded}}$-good, i.e., for every name τ for an element of ω^ω there exists $g \in \omega^\omega$ such that for any $x \in \omega^\omega$ if there exists $p \in \mathbb{P}(\mathcal{U})$ such that $p \Vdash$ "$\forall^\infty n\ x(n) < \tau(n)$", then $\forall^\infty n\ \ x(n) < g(n)$.*

Proof
Let $k_n, (U^n_m\ :\ m < N_n)$ for $n < \omega$ list with infinitely many repetitions all pairs of $k < \omega$ and finite sequences from $\mathcal{U}$. Using Lemma 8 repeatedly we can construct $g \in \omega^\omega$ such that for every $l < \omega$:
for any $n < l$ and $p \in \mathbb{P}(\mathcal{U})$ with

$$|F^p| \leq k_n \text{ and } (U^n_m : m < N_n) = (U^p_m : m < N^p)$$

there exists $q \leq p$ such that $q \Vdash \tau(l) < g(l)$.

Now suppose $p \Vdash \forall^\infty n\ x(n) < \tau(n)$. By extending p (if necessary) we may assume there exists n_0 such that $p \Vdash \forall n > n_0\ x(n) < \tau(n)$. By making n_0 larger (if necessary) we may assume that

$$|F^p| = k_{n_0} \text{ and } (U^{n_0}_m : m < N_{n_0}) = (U^p_m : m < N^p).$$

Claim $\forall n > n_0\ \ x(n) < g(n)$.
Proof

Suppose not and $x(l) \geq g(l)$ for some $l > n_0$. By our construction of g we have that there exists $q \leq p$ such that $q|\vdash\tau(l) < g(l)$. But this means that $q|\vdash\tau(l) < x(l)$ which contradicts the fact that $p|\vdash\forall n > n_0\tau(n) > x(n)$. This proves the Claim and the Lemma.
QED

It follows (see Bartoszynski and Judah [3] Theorem 6.5.4) that the finite support iteration using $\mathbb{P}(\mathcal{U}_\alpha)$ at stage α does not add a dominating real and so over a ground model which satisfies CH we have that $V[G_{\omega_2}]$ satisfies that $\mathfrak{b} = \omega_1$ and hence $\mathfrak{p}(\omega^\omega) = \omega_1$ by Lemma 7. This proves Theorem 5 part (a), the consistency of $\mathfrak{p} = \mathfrak{p}(\omega^\omega) < \mathfrak{p}(2^\omega)$.

Part (b) (the consistency of $\mathfrak{p} < \mathfrak{p}(\omega^\omega) = \mathfrak{p}(2^\omega)$) is simpler. It is well known that $\mathfrak{p} > \omega_1$ implies that $2^{\omega_1} = 2^\omega$. For example, see Rothberger [14]. Now starting with a ground model V which satisfies $2^\omega = \omega_2$ and $2^{\omega_1} = \omega_3$, do a finite support iteration using $\mathbb{P}(\mathcal{U}_\alpha)$ at stage $\alpha < \omega_2$ where $\mathcal{U}_\alpha$ is an ω-cover of $V[G_\alpha] \cap \omega^\omega$. Dovetail so that $\mathcal{U}_\alpha$ for $\alpha < \omega_2$ lists all countable ω-covers of ω^ω in the final model $V[G_{\omega_2}]$. This can be done since in all these models the continuum is ω_2. The analogue of Lemma 6 holds for ω^ω in place of 2^ω so in the final model we have that $\mathfrak{p}(\omega^\omega) = \omega_2$. Also we get $\mathfrak{p} = \omega_1$ since $2^{\omega_1} = \omega_3 > \omega_2 = 2^\omega$. This finishes the proof of Theorem 5.
QED

One obvious question is

Question 10 *Is it consistent to have $\mathfrak{p} < \mathfrak{p}(\omega^\omega) < \mathfrak{p}(2^\omega)$?*

Question 11 *(Scheepers) Are either $\mathfrak{p}(\omega^\omega)$ or $\mathfrak{p}(2^\omega)$ the same as some other well-known small cardinal? See Vaughan [16] for a plethora of such cardinals.*

In Laver's model [11] for the Borel conjecture, we have that $\mathfrak{b} = \mathfrak{d} = \omega_2$ and $\mathfrak{p}(2^\omega) = \mathfrak{p}(\omega^\omega) = \omega_1$. In Laver's model there is a set of reals of size ω_1 which does not have measure zero, i.e., non(measure)=ω_1, Judah and Shelah [7], see also Bartoszynski and Judah [3] or Pawlikowski [13]. But it is easy to see that $\mathfrak{p}(2^\omega) \leq$ non(measure), i.e., if $X \subseteq 2^\omega$ and $|X| < \mathfrak{p}(2^\omega)$ then X has measure zero. Let $\{x_n : n < \omega\} \subseteq X$ be distinct and look at

$$\mathcal{U} = \{C \subseteq 2^\omega \ : \ \exists n \ \ x_n \notin C \text{ is clopen and } \mu(C) < \frac{1}{2^n}\}.$$

This is an ω-cover of 2^ω and so there exists a sequence $C_n \in \mathcal{U}$ with $X \subseteq \cup_n \cap_{m>n} C_m$. For any n at most finitely many C_n have measure $> \frac{1}{2^n}$ which shows that X has measure zero.

It is also true that $\mathfrak{p}(2^\omega) \leq \text{non(SMZ)}$, i.e., if $|X| < \mathfrak{p}(2^\omega)$ then X has strong measure zero. The result of Gerlits and Nagy [5], that γ-sets have the Rothberger property C'', relativizes to show that if $X \subseteq 2^\omega$ and $|X| < \mathfrak{p}(2^\omega)$, then X has the relative Rothberger property and this implies that X has strong measure zero.

Question 12 [2] *Suppose that $Y = \cup_{n<\omega} X_n$ is an increasing union where Y is a separable metric space. If each X_n is relatively γ in Y, is Y a γ-set? If not, suppose each X_n is a γ-set, then is Y a γ-set?*

Tsaban [15] Lemma 22 shows that the answer to this question in the Borel cover case is yes. It is also connected to the existence of a group which is a γ-set, [15] Theorem 20.

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[2]This has been answered. See appendix.

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The appendix is not intended for final publication but for the electronic version only.

Appendix
Scheepers Remarks

Def. $X \subseteq Y$ is C'' (Rothberger) in Y iff for every sequence $(\mathcal{U}_n : n < \omega)$ of open covers of Y there exists a cover $(U_n \in \mathcal{U}_n : n < \omega)$ of X.

Prop.
(a) non(C'' in ω^ω)=non(C'')=cov(meager)= non(SMZ in ω^ω)
(b) non(C'' in 2^ω) = non(SMZ in 2^ω)= non(SMZ in $\mathbb{R}$)
Proof
Here we mean strong measure zero in the usual metric on the reals and for ω^ω or 2^ω the metric $d(x,y) = \frac{1}{n+1}$ where n is minimal such that $x(n) \neq y(n)$.

(a) Fremlin and Miller (1988) prove:
non(C'')=cov(meager)=non(SMZ in ω^ω)
non(C'') $\leq$non(C'' in ω^ω) since if X is not relatively C'' it is not C''.
non(C'' in ω^ω) $\leq$ non(SMZ in ω^ω) since $C'' \subseteq$ SMZ.

(b) Suppose $X \subseteq 2^\omega$ fails to be relatively C''. Note that by compactness of 2^ω we may assume there is a sequence $(\mathcal{U}_n : n < \omega)$ of finite clopen covers of 2^ω for which there is no $(U_n \in \mathcal{U}_n : n < \omega)$ which covers X. Now choose $\epsilon_n > 0$ so that any interval $[s]$ with diameter less than ϵ_n is a subset of some U_n. For the converse, suppose X fails to have SMZ in 2^ω which is witnessed by $(\epsilon_n : n < \omega)$. Then the sequence $\mathcal{U}_n = \{C \subseteq 2^\omega : \text{diam}(C) < \epsilon_n\}$ witnesses that it is not C''.
non(SMZ in 2^ω)=non(SMZ in $[0,1]$)=non(SMZ in $\mathbb{R}$) is easy to prove.
QED
These cardinals can be different, for example, in the iterated modified Silver reals model, see Miller (1981), cov(meager)= ω_1 while non(SMZ in 2^ω) $= \omega_2$.

Def. X has the Menger property M iff for every sequence $(\mathcal{U}_n : n < \omega)$ of open covers there exists $(\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega} : n < \omega)$ such that $X \subseteq \cup_n \cup \mathcal{V}_n$.
Def. X has the Hurewicz property H iff for every sequence $(\mathcal{U}_n : n < \omega)$ of open covers there exists $(\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega} : n < \omega)$ such that $X \subseteq \cup_m \cap_{n>m} \cup \mathcal{V}_n$.

Then it is known, see Miller and Fremlin (1988) that

$$\mathfrak{d} = \text{non}(M) \qquad \mathfrak{b} = \text{non}(H).$$

Relativizing H or M to 2^ω doesn't work since 2^ω has property H and M. For ω^ω it is easy to see:

Prop.
(a) non(M) $\leq$ non(M in ω^ω) $\leq \mathfrak{d} \leq$ non(M)
(b) non(H) $\leq$ non(H in ω^ω) $\leq \mathfrak{b} \leq$ non(H)

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Question 12

This was answered by Francis Jordan (There are no hereditary productive γ-spaces, eprint Spring 08). He proves that the increasing countable union of γ-sets is a γ-set.

Lemma 13 *For any γ-set Y and sequence $(\mathcal{U}_n : n < \omega)$ of ω-covers of Y there exists $(\mathcal{V}_n \in [\mathcal{U}_n]^\omega : n < \omega)$ such that $\bigcup_{n<\omega} \mathcal{V}_n$ is a γ-cover of Y.*

The proof is left to the reader. (Gerlits-Nagy)

Suppose X_n for $n < \omega$ are γ-sets, $X_n \subseteq X_{n+1}$ for each n and $X = \bigcup_{n<\omega} X_n$. Given any ω-cover of X we may extract from it a sequence $\mathcal{U}_n$ such that each $\mathcal{U}_n$ is a γ-cover of X_n. Let $\mathcal{U}_n^0 = \mathcal{U}_n$. Using the Lemma construct a sequence $(\mathcal{U}_n^p : p \leq n < \omega)$ by induction on p such that $\mathcal{U}_n^{p+1} \in [\mathcal{U}_n^p]^\omega$ and $\bigcup\{\mathcal{U}_n^{p+1} : p+1 \leq n < \omega\}$ is a γ-cover of X_p. Any sequence $(U_n \in \mathcal{U}_n^n : n < \omega)$ is a γ-cover of X.

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