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7.15 of Ash-Knight

This answers a question raised by Charlie McCoy when he was giving a topics course on a prepublication copy of the book[1] by Ash and Knight.

Fix a recursive language L. Define Symb to be the infinite (recursive) set consisting of

1. all atomic formulas of L
2. $\neg$, $\wedge$, $\rightarrow$
3. $\exists x$ for each variable x, and
4. $\bigvee\!\!\!\bigvee$, $\bigwedge\!\!\!\bigwedge$

For $a \in \mathcal{O}$ define S_a be the set of all triples (s, a, e) with $s \in$ Symb and $e \in \omega$. Also define $S_{<a}$ to be the union of all S_b for $b <_o a$.

For each $H \subseteq \omega$ and $e \in \omega$ define

$$H_e = \{n : (e, n) \in H\}$$

and let W give the usual enumeration of the recursively enumerable sets.

For $i \in S_a$ inductively define ψ_i^H as follows:

1. If $i = (\rho, a, j)$ and ρ atomic, then $\psi_i^H = \rho$
2. If $i = (\neg, a, j)$ and $j \in S_{<a}$ then $\psi_i^H = \neg\psi_j^H$
3. If $i = (\wedge, a, (n, m))$ and $n, m \in S_{<a}$, then $\psi_i^H = (\psi_n^H \wedge \psi_m^H)$
4. If $i = (\rightarrow, a, (n, m))$ and $n, m \in S_{<a}$, then $\psi_i^H = (\psi_n^H \rightarrow \psi_m^H)$
5. If $i = (\exists x, a, j)$ and $j \in S_{<a}$, then $\psi_i^H = \exists x \ \psi_j^H$
6. If $i = (\bigvee\!\!\!\bigvee, a, e)$, then $\psi_i^H = \bigvee\!\!\!\bigvee\{\psi_j^H : j \in H_e \cap S_{<a}\}$
7. If $i = (\bigwedge\!\!\!\bigwedge, a, e)$, then $\psi_i^H = \bigwedge\!\!\!\bigwedge\{\psi_j^H : j \in H_e \cap S_{<a}\}$

[1]Ash, C. J.; Knight, J.; Computable structures and the hyperarithmetical hierarchy. Studies in Logic and the Foundations of Mathematics, 144. North-Holland Publishing Co., Amsterdam, 2000. xvi+346 pp. ISBN: 0-444-50072-3

8. Otherwise, define ψ_i^H to be F (the symbol for false), e.g., in case 2 if $j \notin S_{<a}$. Or T it makes no difference. Similarly, empty infinite conjunctions or disjunctions can be assigned T or F.

Main Lemma.

Suppose $a \in \mathcal{O}$ and H is hyperarithmetic set which is effectively defined by S_a, i.e. there exists a recursive $h : \omega \to S_a(T,F)$ (codes for infinitary propositional language on T,F) such that for all n

$$n \in H \quad \text{iff} \quad \psi^W_{h(n)} \text{ is true}$$

Assume that the order type of a is a limit ordinal and $b+b<a$ for all $b<a$.

Then there exists a recursive $f : S_{<a} \to S_{<a+a}$ so that for every $i \in S_{<a}$ we have that ψ_i^H and $\psi^W_{f(i)}$ are logically equivalent, i.e., $\psi_i^H \equiv \psi^W_{f(i)}$

Proof. In fact, we construct f with the additional property that if $i \in S_b$, then $f(i) \in S_{a+3(b+1)}$. Addition here is the usual $+_o$ operation on the elements of Kleene's $\mathcal{O}$.

The steps in the definition of $f(i)$ are all trivial except for the infinite disjunction or conjunction cases. For example:
If $i = (\wedge, b, (n,m))$, then $f(i) = (\wedge, a+3(b+1), (f(n), f(m)))$.
If $i = (\rho, b, e)$ where ρ atomic, then $f(i) = (\rho, a+3(b+1), e)$.

Now suppose $i = (\bigvee, b, e)$. Note that

$$\begin{aligned}\psi_i^H &= \bigvee\{\psi_j^H : j \in H_e \cap S_{<b}\} \equiv \bigvee\{(\psi^W_{h(e,j)} \wedge \psi^H_j) : j \in S_{<b}\} \\ &\equiv \bigvee\{(\psi^W_{h(e,j)} \wedge \psi^W_{f(j)}) : j \in S_{<b}\}\end{aligned}$$

We construct g recursive so that

$$(\psi^W_{h(e,j)} \wedge \psi^W_{f(j)}) \equiv \psi^W_{g(j)}$$

as follows:

Suppose $h(e,j) = (s_1, a, e_1)$ and $f(j) = (s_2, a+3(c+1), e_2)$. Then define

$$g(j) = (\wedge, a+3(c+1)+1, (h(e,j), f(j)))$$

and define $f(i) = (\bigvee, a+3(b+1), e)$ where $W_e = \{g(j) : j \in S_{<b}\}$. Note that $c+1 \leq_o b$ implies $3(c+1)+1 \leq_o 3b+1 <_o 3(b+1)$ and hence $W_e \subseteq S_{<a+3(b+1)}$ as we needed to show the logical equivalence:

$$\psi^W_{f(i)} = \bigvee\{\psi^W_k : k \in W_e\} \equiv \bigvee\{\psi^W_{g(j)} : j \in S_{<b}\}$$

The infinite conjunction case is similar except we use
$\bigwedge\!\!\!\bigwedge\{(\psi^W_{h(e,j)} \rightarrow \psi^W_{f(j)}) : j \in S_{<b}\}$

This proves the Main Lemma.

Given $K \subseteq \bigcup_{a \in \mathcal{O}} S_a$ hyperarithmetic, it is easy to construct H hyperarithmetic and j so that $\bigvee\!\!\!\bigvee\{\psi^W_i : i \in K\} \equiv \psi^H_j$. By the main lemma we can find k with $\psi^H_j \equiv \psi^W_k$. Hence the recursive infinitary formulas are closed under hyperarithmetic disjunctions.

I think the usual "change into normal form" arguments allow for an effective translation of these codes into the codes that Ash-Knight use (and back).