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RATIONAL PERFECT SET FORCING

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ABSTRACT. This paper introduces a notion of forcing intermediate between Sacks perfect set forcing and Laver forcing. We say that a perfect subset of the real line is a rational perfect set if the rationals in it are dense in it. This forcing is equivalent to forcing with infinitely branching $\omega^{<\omega}$ trees. The real added is not dominated by the ground model and is of minimal degree. Also, any nondominated real in the extension is itself generic with respect to the same partial order. In the extension there is no independent subset of ω, i.e. every subset of ω in the extension contains or is disjoint from some infinite subset of ω in the ground model.

§1. DEFINITIONS AND EQUIVALENCES. In this section we present two partial orders and then show that they are isomorphic. For T a subtree of $2^{<\omega}$, the tree of finite sequences of 0's and 1's, let

$$[T] = \{x \in 2^{\omega} : \forall n \quad x \upharpoonright n \in T\} .$$

Such a tree is perfect if every node has two incomparable extensions. Let Q be the set of all $x \in 2^{\omega}$ which are eventually zero. The rational perfect trees are the perfect tree $T \subseteq 2^{<\omega}$ such that $Q \cap [T]$ is dense in $[T]$, i.e. for all $s \in T$ there exists $x \in Q \cap [T]$ with $s \subseteq x$. The trees are partially ordered by containment.

A subtree T of $\omega^{<\omega}$, the finite sequences from ω, is infinitely branching if for all $s \in T$ there exists $t \supseteq s$ such that for infinitely many $n < \omega$, $t^\frown\langle n\rangle \in T$. These trees are also ordered by containment.

1. PROPOSITION. The partial order of rational perfect trees is isomorphic to the partial order of infinitely branching trees.

Proof. Define the map $f: \omega^{<\omega} \to 2^{<\omega}$ by induction on the length of the sequence:

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(a) $f(\emptyset) = \emptyset$; and

(b) $f(s^\frown\langle n\rangle) = f(s)^\frown\langle 000\ldots0\rangle^\frown\langle 1\rangle$, where $\langle 000\ldots0\rangle$ is a string of n zeros.

The isomorphism of the partial order is given by

$$\hat{f}(T) = \{s \in 2^{<\omega} : \exists t \in T \ \ s \subseteq f(t)\} .$$

Note that the function f has the following properties:

(a) for $s,t \in \omega^{<\omega}$

$$s \subseteq t \ \text{ iff } \ f(s) \subseteq f(t) ;$$

(b) f is 1-1; and

(c) f is onto the sequences ending in a 1.

Now it is easy to check that $\hat{f}$ is the desired isomorphism. □

REMARK. We did not demand that in our infinitely branching trees every node either branched infinitely or not at all, although clearly these trees would form a dense subset. They correspond to the binary branching trees in which the left most branch through each node is eventually zero.

§2. CHARACTERIZING THE GENERIC REAL. Let $\mathbb{P}$ be the partial order of infinitely branching trees with the property that every node either has infinitely many immediate extensions or only one immediate extension. Suppose G is $\mathbb{P}$-generic over the ground model M. Let $f \in \omega^\omega$ be such that for all $p \in G$ and $n<\omega$, $f\restriction n \in p$. It is easy to see that f is a total function and

$$G = \{p \in \mathbb{P} \mid \forall n \ \ f\restriction n \in p\} .$$

Thus we say $f \in \omega^\omega$ is $\mathbb{P}$-generic if $\{p \in \mathbb{P} \mid \forall n \ \ f\restriction n \in p\}$ is a $\mathbb{P}$-generic filter. An easy density argument shows that any $f \in \omega^\omega$ which is $\mathbb{P}$-generic over M weakly dominated M. That is to say, for any $g \in \omega^\omega \cap M$ there are infinitely many $n < \omega$ such that $g(n) < f(n)$. The following proposition is the converse.

2. PROPOSITION. Suppose G is $\mathbb{P}$-generic over M and $f \in \omega^\omega \cap M[G]$ is any real which weakly dominates M. Then f itself is $\mathbb{P}$-generic (although it may generate a different G).

Proof. The proof will follow easily from the following claims. Before that we make some definitions.

DEFINITIONS. (a) for $p \in \mathbb{P}$ and $s \in p$ let

$$X_s^p = \{n < \omega: s^\frown\langle n\rangle \in p\} ;$$

(b) for $p \in \mathbb{P}$ let

$$\text{split}(p) = \{s \in p: X_s^p \text{ is infinite}\} ;$$

(c) for $p \in \mathbb{P}$ and $s \in p$ let

$$p_s = \{t \in p: t \subseteq s \text{ or } s \subseteq t\} ;$$

(d) for $s,t \in \omega^{<\omega}$ let $s \ll t$ if for all $n < \text{length}(s)$, $s(n) < t(n)$;

and

(e) $s \in \text{split}(p)$ is at level n if there are exactly n proper initial segments of s in $\text{split}(p)$.

2.1. CLAIM. Suppose $p \Vdash$ "$\tau \in \omega^\omega$". Then there exists $q \leq p$ such that for all $s \in \text{split}(q)$ there exists $t_n^s \in \omega^n$ such that for all $n \in X_s^q$

$$q_{s^\frown\langle n\rangle} \Vdash \text{"}\tau \restriction n = t_n^s\text{"} .$$

Proof. This will follow from a fusion argument. Suppose $p_{n+1} \leq p_n$ for $n < \omega$ and the first n levels of $\text{split}(p_n)$ are still in $\text{split}(p_{n+1})$. Then the fusion

$$q = \bigcap_{n<\omega} p_n$$

will be in $\mathbb{P}$. Other proofs will require that only finitely many nodes are kept at each step. Let $p_0 = p$. At stage n look at each $s \in \text{split}(p_n)$ at level n. For each $m \in X_s^{p_n}$ extend $p_{n,s^\frown\langle m\rangle}$ to decide $\tau \restriction m$. Paste all such extensions together to get p_{n+1}. □

2.2. CLAIM. Given $t_n \in \omega^n$ for $n \in X \in [\omega]^\omega$ either:

(1) $\exists Y \in [X]^\omega\ \exists m < \omega\ \forall k,\ell \in Y\quad t_k \restriction m = t_\ell \restriction m$ and $\{t_n(m): n \in Y\}$ are distinct; or

(2) $\exists g \in \omega^\omega\ \forall n \in X\quad t_n \ll g \restriction n.$

Proof. If for some m_0 $\{t_n(m_0): m_0 < n\}$ is infinite, then (1) holds for m the least such m_0. Otherwise (2) holds. □

2.3. CLAIM. Suppose $p \Vdash "\tau \in \omega^\omega"$. Then there exists $q \leq p$ such that for all $s \in \mathrm{split}(q)$ either:

(1) $\exists m \; \exists t \in \omega^m \; \exists h: X^q_s \xrightarrow{1-1} \omega \; \forall n \in X^q_s$
$q_{s\frown\langle n\rangle} \Vdash "\tau \restriction m+1 = t\frown h(n)"$; or

(2) $\exists g \in \omega^\omega \; \forall n \in X^q_s \; \exists s \in \omega^n \; s \ll g \restriction n$
$q_{s\frown\langle n\rangle} \Vdash "\tau \restriction n = s"$.

Proof. First apply Claim 2.1 and then use Claim 2.2 at each stage in a fusion argument.

2.4. CLAIM. Suppose $\mathrm{split}(p) = A \cup B$. Then there exists $q \leq p$ such that either $\mathrm{split}(q) \subseteq A$ or $\mathrm{split}(q) \subseteq B$.

Proof. Either there exists $s \in \mathrm{split}(p)$ such that $\mathrm{split}(p_s) \subseteq A$ or for every $s \in \mathrm{split}(p)$ there exists $t \supseteq s$ $t \in B$. In the first case we are done and in the second we do a fusion argument. □

2.5. CLAIM. Suppose $p \Vdash "\tau \in \omega^\omega"$. Then there exists $q \leq p$ such that either:

(1) for all $s \in \mathrm{split}(q) \; \exists m \; \exists t \in \omega^m \; \exists h: X^q_s \xrightarrow{1-1} \omega \; \forall n \in X^q_s$
$q_{s\frown\langle n\rangle} \Vdash "\tau \restriction m+1 = t\frown h(n)"$; or

(2) for all $s \in \mathrm{split}(q) \; \exists g \in \omega^\omega \; \forall n \in X^q_s \; \exists r \in \omega^n \; r \ll g \restriction n$ and $q_{s\frown\langle n\rangle} \Vdash "\tau \restriction n = r"$.

Proof. This follows from Claims 2.3 and 2.4. □

DEFINITION. For $q \in \mathbb{P}$ let $\mathbb{P}_q$ be the partial order of conditions $\leq q$.

2.6. CLAIM. Suppose q and τ are as in Claim 2.5(1). Then there exists $\tilde{q} \in \mathbb{P}$ and an order isomorphism $\rho: \mathbb{P}_q \to \mathbb{P}_{\tilde{q}}$ such that $q \Vdash "\tau$ is the generic real associated with the $\mathbb{P}_{\tilde{q}}$-generic filter $\rho(G)$".

Proof. Define $\rho^*: q \to \omega^{<\omega}$ by $\rho^*(s) = t$ iff $q_s \Vdash "\tau \restriction m = t"$ and q_s does not decide $\tau(m)$. Then $\tilde{q}$ is the tree generated by the image of q under ρ^* and ρ is defined similarly. □

2.7. CLAIM. Suppose q and τ are as in Claim 2.5(2). Then there exists $\tilde{q} \leq q$ and $g \in \omega^\omega$ such that

$$\tilde{q} \Vdash "\forall n \;\; \tau(n) < g(n)" .$$

Proof. Obtain $\tilde{q}$ by prunning q if necessary so that for all $s \in \mathrm{split}(\tilde{q})$ at level n, $\tilde{q}_s$ decides $\tau(n)$. To obtain this it is only necessary to make sure that $X_t^q \subseteq \omega \setminus (n+1)$ for all t at level $n-1$. To finish the proof it is enough to note that for every n $\{m: \exists s \in \tilde{q}$ $\tilde{q}_s \Vdash "\tau(n) = m"\}$ is finite. For suppose not. There must then exist s_i at level n of $\tilde{q}$ and distinct $k_i \in \omega$ such that for each $i \in \omega$

$$\tilde{q}_{s_i} \Vdash "\tau(n) = k_i" .$$

Look at $T = \{s: \exists i \ \ s \subseteq s_i\}$. Since the s_i are all on the n^{th} level of $\tilde{q}$ there must be some $s \in T$ such that there are infinitely many $m \in \omega$ with $s^\frown\langle m\rangle \in T$. But by hypothesis there exist $g \in \omega^\omega$ for all $m \in X_s^q$

$$\tilde{q}_{s^\frown\langle m\rangle} \Vdash "\tau \restriction m \ll g \restriction m" .$$

This would force infinitely many $k_i < g(n)$, a contradiction. □

The last three claims finish the proof of Proposition 2.

REMARK. Claim 2.4 is the only one that does not work for Laver forcing (1976). This claim is analogous to the fact that if the rationals are split into two pieces, then one of the two pieces contains a subset order isomorphic to the rationals.

REMARK. Similar to Sacks forcing (1970), it is not difficult to see that rational perfect set forcing produces a real of minimal constructibility degree. It is also known (Gray (1980)) that a Laver real has minimal degree.

REMARK. In the spirit of the first section, for any $F \subseteq 2^\omega$ a countable dense-in-itself set, define

$$\mathbf{P}_F = \{Q \subseteq 2^\omega: Q \text{ perfect and } F \cap Q \text{ is dense in } Q\} .$$

Then for any $x \in 2^\omega$ in a rational perfect set forcing extension which is not in the ground model, there exist F such that x is $\mathbf{P}_F$-generic. Or we might define

$$\mathbf{P} = \{(Q,F): Q \in \mathbf{P}_F\}$$

and order $\mathbb{P}$ by

$$(\hat{Q},\hat{F}) \leq (Q,F) \quad \text{iff} \quad \hat{Q} \subseteq Q \quad \text{and} \quad \hat{F} \subseteq F .$$

Then just like Sacks forcing any real in $M[G] \setminus M$ for G $\mathbf{P}$-generic over M is itself "$\mathbf{P}$-generic" over M.

§3. INDEPENDENT SETS. In this section fix M a transitive model of ZFC which we will refer to as our ground model. We think of V as a generic extension of M.

DEFINITIONS. (a) $X \subseteq \omega$ is an independent set iff for all $Y \in [\omega]^\omega \cap M$ both $X \cap Y$ and $Y \setminus X$ are infinite.

(b) $f \in \omega^\omega$ is a dominating real iff for all $g \in \omega^\omega \cap M$ for all but finitely many $n \in \omega$, $g(n) < f(n)$.

(c) $f \in \omega^\omega$ is a weak dominating real iff for all $g \in \omega^\omega \cap M$ there are infinitely many $n \in \omega$, $g(n) < f(n)$.

3.1. PROPOSITION. (Folklore) If there exists a dominating real, then there exists an independent set.

Proof. Suppose f is a dominating real and strictly increasing. Define $x_0 = 0$ and $x_{n+1} = f(x_n)$. And let

$$X = \bigcup_{n=0}^{\infty} [x_{2n}, x_{2n+1}) ,$$

($[a,b) = \{n \in \omega : a \leq n < b\}$). Given $Y \in M \cap [\omega]^\omega$ define

$$g(n) = \mu m \in Y \quad m > n .$$

Then for all but finitely many n $g(n) < f(n)$, hence for all but finitely many n, $x_n < g(x_n) < f(x_n) = x_{n+1}$. So both $X \cap Y$ and $Y \setminus X$ are infinite. □

REMARK. A slightly weaker version of Proposition 3.1 was proved by Solomon (1977). This version was noticed by several people including Nyikos, Galvin, Gruenhage, and the author.

REMARK. If $r \subseteq \omega$ is a random real over M, then in $M[r]$ there are no weak dominating reals although r is itself an independent set. If $f \in \omega^\omega$ is a Cohen real, then f is a weak dominating real, it is easy to find an independent set, and there are no dominating reals in $M[f]$.

The next proposition is a weak version of a result of Baumgartner and Laver (1979). It is included here for mainly heuristic reasons.

3.2. PROPOSITION. (Baumgartner and Laver) Perfect set forcing does not add an independent set (i.e. if G is Sacks generic over M, then for any $Y \in [\omega]^\omega \cap M[G]$ there exists $X \in [\omega]^\omega \cap M$ such that $X \subseteq Y$ or $X \cap Y = \emptyset$).

Proof. Suppose

$$p \Vdash "Y \subseteq \omega" .$$

CASE 1. $\exists q \leq p \quad |\{n: q \Vdash n \in Y\}| = \omega$.

CASE 2. $\forall q \leq p \ \forall^{\infty} n \ \exists r \leq q \quad r \Vdash "n \notin Y"$.

($\forall^{\infty} n$ means "for all but finitely many n".)

If Case 1 occurs, we are done. Case 2 requires a fusion argument. Define $p \leq_n q$ iff $p \leq q$ and the first n splitting levels of q are still in p. Then Case 2 implies

$$\forall q \leq p \ \forall m \ \forall^{\infty} n \ \exists r \leq_m q \quad r \Vdash "n \notin X" .$$

This is true because the finite intersection of cofinite sets is cofinite. But now build two sequences $a_1 < a_2 < a_3 < \ldots < a_n < \ldots$ and $p = p_0 \underset{0}{\geq} p_1 \underset{1}{\geq} p_2 \cdots \underset{n-1}{\geq} p_n \underset{n}{\geq} \cdots$ so that for each n

$$p_n \Vdash "a_n \notin X" .$$

Then the fusion $p^* = \bigcap_{n=0}^{\infty} p_n$ forces that "$\{a_n : n = 1, 2, \ldots\} \cap X = \emptyset$". □

REMARK. This proposition is also true for the finite product of Sacks forcing. In fact, it is not hard to see it is equivalent to the Halpern-Läuchli theorem (1966). Laver (1982) has shown that it holds for the ω-product of Sacks forcing (and therefore for arbitrarily large products with countable support).

REMARK. Like random reals, Sacks reals do not add weak dominating reals. Rational perfect set forcing does, of course, add a weak dominating real.

3.3. PROPOSITION. Rational perfect set forcing does not add independent sets.

Proof. The proof breaks down into the following steps. Suppose $p \Vdash "Y \subseteq \omega"$.

3.3.1. STEP. Construct $q \leq p$ and $\langle Y_s : s \in \text{split}(q) \rangle$ so that for all $s \in \text{split}(q)$, for all m and for all but finitely many $n \in X_s^q$

$$q_{s^\frown\langle n \rangle} \Vdash "Y \cap m = Y_s \cap m" .$$

CONSTRUCTION. We use a straightforward fusion argument. Suppose s is the lowest level splitting node of q and we want to retain it. For each $n \in X_s^q$ extend $q_{s^\frown\langle n\rangle}$ to $q^*_{s^\frown\langle n\rangle}$ to decide $Y \cap n$. Choose $a_n \subseteq \omega$ arbitrarily so that

$$q^*_{s^\frown\langle n\rangle} \Vdash \text{"}Y \cap n = a_n \cap n\text{"} .$$

By compactness, $\{a_n : n < \omega\}$ has a convergent subsequence, say $\{a_{k_n} : n < \omega\}$. Suppose $a_{k_n} \to Y_s$ as $n \to \infty$. Let

$$q^* = \{t \in q : t \leq s \text{ or } \exists n \ s^\frown\langle k_n\rangle \leq t\} .$$

Now since $a_{k_n} \to Y_s$, for any $m < \omega$ for all but finitely many n

$$q^*_{s^\frown\langle k_n\rangle} \Vdash \text{"}Y \cap m = Y_s \cap m\text{"} .$$

The fusion argument finishes the construction.

3.3.2. STEP. Obtain $\tilde{p} \leq q$ such that either

(a) $\{Y_s : s \in \text{split}(\tilde{p})\}$ has the finite intersection property; or

(b) $\{\omega \setminus Y_s : s \in \text{split}(\tilde{p})\}$ has the finite intersection property.

CONSTRUCTION. Let U be any nonprincipal ultrafilter on ω. Partition split(q) according to whether $Y_s \in U$. Now apply Claim 2.4.

3.3.3. STEP. Suppose $\{Y_s : s \in \text{split}(\tilde{p})\}$ has the finite intersection property. (The other case is handled analogously.) Build two sequences $a_n < a_{n+1} < \cdots$ and $\tilde{p} \geq q_n \geq q_{n+1} \geq \cdots$ so that for each $n = 1,2,\ldots$

$$q_n \Vdash \text{"}a_n \in Y\text{"}$$

and $\bigcap_{n=0}^{\infty} q_n$ is a condition.

CONSTRUCTION. Along with our two sequences above we build another sequence

$$F_n \subseteq F_{n+1} \subseteq \cdots$$

such that each F_n is a finite subset of split(q_n). Our construction will guarantee that

$$\bigcup_{n=0}^{\infty} F_n = \text{split}\Big(\bigcap_{n=0}^{\infty} q_n\Big) .$$

STAGE. Suppose we have q_{n-1}, a_{n-1}, and F_{n-1}. We first get q_n and $a_n > a_{n-1}$ by pruning finitely nodes from q_{n-1} and retaining $F_{n-1} \subseteq \mathrm{split}(q_n)$ and making $q_n \Vdash$ "$a_n \in Y$". This is done as follows. Choose $a_n \in \bigcap_{s \in F_{n-1}} Y_s$ with $a_n > a_{n-1}$ arbitrarily. (This can be done because of the finite intersection property.) Now let T be the (finite) tree generated by F_{n-1}. Now suppose $s \in T$ is a terminal node (necessarily $s \in F_{n-1}$). Then since $a_n \in Y_s$ for all but finitely many $n \in X_s^{q_{n-1}}$

$$q_{n-1, s^\frown\langle m\rangle} \Vdash \text{"} a_n \in Y \text{"} .$$

Prune q_{n-1} by throwing out the finitely many exceptions. For $s \in F_{n-1}$ which is an interior node of T do exactly the same except also retain in $X_s^{q_n}$ all elements of T which might have been in $X_s^{q_{n-1}}$. Then $q_n \Vdash$ "$a_n \in Y$" and also $F_{n-1} \subseteq \mathrm{split}(q_n)$. Now choose a finite $F_n \supseteq F_{n-1}$ such that $F_n \subseteq \mathrm{split}(q_n)$ and for each $s \in F_{n-1}$ let F_n contain a new "witness" to the fact s is to be a splitting node of the fusion, i.e. for some $i \in \omega$ $s^\frown\langle i\rangle \in q_n \setminus T$ and there exists $t \in F_n$ such that $s^\frown\langle i\rangle \leq t$.

Let $\tilde{q} = \bigcap_{n=0}^{\infty} q_n$ and $X = \{a_n : n = 1, 2, \ldots\}$. Then

$$\tilde{q} \Vdash \text{"} X \subseteq Y \text{"} .$$

In the analogous case ((b) of Step 3.3.2) we would have

$$\tilde{q} \Vdash \text{"} X \cap Y = \emptyset \text{"} .$$

This ends the proof of Proposition 3.3.

REMARK. This proposition is false for the product of rational perfect set forcing. To see this, let $\lhd$ well order $\omega^{<\omega}$. Define

$$f \colon \bigcup_{n<\omega} (\omega^n \times \omega^n) \longrightarrow 2^{<\omega}$$

by

$$f(s^\frown\langle n\rangle, t^\frown\langle m\rangle) = f(s,t)^\frown i^{(n+m)}$$

where

$$i = \begin{cases} 0 & \text{if } s \lhd t \\ 1 & \text{if } t \trianglelefteq s \end{cases}$$

and $i^{(n+m)}$ is a string of $n+m$ i's. Now suppose $g, h \in \omega^\omega$ are $\mathbb{P} \times \mathbb{P}$ generic over M. Let $X \subseteq \omega$ have as its characteristic function

$$\bigcup_{n<\omega} f(g \restriction n, h \restriction n) .$$

I claim that for any infinite $Y \subseteq \omega$ in M both $X \cap Y$ and $Y \setminus X$ are infinite. Note that for any $(p,q) \in \mathbb{P} \times \mathbb{P}$ there is an n such that there exists $s \in \text{split}(p) \cap \omega^n$ and $q \cap \omega^n$ is infinite, hence there exists $t \in q \cap \omega^n$ $s \lhd t$, and therefore X can be made arbitrarily thin. Conversely, there is an n such that there exists $t \in \text{split}(q) \cap \omega^n$ and $p \cap \omega^n$ is infinite, hence there exists $s \in p \cap \omega^n$ $t \ntriangleleft s$, and therefore X can be made arbitrarily thick.

§4. P-POINTS. Recall that an ultrafilter U on ω is a P-point iff whenever given $X_n \in U$ for $n < \omega$ there exists $X \in U$ such that for all $n < \omega$

$$X \subseteq^* X_n$$

(i.e. contained in mod finite). Our next proposition gives a stronger version of Proposition 3.3 assuming there is a P-point in the ground model. In fact, it will give an alternative proof (though less direct).

4.1. PROPOSITION. Suppose U is a P-point and $p \Vdash$ "$Y \subseteq \omega$" (rational perfect set forcing). Then there exists $q \leq p$ and $X \in U$ such that either

$$q \Vdash \text{"}X \subseteq Y\text{"} \quad \text{or}$$

$$q \Vdash \text{"}X \cap Y = \emptyset\text{"} .$$

Proof. Without loss we may assume there exists $Y_s \subseteq \omega$ for $s \in \text{split}(p)$ such that for every $s \in \text{split}(p)$ and $n < \omega$ for all but finitely many $m \in X_s^p$

$$p \Vdash \text{"}Y_s \cap n = Y \cap n\text{"} .$$

Also without loss we can assume there exists $X \in U$ such that for all $s \in \text{split}(p)$, $X \subseteq^* Y_s$. (In the alternative case we would have an $X \in U$ such that $X \subseteq^* (\omega \setminus Y_s)$ for all $s \in \text{split}(p)$.)

We will build a sequence $s_n \in \text{split}(p)$ and $k_n < \omega$ an increasing sequence. In the end both

$$\{s_n : n \text{ even}\} \quad \text{and}$$

$$\{s_n : n \text{ odd}\} ,$$

will be the splitting nodes of some condition. To begin with let $s_0 = s_1 =$ first splitting node of p, and let $k_0 = 0$. Also modify X if necessary so that $X \subseteq Y_{s_0} = Y_{s_1}$.

STAGE n, n even. Find some even $m < n$ such that s_m would like another splitting node beneath it at this stage (dovetailing so that every m is given attention infinitely often). Choose a new $s_n \in \mathrm{split}(p)$ extending s_m and sufficiently far out so that

$$Y_{s_n} \cap k_{n-1} = Y_{s_m} \cap k_{n-1} .$$

Remember that for all but finitely many $k \in X_p^{s_m}$, $p_{s^\frown\langle k\rangle} \Vdash$ "$Y \cap k_{n-1} = Y_{s_m} \cap k_{n-1}$", so that $s_n \geq s_m^\frown\langle k\rangle$ would make

$$Y_{s_n} \cap k_{n-1} = Y_{s_m} \cap k_{n-1} .$$

Now choose $k_n > k_{n-1}$ so that

$$X \setminus k_n \subseteq Y_{s_n} .$$

STAGE n, n odd. Exactly the same but find some odd $m < n$.

Now let $q_0 \leq p$ be the condition whose splitting nodes are $\{s_n : n \text{ even}\}$ and let $q_1 \leq p$ be the condition whose splitting nodes are $\{s_n : n \text{ odd}\}$. And let

$$X_0 = X \cap \bigcup \{[k_n, k_{n+1}) : n \text{ even}\} , \quad \text{and}$$

$$X_1 = X \cap \bigcup \{[k_n, k_{n+1}) : n \text{ odd}\} .$$

Without loss let us assume that $X_0 \in U$. We claim that for every $s \in \mathrm{split}(q_0)$

$$X_0 \subseteq Y_s .$$

(Note that $s = s_n$ for some even n.) We will prove that $X_0 \subseteq Y_{s_n}$ by induction on n. If $n = 0$ we are done by the choice of X. Otherwise we have that $X_0 \setminus k_n \subseteq Y_{s_n}$, $X_0 \cap k_n = X_0 \cap k_{n-1}$, and $Y_{s_n} \cap k_{n-1} = Y_{s_m} \cap k_{n-1}$ for some even $m < n$. But by induction we know that $X_0 \subseteq Y_{s_m}$, so $X_0 \subseteq Y_{s_n}$.

Now we claim that

$$q_0 \Vdash \text{"}X_0 \subseteq Y\text{"} .$$

For suppose otherwise. Then there must be some $\tilde{p} \leq q_0$ and $m \in X_0$ such that

$$\tilde{p} \Vdash \text{"}m \notin Y\text{"} .$$

But let s_n be the first splitting node of $\tilde{p}$. Then for all but finitely many $k \in X_{s_n}^{\tilde{p}}$

$$\tilde{p} \Vdash "Y \cap (m+1) = Y_{s_n} \cap (m+1)" ,$$

contradicting that $X_0 \subseteq Y_{s_n}$. This proves Proposition 4.1. $\square$

REMARK. A similar proof would show the following. Identify $P(\omega)$ with 2^{ω} under the usual topology. Suppose that U is a P-point and $F \subseteq U$ is dense-in-itself. Then there exist $X \in U$ and $\hat{F} \subseteq F$ dense-in-itself such that for all $Y \in \hat{F}$, $X \subseteq Y$. In fact, if we keep in mind the last remark of section two, it is really the same proposition. This proposition gives an alternative proof of a theorem of Kanamori (1978) that for any P-point U if we are given $A_\alpha \in U$ for $\alpha < \omega_1$, then there exists α_n for $n < \omega$ and $X \in U$ such that $X \subseteq A_{\alpha_n}$ for each $n < \omega$. This is true because any uncountable subset of 2^{ω} must contain a subset dense-in-itself.

For G $\mathbb{P}$-generic over M and U an ultrafilter in M define $U^* = \{A \subseteq \omega : \exists B \in U \; B \subseteq A\}$. Then Proposition 4.1 simply says that if $\mathbb{P}$ is rational perfect set forcing and U is a P-point in M, then U^* is an ultra-filter in $M[G]$. In fact, it is a P-point in $M[G]$.

4.2. PROPOSITION. If U is a P-point in M and G is $\mathbb{P}$-generic over M (where $\mathbb{P}$ is rational perfect set forcing), then $U^* = \{A \subseteq \omega : \exists B \in U \; B \subseteq A\}$ is a P-point in $M[G]$.

<u>Proof</u>. Suppose $p \Vdash "\{A_n : n < \omega\}$ is a partition of ω and for all n, $A_n \notin U^*"$. We will find $q \leq p$ and $X \in U$ such that for all n

$$q \Vdash "A_n \cap X \text{ is finite}" .$$

Build a fusion sequence $p_{n+1} \leq p_n$ where the first n splitting levels of p_n (i.e. the splitting levels naturally isomorphism to $\omega^{\leq\omega}$) are still splitting levels of p_{n+1}. Using Proposition 4.1 find $X_s \in U$ and p_n so that for every s on the n^{th} splitting level of p_n:

$$p_{n,s} \Vdash "X_s \cap (A_0 \cup A_1 \cup \ldots \cup A_n) = \emptyset" .$$

Now let $q = \bigcap_{n<\omega} p_n$ be the fusion and find $X \in U$ such that $X \subseteq^* X_s$ for each $s \in \text{split}(q)$. $\square$

REMARKS. These results are analogous to a theorem of Baumgartner and Laver (1979) that under perfect set forcing selective ultrafilters generate selective ultrafilters in the extension. A. Blass pointed out to me that the same is true for P-points by using the above arguments and viewing rational perfect set forcing as "a kind of" subset of perfect set forcing. One big difference is that while Baumgartner and Laver showed that their result continues to hold when perfect set forcing is iterated, I don't know whether P-points continue to generate P-points when either perfect set forcing or rational perfect set forcing is iterated. I would conjecture that they do.

Does there always exist an ultrafilter in the ground model which witnesses the non existence of an independent set by generating an ultrafilter in the extension? The answer is no. This follows from the next proposition which is implicit in Ketonen (1976), and a theorem of Shelah (1982) that it is consistent that there are no P-points.

4.3. PROPOSITION. (Ketonen) Suppose U is an ultrafilter in the ground model M, $f \in \omega^\omega$ is a weak dominating real (i.e. for all $g \in \omega^\omega \cap M$ there are infinitely many n such that $g(n) < f(n)$), and U^* the filter generated by U in $M[G]$ is an ultrafilter. Then U is a P-point in M.

Proof. Suppose $\{X_n : n \in \omega\} \in M$ is a partition of ω such that $X_n \notin U$ for all n. In $M[G]$ let

$$A = \bigcup_{n<\omega} (X_n \cap f(n)) .$$

Since U^* is an ultrafilter there exists $X \in U$ such that either $X \subseteq A$ or $X \cap A = \emptyset$. In the former case we have found $X \in U$ meeting each X_n in a finite set. The latter case cannot happen because f is a weak dominating real. Since X and $\{X_n : n \in \omega\}$ are in M we could define $g \in \omega^\omega$ in M by

$$g^*(n) = \text{least } m \qquad m \in (X \cap X_n)$$

and $g(n) = g^*(m)$ where m is least element if the domain of g^* is greater than or equal to n. Clearly g^* dominates f on its domain and assuming that f is strictly increasing g dominates f everywhere. □

REMARK. Clearly this proposition shows that in the rational perfect set forcing extension the only ultrafilters which remain ultrafilters are the P-points. This is not true for Sacks forcing. Suppose G is Sacks generic over M and U and V are ultrafilters in M such that U^* and V^* (the filters generated by U and V) are ultrafilters in $M[G]$. Then $U \otimes V$ generates an ultrafilter in $M[G]$. Recall that $U \otimes V$ is the ultrafilter on $\omega \times \omega$ defined by $A \in U \otimes V$ iff $\{n \mid \{m : (n,m) \in A\} \in V\} \in U$. It is neither a P-point nor a Q-point.

§5. Q-POINTS AND THE BOREL CONJECTURE. An ultrafilter U on ω is a Q-point iff for any partition $\{X_n : n \in \omega\}$ of ω into finite sets there exists an $X \in U$ such that for each $n < \omega$ $|X_n \cap X| \leq 1$. If Laver or Mathias forcing is iterated, there are no Q-points (see Miller (1980)). This is also true when rational perfect set forcing is iterated. The next proposition shows what can be shown with respect to the simple extension.

5.1. PROPOSITION. Suppose G is $\mathbb{P}$-generic over M where $\mathbb{P}$ is rational perfect set forcing. Then no non principal ultrafilter in M extends to a Q-point in $M[G]$.

Proof. We can assume without loss of generality that the generic real $f \in \omega^\omega$ which generates G is strictly increasing. Consider the partition of ω into finite sets $X_n = [f(n), f(n+1))$. The proposition will follow easily from the following claim.

5.1.1. CLAIM. Suppose $p \Vdash$ "$\forall n\ |X_n \cap Y| \leq 1$". Then there exists Y_0, Y_1, p_0, p_1 such that $Y_0 \cap Y_1$ is finite and for each $i = 0,1$, $p_i \leq p$ and $p_i \Vdash$ "$Y \subseteq Y_i$".

Proof.

Step 1. By an easy fusion argument construct $q \leq p$ such that for every $s \in q$ if $s \in \omega^{n+1}$ then q_s decides $Y \cap s(n)$.

Step 2. Refine q to $\hat{q} \leq q$ which has the property that for each $s \in \text{split}(\hat{q})$ (say $s \in \omega^{n+1}$) wither there exists $a_s \in \omega$ such that for all $n \in X_s^{\hat{q}}$

$$\hat{q}_{s\frown\langle n\rangle} \Vdash \text{"}Y \cap [f(n-1), f(n)) = \{a_s\}\text{"}$$

or any $m \in \omega$ for all but finitely many $n \in X_s^{\hat{q}}$

$$\hat{q}_{s\frown\langle n\rangle} \Vdash \text{"}Y \cap [f(n-1), m) = \emptyset\text{"}.$$

Step 3. Using a finite pruning argument obtain the required $p_0, p_1 \leq \hat{q}$ and Y_0 and Y_1. At stage n suppose we have determined finitely many splitting nodes for each of p_0 and p_1, p_0^n and p_1^n, and finite subsets of ω, Y_0^n and Y_1^n. Suppose we want to find another splitting node beneath a splitting node s of p_0^n. First pick any $m \in X_s^{\hat{q}}$ greater than $\max(Y_1^n)$. Then choose $t \in \text{split}(\hat{q})$ such that $t \geq s\frown\langle m\rangle$. If $\hat{q}_t \Vdash$ "$Y \cap \max(\text{range}(t)) = F$" then let $Y_0^{n+1} = Y_0^n \cup F \cup \{a_t\}$. Now let $Y_1^{n+1} = Y_1^n$, $p_1^{n+1} = p_1^n$, and $p_0^{n+1} = p_0^n \cup \{t\}$. This proves the claim. To prove the proposition proceed as follows. Suppose U is an ultrafilter in M and U^* is a term for any ultrafilter in $M[G]$ which extends U. Suppose for contradiction that $p \Vdash$ "$Y \in U^*$ and $\forall n\ |X_n \cap Y| \leq 1$". Using the claim obtain p_0, p_1, Y_0, Y_1. Since $Y_0 \cap Y_1$ is finite either $Y_0 \notin U$ or $Y_1 \notin U$. Suppose $Y_0 \notin U$. Then

$$p_0 \Vdash \text{"}Y \subseteq Y_0 \wedge Y \in U^*\text{"}$$

a contradiction.

The above argument is essentially the same as for Laver forcing. Unlike Laver or Mathias forcing, the Borel conjecture fails in the rational perfect set forcing model. Recall that a set of reals X is concentrated iff X is uncountable and there exists a countable set of reals Q such that for any open set $U \supseteq Q$ $X \setminus U$ is countable. A concentrated set has strong measure zero.

5.2. PROPOSITION. Suppose $M \models CH$ and $\mathbb{P}$ is rational perfect set forcing. Then there exist $X \in M$ such that for all G $\mathbb{P}$-generic over M

$$M[G] \models \text{"}X \text{ is concentrated"}.$$

Proof. Identify 2^ω and $P(\omega)$ and let $Q \subseteq 2^\omega$ be the set of sequences which are eventually 0. Working in M we will build a set $X = \{A_\alpha : \alpha < \omega_1\} \subseteq P(\omega)$ such that X is concentrated on Q and remains so in $M[G]$.

5.2.1. CLAIM. Suppose $k < \omega$ and $p \Vdash \text{"}Q \subseteq U^{open}\text{"}$. Then there exists $q \leq p$ and $\hat{k} > k$ such that for every $t \in 2^k$

$$q \Vdash \text{"}[t^\frown 0^{\hat{k}-k}] \subseteq U\text{"}$$

where

$$[t^\frown 0^{\hat{k}-k}] = \{x \in 2^\omega : x \restriction \ell = t \wedge x(i) = 0 \text{ for } k \leq i \leq \hat{k}\}.$$

5.2.2. CLAIM. Suppose $A \in [\omega]^\omega$ and $p \Vdash \text{"}Q \subseteq U^{open}\text{"}$. Then there exists $q \leq p$ and $B \in [A]^\omega$ such that for all $C \subseteq^* B$

$$q \Vdash \text{"}C \in U\text{"}.$$

Proof. Construct two sequences $k_n < k_{n+1} \in A$ and $p \geq q_0 \geq q_1 \geq q_2 \cdots$, so that $q_{n,s_n} \Vdash \text{"}\forall t \in 2^{k_{n-1}} [t^\frown 0^{k_n - k_{n-1}}] \subseteq U\text{"}$, where s_n is the n^{th} splitting node of q_n and q_n retains the first $n-1$ splitting nodes of q_{n-1}. Now let $q = \bigcap_{n<\omega} q_n$ and let $B = \{k_n : n < \omega\}$. Suppose for contradiction there exists $C \subseteq^* B$ and $r \leq q$

$$r \Vdash \text{"}C \notin U\text{"}.$$

Then we can find some $n < \omega$ such that $C \setminus k_n \subseteq B$ and the n^{th} splitting node s_n of q is still in r. But we know that if $t = C \restriction k_n$, then

$$q_{s_n} \Vdash \text{"}[t\hat{\ }0^{k_{n+1}-k_n}] \subseteq U\text{"}$$

and $C \in [t\hat{\ }0^{k_{n+1}-k_n}]$, a contradiction.

Now we prove Proposition 5.2. Let $\{(p_\alpha, U_\alpha): \alpha<\omega_1\}$ be such that for every $p \in \mathbb{P}$ and term U if

$$p \Vdash \text{"}Q \subseteq U^{\text{open}}\text{"}$$

then there exists $\alpha < \omega_1$ such that $p_\alpha \leq p$ and $p_\alpha \Vdash \text{"}U = U_\alpha\text{"}$. Using Claim Claim 5.2.2 construct $X = \{X_\alpha: \alpha < \omega_1\} \subseteq P(\omega)$ so that $\alpha < \beta$ implies $X_\alpha \supseteq^* X_\beta$ and for each $\alpha < \omega_1$ there exists $q \leq p$ such that for all $C \subseteq^* X_{\alpha+1}$

$$q \Vdash \text{"}C \in U_\alpha\text{"} .$$

It is easy to see that X remains concentrated on Q in $M[G]$.

REMARK. It is not true that every meager set in $M[G]$ is covered by a meager set coded in M. This follows from the fact that there is a weak dominating real (see Miller (1983), wD $\rightarrow$ C(c)). It is not hard to show however that the ground model reals do not have first category.

Proposition 5.2 is also true when rational perfect set forcing is iterated with countable support.

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