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ON Q SETS

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ABSTRACT. A Q set is an uncountable set X of the real line such that every subset of X is an F_σ relative to X. It is known that the existence of a Q set is independent of and consistent with the usual axioms of set theory. We show that one cannot prove, using the usual axioms of set theory: 1. If X is a Q set then any set of reals of cardinality less than the cardinality of X is a Q set. 2. The union of a Q set and a countable set is a Q set.

The existence of a Q set is a fundamental question of set theory considered by Hausdorff [1], Sierpinski [2] and Rothberger [3] over thirty years ago, and by many others since [4]–[7], [11]. A Q set is an uncountable subset X of the real line $\mathbf{R}$ such that every subset of X is an F_σ relative to X. Precisely, for every $A \subset X$, there are countably many closed subsets H_n, $n \in \omega$, of $\mathbf{R}$ such that $\bigcup\{H_n: n \in \omega\} \cap X = A$.

The fundamental nature of the existence of Q sets is illustrated by its equivalence with and implications with varied and apparently unrelated questions. Theorems 1–3 are a sample of the work cited above.

THEOREM 1. *The following statements about a subset X of $\mathbf{R}$ are equivalent.*

(a) *X is a Q set.*

(b) *There is a countable family $\mathcal{F}$ of continuous real-valued functions defined on X such that every real-valued function defined on X is the pointwise limit of a sequence of functions from $\mathcal{F}$.*

(c) *The "bubble space" constructed from X is a separable normal nonmetrizable Moore space.*

THEOREM 2. *Each of the following statements implies the next.*

(a) *Martin's Axiom plus the negation of the Continuum Hypothesis.*

(b) *There are no Ω limits in the partial order $\langle \mathcal{P}(\omega), \subseteq \bmod \textit{finite}\rangle$ (in the sense of Hausdorff).*

(c) *Every subset of $\mathbf{R}$ of cardinality ω_1 is a Q set.*

(d) *There is a Q set X such that for all $n \in \omega$ X^n is homeomorphic to a Q set.*

(e) *There is a normal not separable Moore space in which every collection of pointwise disjoint open sets is countable.*

(f) $2^\omega = 2^{\omega_1}$.

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THEOREM 3. *Suppose for some subset X of $\mathbf{R}$ of cardinality ω_1 there is a countable family $\mathcal{E}$ of subsets of X such that every subset A of X can be expressed in the form $A = \bigcap \{\bigcup \{E_j: i < j < \omega\}, i < \omega\}$ where each $E_j \in \mathcal{E}$. Then* (a) *every subset Y of $\mathbf{R}$ of cardinality ω_1 has this property, and* (b) *there is a Q set.*

It is tempting to conjecture that Theorem 3 can be strengthened by replacing (b) with (b′) every subset Y of $\mathbf{R}$ of cardinality ω_1 is a Q set.

Let us consider the set I of subsets X of $\mathbf{R}$ such that every subset A of X is an F_σ relative to X. That is, I is the set of subsets of $\mathbf{R}$ which are either countable or Q sets. In several senses, I is a family of small sets. Every member of I is first category, measure zero, and has cardinality less then 2^ω.

Assuming $2^\omega < 2^{\omega_1}$, or Martin's Axiom, or certain modifications of Martin's Axiom, I is the ideal of subsets of $\mathbf{R}$ of cardinality less than some cardinal. In this paper, we answer negatively the following two questions: 1. If $X \in I$, must every set of smaller cardinality be in I? 2. Must I be an ideal? Our answers will be in the form of describing how to extend a countable transitive model of set theory to a model in which: 1. There is a Q set of cardinality ω_2 and a non Q set of cardinality ω_1 and/or 2. There is a Q set X and a countable set F such that $X \cup F$ is not a Q set. Using standard techniques our answers can be recast as relative consistency results or proofs of certain statements of arithmetic.

Our terminology and notation conform with current set theoretic practice. We assume familiarity with Cohen's method of forcing [**8**], and the method of iterated forcing [**9**].

We begin by describing the basic step which is iterated. Let $\mathcal{B} = \{B_n: n \in \omega\}$ be a basis for the usual topology on $\mathbf{R}$, with $\mathbf{R} \in \mathcal{B}$. Let $A \subset X \subset \mathbf{R}$. We will define a notion of forcing which makes A a relative F_σ in X. Let $\mathbf{P}(A, X)$ be the set of r satisfying

1. r is finite.

2. $r \subset \omega \times (\mathcal{B} \cup A)$.

3. If $\langle n, B\rangle, \langle n, x\rangle \in r$ (where $B \in \mathcal{B}$ and $x \in A$) then $x \notin B$. (We think of $\langle n, B\rangle \in r$ as saying that $B \subseteq U_n$ and $\langle n, a\rangle \in r$ as saying that $a \notin U_n$.)

We say that r' extends r if $r' \supseteq r$.

Let M be a countable transitive model of set theory, and let G be an M-generic ultrafilter on $\mathbf{P}(A, X)$. In the extension, $M[G]$, set for each $n \in \omega$ $U_n = \{x \in X: \exists r \in G\ \exists B \in \mathcal{B}, \langle n, B\rangle \in r$ and $x \in B\}$, and set $K = \bigcup \{(X - U_n): n \in \omega\}$. Then each U_n is open relative to X, and K is an F_σ relative to X. We verify that $K = A$, using the fact that the generic ultrafilter G meets every dense set of $\mathbf{P}(A, X)$ in M. For all $a \in A$, $r \in \mathbf{P}(A, X)$, there are $r' \supseteq r$ and $n \in \omega$ such that $\langle n, a\rangle \in r'$. Then $a \in X - U_n \subset K$. For all $x \in X - A$, $r \in \mathbf{P}(A, X)$, and $n \in \omega$ there are $r' \supseteq r$ and $B \in \mathcal{B}$ such that $\langle n, B\rangle \in r'$ and $x \in B$. Then for all n, $x \in U_n$. Note that for any $r, s \in \mathbf{P}(A, X)$ if $r \cap (\omega \times \mathcal{B}) = s \cap (\omega \times \mathcal{B})$ then $r \cup s \in \mathbf{P}(A, X)$. It follows that $\mathbf{P}(A, X)$ has the countable antichain condition–any set of incompatible elements of $\mathbf{P}(A, X)$ is countable.

We next discuss how to iterate the basic step, described above. We will explicitly describe how to construct an extension in which there is a Q set X and a countable set F such that $X \cup F$ is not a Q set. Afterwards, we will explain the modifications

necessary to construct an extension in which there is a Q set of cardinality ω_2 and a non Q set of cardinality ω_1.

Let F be a countable dense subset of $\mathbf{R}$, disjoint from the frontier of each $B_n \in \mathcal{B}$. The iterated forcing will be first to add a set Y of ω_1 Cohen reals and then iterate $\mathbf{P}(A_\alpha \cup F, Y \cup F)$ forcing, where A_α, $\alpha < \omega_2$, is chosen to list all the subsets of Y *in the extension.*

One's first attempt might be to do this iteration in exact analogy with [**9**] to get a Boolean algebra $\mathbf{B}$. This works, but in order to show that $Y \cup F$ is not a Q set we need to work with forcing conditions which display the basic steps. We will construct a partial order $\mathbf{P}$ which can be embedded as a dense subset in such a $\mathbf{B}$.

In our particular situation, we can and do avoid the machinery of iterated forcing. To be more precise, we do define $\mathbf{P}$ by induction, but we do not use analogues of §5 of [**9**] nor do we explicitly consider intermediate models.

Assume that $2^{\omega_1} = \omega_2$ in M, the ground model. A forcing condition will be a pair $\langle p, r\rangle$ where $p = \langle p_\beta : \beta < \omega_1\rangle$ is a sequence of basic open sets from B and where $r = \langle r_\gamma : \gamma < \omega_2\rangle$, with each r_γ in something like $\mathbf{P}(A, X)$ above. In the extension, $M[G]$, we will set $\{y_\beta\} = \bigcap \{p_\beta : \langle p, r\rangle \in G\}$ and will set $Y = \{y_\beta : \beta < \omega_1\}$. Thus the p_β's add the Cohen real y_β, and the r's will make Y into a Q set because each r_γ will be in a partial order making a particular subset of Y a relative F_σ. Because the y_β's are not in the ground model, we will not use $\mathbf{P}(A, X)$ as defined above but rather a version with y_β replaced by β.

We begin by defining a partial order $\mathbf{Q}$ which will contain the desired $\mathbf{P}$. Let $\mathbf{Q}$ be the set of pairs $\langle p, r\rangle$ where

4. $p = \langle p_\beta : \beta < \omega_1\rangle, p_\beta \in \mathcal{B}$,

5. $r = \langle r_\gamma : \gamma < \omega_2\rangle$, r_γ a finite subset of $\omega \times (\mathcal{B} \cup \omega_1 \cup F)$,

6. $\{\beta < \omega_1 : p_\beta \neq \mathbf{R}\} \cup \{\gamma < \omega_2 : r_\gamma \neq \varnothing\}$ is finite,

7. p forces that each $r_\gamma \in \mathbf{P}(Y \cup F, Y \cup F)$; explicitly:

(a) If $\langle n, B\rangle, \langle n, f\rangle \in r_\gamma$, where $n \in \omega$, $B \in \mathcal{B}$, $f \in F$, then $f \notin B$.

(b) If $\langle n, B\rangle, \langle n, \beta\rangle \in r_\gamma$, where $n \in \omega$, $B \in \mathcal{B}$, $\gamma < \omega_2$, then $p_\beta \cap B = \varnothing$.

We say that $\langle p', r'\rangle$ extends $\langle p, r\rangle$ if $p'_\beta \subseteq p_\beta$ for all $\beta < \omega_1$ and $r'_\gamma \supseteq r_\gamma$ for all $\gamma < \omega_2$.

We now attack the problem of listing in M all subsets of ω_1 in the extension $M[G]$. We call $\{q_{i\beta} : i < \omega, \beta < \omega_1\}$ a shadow if for each $\beta < \omega_1$, $\{q_{i\beta} : i < \omega\}$ is a set of pairwise incompatible elements of $\mathbf{Q}$. (Intuitively, the meaning is "$q_{i\beta}$ forces $\beta \in A$".) Since $2^{\omega_1} = \omega_2$, there is $\langle q^\eta_{i\beta} : i < \omega, \beta < \omega_1, \eta < \omega_2\rangle$ which lists each shadow ω_2 times.

By induction on $\alpha \leqslant \omega_2$ we define subsets $\mathbf{P}_\alpha$ of $\mathbf{Q}$. $\mathbf{P}_0 = \{\langle p, r\rangle \in \mathbf{Q}$: for all $\gamma < \omega_2$, $r_\gamma = \varnothing\}$. Assume that $\mathbf{P}_\delta$ has been defined for all $\delta < \alpha$. If α is a limit ordinal, let $\mathbf{P}_\alpha = \bigcup \{\mathbf{P}_\delta : \delta < \alpha\}$. If $\alpha = \eta + 1$ there are two cases. First, if $\{q^\eta_{i\beta} : i < \omega, \beta < \omega_1\}$ is *not* a subset of $\mathbf{P}_\eta$, let $\mathbf{P}_\alpha = \mathbf{P}_\eta$. Second, if $\{q^\eta_{i\beta} : i < \omega, \beta < \omega_1\}$ is a subset of $\mathbf{P}_\eta$ define $\mathbf{P}_\alpha$ to be those pairs $\langle p, r\rangle$ in $\mathbf{Q}$ such that

8. $\forall \gamma > \eta$, $r_\gamma = \varnothing$ and $\langle p, r'\rangle \in \mathbf{P}_\eta$ where $r'_\gamma = r_\gamma$ for $\gamma \neq \eta$ and $r'_\eta = \varnothing$.

9. If for some $n < \omega$, $\beta < \omega_1$, $\langle n, \beta\rangle \in r_\eta$, then for some $i < \omega$, $\langle p, r\rangle$ extends $q^\eta_{i\beta}$.

Set $\mathbf{P} = \mathbf{P}_{\omega_2}$; let G be M-generic over $\mathbf{P}$. A standard counting argument shows that

10. Every set of pairwise incompatible elements of $\mathbf{P}$ is countable.

One consequence of 10 is that M and $M[G]$ have the same cardinals. A second consequence is that for every subset A of ω_1 in $M[G]$ there is an $\eta < \omega_2$ such that for each $\beta < \omega_1$, $\{q^\eta_{i\beta}: i < \omega\} \subseteq \mathbf{P}_\eta$ and $\beta \in A$ iff for some i, $q^\eta_{i\beta} \in G$. For such an η, set for each n

$$U_n = \{y \in Y: \exists \langle p, r\rangle \in G, \langle n, B\rangle \in r_\eta \text{ and } y \in B\}.$$

We can verify as in the basic step that $\{y_\beta: \beta \in A\} = \bigcup \{Y - U_n: n \in \omega\}$. Thus Y is a Q set.

We must show that $Y \cup F$ is not a Q set. First we demonstrate that it will be sufficient to establish

11. If U is an open set containing F, then $Y - U$ is countable. For then if H is a G_δ containing F, then $Y - H$ is countable. This establishes that Y is not an F_σ relative to $Y \cup F$, and hence that $Y \cup F$ is not a Q set.

Towards establishing 11, let U be an open set containing F. For each $n \in \omega$, let W_n be a maximal pairwise incompatible set of conditions forcing $B_n \subset U$. By 10, 1, and 6, there is a countable set J such that if anything at all is said about y_β in any element of $\bigcup_{n<\omega} W_n$ then $\beta \in J$–that is if $\langle p, r\rangle \in \bigcup_{n<\omega} W_n$ and $p_\beta \neq \mathbf{R}$ or there is $\gamma < \omega_2$ and $m < \omega$ such that $\langle m, \beta\rangle \in r_\gamma$, then $\beta \in J$. We will show that $Y - J \subset U$, establishing 11.

Let $y_\delta \in Y$, $\delta \notin J$, and $\langle p, r\rangle \in \mathbf{P}$. Because F is an infinite dense set, there is $f \in F$ such that $f \in p_\delta$ and $\langle n, f\rangle \notin r_\alpha$ for all α and n. Extend $\langle p, r\rangle$ to $\langle p^1, r^1\rangle$ by setting $p^1_\beta = p_\beta$, and $r^1_\alpha = r_\alpha \cup \{(n, f): (n, \delta) \in r_\alpha\}$. Because U is forced to contain F, some extension of $\langle p^1, r^1\rangle$ forces $f \in B_k \subset U \cap p_\delta$ for some $k \in \omega$. Because W_k is maximal, some $\langle p^2, r^2\rangle \in W_k$ is compatible with $\langle p^1, r^1\rangle$. We can verify that there is in $\mathbf{P}$ a condition $\langle p^3, r^3\rangle$ satisfying $p^3_\beta \subseteq p^1_\beta \cap p^2_\beta$, $r^3_\alpha = r^1_\alpha \cup r^2_\alpha$. (We cannot say $p^3_\beta = p^1_\beta \cap p^2_\beta$ because "p^3_β's knowledge of y_β is incomplete" and p^3_β must force

$$\text{if } \langle n, \beta\rangle \in r^1_\alpha \cup r^2_\alpha \text{ and } \langle n, B\rangle \in r^1_\alpha \cup r^2_\alpha \text{ then } y_\beta \notin B. \qquad (*)$$

If $p^3_\delta \neq p^1_\delta \cap p^2_\delta$ it may happen that $p^3_\delta \cap B_k = \varnothing$.)

We choose $B_j \subseteq B_k$ so that whenever $\langle n, f\rangle \in r^1_\alpha$ and $\langle n, B_i\rangle \in r^2_\alpha$ then $B_j \cap B_i = \varnothing$. Define $\langle p^4, r^4\rangle$ by setting $p^4_\delta = B_j$, $p^4_\beta = p^3_\beta$ if $\beta \neq \delta$, and $r^4 = r^3$. We verify that $\langle p^4, r^4\rangle \in \mathbf{P}$. Everything follows from the fact that $\langle p^3, r^3\rangle \in \mathbf{P}$ except that $(*)$ above. We check this last case by noting that $\langle n, \delta\rangle \notin r^2_\alpha$ because $y_\delta \notin J$, and if $\langle n, \delta\rangle \in r^1_\alpha$ then $B_j \cap B = \varnothing$.

Now $\langle p^4, r^4\rangle$ extends $\langle p, r\rangle$ and $\langle p^2, r^2\rangle$ since $B_j \subset B_k \subset p_\delta$ and $p^2_\delta = \mathbf{R}$ since $\delta \notin J$. So $\langle p^4, r^4\rangle$ forces $y_\delta \in U$, verifying 11.

To construct a model in which there is a Q set of cardinality ω_2 and a non Q set of cardinality ω_1, we start with a model M in which $2^\omega = \omega_2$, $2^{\omega_2} = \omega_3$. Choose a set $X \in M$ of cardinality ω_2. The extension is to first add a set Y of ω_1 Cohen reals and then to iterate $\mathbf{P}(A_\alpha, X)$ forcing, where A_α, $\alpha < \omega_3$, lists all subsets of X in the extension. Then in the extension X will be a Q set of cardinality ω_2. That Y is not a Q set in this extension follows from the fact that if U is a dense open set, then

$Y - U$ is countable. This fact is verified in the same manner as 11 was. This verification is simpler because we can choose $p_\beta^3 = p_\beta^1 \cap p_\beta^2$.

For concreteness, we have constructed the extension using specific cardinals. Of course similar extensions may be constructed with ω_{17}, say, in place of ω_2. Further, the two extensions discussed in this paper can be combined. Some other applications of our technique are to show that (d) of Theorem 2 does not imply (c), and combined with the techniques of [13] we can show that it is consistent that there is a concentrated space of Baire order ω_1 (see [12]).

The following question remains open (see Rudin [10]): Is the product of two Q sets a Q set? (Since this paper was written, the first author has used the techniques of this paper to answer "not necessarily".)

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