A product of γ-sets which is not Menger.

A. Miller
Dec 2009

Theorem. Assume CH. Then there exists γ-sets $A_0, A_1 \subseteq 2^\omega$ such that $A_0 \times A_1$ is not Menger.

We use perfect sets determined by Silver forcing (see Grigorieff [3]) and construct an Aronszajn-tree of such perfect sets in the style of Todorcevic (see Galvin-Miller [2]).

Define $p \in \mathbb{P}$ iff $p : D \to 2$ where $D \subseteq \omega$ is co-infinite, i.e., $\overline{D} = \omega \setminus D$ is infinite. For $p \in \mathbb{P}$ define

$$[p] = \{x \in 2^\omega \ : \ p \subseteq x\}.$$

Define $p \leq q$ iff[1] $p \supseteq q$ or equivalently $[p] \subseteq [q]$.

For $n < \omega$ define $p \leq_n q$ iff $p \leq q$ and the first n elements of $\overline{D_p}$ are the same as the first n elements of $\overline{D_q}$. The fusion property is important:

> **Fusion Lemma**. Suppose $(p_n \in \mathbb{P} : n < \omega)$ has the property that $p_{n+1} \leq_n p_n$ for every $n < \omega$. Then the fusion $q = \bigcup_{n<\omega} p_n$ is in $\mathbb{P}$ and $q \leq_n p_n$ for every n.

Define

$$Q^0(p) = \{x \in [p] : \forall^\infty n \in \overline{D_p} \ \ x(n) = 0\}$$

and

$$Q^1(p) = \{x \in [p] : \forall^\infty n \in \overline{D_p} \ \ x(n) = 1\}.$$

Each of these are countable dense subsets of $[p]$.

Define $q \leq_n^0 p$ iff $q \leq_n p$ and q restricted to $D_q \setminus D_p$ is identically zero.

Lemma 1 *Given $\mathcal{U}$ an ω-cover of $Q^0(p)$ and $n < \omega$ there exist $U \in \mathcal{U}$ and $q \leq_n^0 p$ such that $[q] \subseteq U$.*

[1] We follow the convention that $p \leq q$ means that p is stronger than q or equivalently that p extends q.

Proof
To see why this is true let F be the first n elements of $\overline{D_p}$. For each $s \in 2^F$ let $x_s \in Q^0(p)$ such that $x_s \restriction F = s$ and $x_s(n) = 0$ for every $n \in \overline{D_p} \setminus F$. Take $U \in \mathcal{U}$ with $\{x_s : s \in 2^F\} \subseteq U$. Since U is open there exists $N < \omega$ with $[x_s \restriction N] \subseteq U$ for all $s \in 2^F$. Define $q \leq_n^0 p$ by

$$q = p \cup \{\langle k, 0\rangle : k < N \text{ and } k \in (\overline{D_p} \setminus F)\}$$

QED

Lemma 2 *Given $p_n \in \mathbb{P}$ and $k_n < \omega$ for $n < N$ and an ω-cover $\mathcal{U}$ of $\bigcup_{n<N} Q^0(p_n)$ there exists $U \in \mathcal{U}$ and $(q_n \leq_{k_n}^0 p_n : n < N)$ such that*

$$\bigcup_{n<N} [q_n] \subseteq U.$$

Proof
Let F_n be the first k_n elements of $\overline{D_{p_n}}$. For $s \in 2^{F_n}$ define $x_s^n \in Q^0(p_n)$ as in the proof of Lemma 1. Let $H \subseteq \bigcup_{n<N} Q^0(p_n)$ be a finite set containing all such x_s^n. Choose $U \in \mathcal{U}$ with $H \subseteq U$ and determine the q_n for $n < N$ as in Lemma 1.
QED

Remark. Note that if $q \leq_k^0 p$ then $Q^0(q) \subseteq Q^0(p)$ and hence any ω-cover of $Q^0(p)$ is still an ω-cover of $Q^0(q)$. In these two lemmas, the q we obtain are also equal mod finite to the p, which also implies this.

Lemma 3 *Given $(p_n, k_n : n \in \omega)$ elements of $\mathbb{P} \times \omega$ and $(\mathcal{U}_n : n < \omega)$ which are ω-covers of $Q = \bigcup_{n<\omega} Q^0(p_n)$ there exists $(U_m \in \mathcal{U}_m : m < \omega)$ and $(q_n \leq_{k_n} p_n : n \in \omega)$ such that*

$$\forall n < \omega \;\; \forall m \geq n \quad [q_n] \subseteq U_m.$$

Proof
Construct $(q_n^m : n, m < \omega)$ and $(U_n \in \mathcal{U}_n : n < \omega)$ inductively. Given $(q_n^m : n < \omega)$ and $(U_n : n < m)$, we construct q_n^{m+1} and $U_m \in \mathcal{U}_m$ so that

1. $q_n^{m+1} = p_n$ for $n \geq m + 1$,

2. $q_n^{m+1} \leq^0_{k_n+m} q_n^m$ for $n \leq m$, and

3. $[q_n^{m+1}] \subseteq U_m$ for $n \leq m$.

Then we let $q_n = \bigcup_{m>n} q_n^m$ be the fusion. We have that $q_n \leq_{k_n} q_n^n = p_n$ and $[q_n] \subseteq U_m$ whenever $m \geq n$.
QED

Remark. Obviously, the analogue of this Lemma for Q^1 is also true.

Lemma 4 *Suppose $(p_n, k_n : n \in \omega)$ are elements of $\mathbb{P} \times \omega$. Then there exists $(q_n \leq_{k_n} p_n : n \in \omega)$ such that for $n \neq m$, q_n and q_m are strongly disjoint, i.e., there are infinitely many $k \in (D_{q_n} \cap D_{q_m})$ with $q_n(k) \neq q_m(k)$.*

Proof
Given p_1, p_2 and n it is easy to find $q_1 \leq_n p_1$ and $q_2 \leq_n p_2$ which are strongly disjoint. A fusion argument produces a sequence $(q_n : n < \omega)$ where all pairs have been considered and made strongly disjoint.
QED

Now we construct the Aronszajn tree of Silver conditions. Let $\mathcal{U}_\alpha$ for $\alpha < \omega_1$ list all ω sequences of countable families of open subsets of 2^ω. Make sure that each such sequence occurs as a $\mathcal{U}_\alpha$ for both α even and α-odd.

We can construct a tree $T \subseteq \omega^{<\omega_1}$ and $(p_s \in \mathbb{P} \ : \ s \in T)$ which has the following properties.

1. $T \subseteq \omega^{<\omega_1}$ is a subtree, i.e., $s \subseteq t \in T$ implies $s \in T$.

2. $T_\alpha = T \cap \omega^\alpha$ is countable for each $\alpha < \omega_1$.

3. $s \subseteq t \in T$ implies $p_t \leq p_s$.

4. If $s, t \in T$ are incomparable, then p_s and p_t are strongly disjoint (as in Lemma 4).

5. For any $\alpha < \beta < \omega_1$ and any $s \in T_\alpha$ and $n < \omega$ there exists $t \in T_\beta$ with $p_t \leq_n p_s$.

6. Define

$$Q^0_\alpha = \bigcup \{Q^0(p_t) : t \in T_{\leq\alpha}\} \text{ and } Q^1_\alpha = \bigcup \{Q^1(p_t) : t \in T_{\leq\alpha}\}.$$

(a) For α an even ordinal, if $\mathcal{U}_\alpha = (\mathcal{U}_n^\alpha : n < \omega)$ is a sequence of ω-covers of Q_α^0 then there exists a sequence $(U_n \in \mathcal{U}_n^\alpha : n < \omega)$ which is a γ-cover of

$$Q_\alpha^0 \cup \bigcup \{[p_s] : s \in T_{\alpha+1}\}.$$

(b) For α odd, the analogous statement is true but with Q_α^1 in place of Q_α^0.

7. Let $\mathcal{D}$ be the family of $f \in \omega^\omega$ such that for some $s \in T$ the function $f : \omega \to \overline{D_{p_s}}$ is the unique order preserving bijection. Then $\mathcal{D}$ is a dominating family, i.e., for all $g \in \omega^\omega$ there exists $f \in \mathcal{D}$ such that $g(n) < f(n)$ for all $n < \omega$.

To construct T_λ and p_s for $s \in T_\lambda$ where λ is a countable limit ordinal, proceed as follows. For any $s \in T_{<\lambda}$ and $N < \omega$ choose a strictly increasing sequence λ_n cofinal in λ with $s = s_0 \in T_{\lambda_0}$. By inductive hypothesis we can find $s_n \in T_{\lambda_n}$ with $p_{s_{n+1}} \leq_{N+n} p_{s_n}$. Take p_t for $t = \bigcup_{n<\omega} s_n$ to be the fusion of this sequence. Repeat countably many times, to take care of all $s \in T_{<\lambda}$ and $N < \omega$.

At successor stages for α even, check to see if $\mathcal{U}_\alpha$ is an ω-sequence of ω-covers of Q_α^0. If it is not, we need never worry about it since the set we are building will contain Q_α^0. If it is, let $\{x_n : n < \omega\} = Q_\alpha^0$ and let

$$\mathcal{U}_n = \{U \in \mathcal{U}_n^\alpha \ : \ \{x_i : i < n\} \subseteq U\}.$$

Let $(p_n, k_n : n < \omega)$ list all elements of

$$\{p_s : s \in T_\alpha\} \times \omega$$

with infinite repetitions and apply Lemma 3 followed by Lemma 4. From the resulting sequence we may find for each $s \in T_\alpha$ and $n < \omega$ a distinct condition $q \leq_n p_s$ which we label $p_{s^\frown\langle n\rangle}$. Obtaining the last condition (7), is easy (in fact, it seems hard to avoid), and is left to the reader.

The γ sets are the sets:

$$A_0 = \bigcup_{s\in T} Q^0(p_s) \qquad \text{and} \qquad A_1 = \bigcup_{s\in T} Q^1(p_s).$$

For $x \in A_0$ and $y \in A_1$ we note that there are infinitely many n with $x(n) \neq y(n)$. To see this note that if $x \in Q^0(p_s)$ and $y \in Q^1(p_t)$, and s and t are incomparable, then p_s and p_t are strongly disjoint. On the other hand, if they are comparable, for example, $s \subseteq t$, then since $p_t \leq p_s$, we have that $\overline{D_{p_t}} \subseteq \overline{D_{p_s}}$. So for all but finitely many $n \in \overline{D_{p_t}}$ we will have that $y(n) = 1$ and $x(n) = 0$.

Condition (7) gives us a continuous map from $A_0 \times A_1$ onto a dominating family $\mathcal{D} \subseteq \omega^\omega$. Namely, if $x_0 \in Q^0(p_s)$ is identically zero on $\overline{D_{p_s}}$ and $x_1 \in Q^1(p_s)$ identically one on $\overline{D_{p_s}}$, then $\overline{D_{p_s}} = \{n : x_0(n) \neq x_1(n)\}$. So the continuous map is $\phi : A_0 \times A_1 \to \omega^\omega$ defined by $\phi(x, y) = f$ where f is the unique order preserving map from ω to the infinite set $\{n : x(n) \neq y(n)\}$.
QED

Remark. Our result also shows that assuming CH there are Borel-cover γ-sets whose product is not Menger. To see this note the following:

Lemma 5 *Suppose $p \in \mathbb{P}$, $n < \omega$, and $B \subseteq 2^\omega$ a Borel set. Then there exists $q \leq_n p$ such that $[q] \cap B$ is clopen in $[q]$.*

Proof

Let F be the first n elements of $\overline{D_p}$ and let $\phi : \omega \to (\overline{D_p} \setminus F)$ be a bijection. For $X \subseteq \omega$ let $\psi_X : (\overline{D_p} \setminus F) \to 2$ be the restriction of the characteristic function of $\phi(X)$. For each $s \in 2^F$ define

$$C_s = \{X \in [\omega]^\omega \ : \ (p \cup s \cup \psi_X) \in B\}.$$

Since these are Borel sets by the Galvin-Prikry Theorem [1] there exists $H \in [\omega]^\omega$ such that for each $s \in 2^F$ either $[H]^\omega \subseteq C_s$ or $[H]^\omega \cap C_s = \emptyset$. Let $H_1 \subseteq H$ be infinite such that $H \setminus H_1$ is also infinite. Let

$$q = p \cup (\phi(\overline{H}) \times \{0\}) \cup (\phi(H_1) \times \{1\}).$$

Note that $\overline{D_q} = F \cup \phi(H \setminus H_1)$. We claim that given any $x, y \in [q]$ if $x \restriction F = y \restriction F = s$, then $x \in B$ iff $y \in B$. Letting $H_x = \phi^{-1}(x^{-1}(1))$ we see that $H_1 \subseteq H_x \subseteq H$ and so H_x is an infinite subset of H. Similarly for H_y. By choice of H we have that $H_x \in C_s$ iff $H_y \in C_s$ and so the claim follows.
QED

Modify our construction in the open cover case as follows. Let B_β for $\beta < \omega_1$ list all Borel sets. At the end of each stage β apply Lemma 5 accross the level to make sure that $[p_s] \cap B_\beta$ is relatively clopen for each $s \in T_{\beta+1}$.

Let $\mathcal{B}_\alpha = (\mathcal{B}_\alpha^n : n < \omega)$ for $\alpha < \omega_1$ all ω sequences of countable families of Borel sets. We may assume that each element of $\bigcup \mathcal{B}_\alpha$ has already been listed as a B_β for some $\beta < \alpha$.

At successor stages for α even, check to see if $\mathcal{B}_\alpha = (\mathcal{B}_n^\alpha : n < \omega)$ is an ω-sequence of countable Borel ω-covers of Q_α^0. If it is not, we need never worry about it since the set we are building will contain Q_α^0. If it is, let $\{x_n : n < \omega\} = Q_\alpha^0$ and let

$$\mathcal{B}_n = \{U \in \mathcal{B}_n^\alpha \ : \ \{x_i : i < n\} \subseteq U\}.$$

Since the elements of each $\mathcal{B}_n$ are relatively open in the $[p_s]$ the rest of the argument is the same as in the open case.

References

[1] Galvin, Fred; Prikry, Karel; Borel sets and Ramsey's theorem. J. Symbolic Logic 38 (1973), 193–198.

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