Math. Proc. Camb. Phil. Soc. (1983), **94**, 399
Printed in Great Britain

Borel and projective sets from the point of view of compact sets

By KENNETH KUNEN

Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706

AND ARNOLD W. MILLER

Department of Mathematics, The University of Texas, Austin, Texas 78712

(*Received* 1 *November* 1982)

In this paper we prove several results concerning the complexity of a set relative to compact sets. We prove that for any Polish space X and Borel set $B \subseteq X$, if B is not $\mathbf{\Pi}^0_\alpha$, then there exists a compact zero-dimensional $P \subseteq X$ such that $P \cap X$ is not $\mathbf{\Pi}^0_\alpha$. We also show that it is consistent with ZFC that, for any $A \subseteq \omega^\omega$, if for all compact $K \subseteq \omega^\omega$ $A \cap K$ is $\mathbf{\Sigma}^1_2$, then A is $\mathbf{\Sigma}^1_2$. This generalizes to $\mathbf{\Sigma}^1_1$ in place of $\mathbf{\Sigma}^1_2$ assuming the consistency of some hypotheses involving determinacy. We give an alternative proof of the following theorem of Saint-Raymond. Suppose X and Y are compact metric spaces and f is a continuous surjection of X onto Y. Then, for any $A \subseteq Y$, A is $\mathbf{\Pi}^0_\alpha$ in Y iff $f^{-1}(A)$ is $\mathbf{\Pi}^0_\alpha$ in X. The non-trivial part of this result is to show that taking pre-images cannot reduce the Borel complexity of a set. The techniques we use are the definability of forcing and Wadge games.

We begin by proving several lemmas which illustrate the techniques to be used in this paper. The first lemma is easy using Wadge games and Borel determinacy.

LEMMA 1. *Suppose A is a Borel subset of ω^ω which is not $\mathbf{\Pi}^0_\alpha$. Then there exists a compact $P \subseteq \omega^\omega$ such that $A \cap P$ is not $\mathbf{\Pi}^0_\alpha$.*

Proof. Let $R \subseteq 2^\omega$ be any $\mathbf{\Sigma}^0_\alpha$ set which is not $\mathbf{\Pi}^0_\alpha$. The Wadge game $G(R, A)$ is played as follows. Players I and II alternately write down reals $x_{\mathrm{I}} \in 2^\omega$ and $x_{\mathrm{II}} \in \omega^\omega$ in the following pattern:

$$\begin{array}{cc} x_{\mathrm{I}}(0) & x_{\mathrm{II}}(0) \\ x_{\mathrm{I}}(1) & x_{\mathrm{II}}(1) \\ x_{\mathrm{I}}(2) & x_{\mathrm{II}}(2) \\ \vdots & \vdots \end{array}$$

At the end of play we say that player II wins if ($x_{\mathrm{I}} \in R$ iff $x_{\mathrm{II}} \in A$). A strategy for either player is a function which tells that player what to do on any play given the previous plays. It is easily seen to give a continuous map. Since the set

$$(R \times A) \cup ((2^\omega - R) \times (\omega^\omega - A))$$

is Borel this game is determined. Player I cannot have a winning strategy since this would give a continuous map $f\colon \omega^\omega \to 2^\omega$ such that $f^{-1}(2^\omega - R) = A$, which would imply A is $\mathbf{\Pi}^0_\alpha$. Thus player II must have a winning strategy, which implies that there exists a continuous map $f: 2^\omega \to \omega^\omega$ with $f^{-1}(A) = R$. Now let $P = f''2^\omega$. Since R is not $\mathbf{\Pi}^0_\alpha$ neither is $P \cap A$. ∎

This lemma generalizes to show that, for example, assuming $\mathbf{\Sigma}^1_n$-determinacy, for every $A \subseteq \omega^\omega$ which is $\mathbf{\Sigma}^1_n$ but not $\mathbf{\Pi}^1_n$ there is a perfect $P \subseteq \omega^\omega$ such that $A \cap P$ is not $\mathbf{\Pi}^1_n$. This lemma was also discovered by Mathias (see Mathias, Ostaszewski, and Talagrand [**10**], third to last paragraph).

For more on Wadge games see van Wesep [**15**]. Next we show how the definability of forcing comes in.

Suppose that M is a model of ZFC (or some reasonably large fragment of ZFC) and G is Cohen generic over M (i.e. obtained by forcing with some countable partial order in M).

LEMMA 2. *Suppose that, in $M[G]$, A is a $\mathbf{\Pi}^0_\alpha$ subset of $[0,1]^\omega$. Then for any Cohen condition p*

$$\{x \in M : p \Vdash \text{`}x \in A\text{'}\}$$

is a $\mathbf{\Pi}^0_\alpha$ set in M.

Proof. This is proved by induction on α for all conditions p. If $p \Vdash$ 'A is a closed set', then trivially $\{x : p \Vdash \text{`}x \in A\text{'}\}$ must be closed. Now suppose $\langle \beta_n : n < \omega \rangle$ is a sequence below α and

$$p \Vdash \text{`}A = \bigcap_{n<\omega} A_n \text{ and each } A_n \text{ is } \mathbf{\Sigma}^0_{\beta_n}\text{'}.$$

But now $$p \Vdash \text{`}x \in A_n\text{'} \quad \text{iff} \quad \exists q \leqslant p, \quad q \Vdash \text{`}x \notin A_n\text{'}.$$

By induction, and since there are only countably many q,

$$\{x : p \Vdash \text{`}x \in A_n\text{'}\} \quad \text{is} \quad \mathbf{\Pi}^0_{\beta_n+1}.$$

Whence $$\{x : p \Vdash \text{`}x \in A\text{'}\} = \bigcap_n \{x : p \Vdash \text{`}x \in A_n\text{'}\}$$

is $\mathbf{\Pi}^0_\alpha$. ∎

Remark. This lemma is not true for $\mathbf{\Sigma}^0_1$. Suppose G is the open subset of R determined by Cohen-generically throwing out one rational from each interval $[n, n+1]$ (n an integer). Then for any condition p there will be some interval $[n, n+1]$ such that

$$\{x \in [n, n+1] : p \Vdash \text{`}x \in G\text{'}\}$$

will be the set of irrationals in $[n, n+1]$.

The next lemma shows how to reduce from the Hilbert cube to the space ω^ω.

LEMMA 3. *Suppose $B \subseteq [0,1]^\omega$ is Borel but not $\mathbf{\Pi}^0_\alpha$. Then there exists a countable dense $D \subseteq [0,1]$ such that $B \cap ([0,1] - D)^\omega$ is not $\mathbf{\Pi}^0_\alpha$.*

Proof. Let M be a countable transitive model of a large portion of ZFC such that M contains a Borel code for B and M knows that B is not $\mathbf{\Pi}^0_\alpha$. Let G be Cohen generic over M. In $M[G]$ let D be any countable dense subset of $[0,1]$ disjoint from M. Let $A = B \cap ([0,1] - D)^\omega$. Since $A \cap M = B \cap M$ we know by Lemma 2 that $M[G] \Vdash$ 'A is not $\mathbf{\Pi}^0_\alpha$'. (Otherwise if $p \Vdash$ 'A is $\mathbf{\Pi}^0_\alpha$', then

$$\{x : p \Vdash \text{`}x \in A\text{'}\} = B \quad \text{is} \quad \mathbf{\Pi}^0_\alpha.)$$

Now we use absoluteness to claim that A is not $\mathbf{\Pi}^0_\alpha$ in the real world. To say that a given Borel set A is not $\mathbf{\Pi}^0_\alpha$ is a $\mathbf{\Pi}^1_2$ statement if it is written in the obvious way, and unfortunately $\mathbf{\Pi}^1_2$ statements may not be absolute between $M[G]$ and the real world. However it is, in fact, a $\mathbf{\Sigma}^1_2$ statement.

Claim. If A is a Borel subset of ω^ω, then 'A is not $\mathbf{\Pi}^0_\alpha$' is $\mathbf{\Sigma}^1_2$.

Proof. By Borel determinacy and Wadge theory, there exists a continuous map $f: \omega^\omega \to \omega^\omega$ such that $f^{-1}(A) = C$ where C is some canonical universal $\mathbf{\Sigma}^0_\alpha$ set (canonical relative to some collapse of α). The statement '$f^{-1}(A) = C$' is $\mathbf{\Pi}^1_1$ and implies that A is not $\mathbf{\Pi}^0_\alpha$. ∎

Since $([0,1]-D)^\omega$ is homeomorphic to ω^ω we are done. ∎

Remark. R. Sami pointed out that in fact the claim can be strengthened to 'A is not $\mathbf{\Pi}^0_\alpha$' is $\mathbf{\Pi}^1_1$. This follows from a theorem of Louveau [**8**] that implies that any Borel set A with Borel code r which is $\mathbf{\Pi}^0_\alpha$ has a Π^0_α code which is hyperarithmetic in r.

Our first theorem answers a question of J. E. Jayne (see [**12**], page 487, problem 44).

THEOREM 4. *Suppose A is a Borel subset of a Polish space X. Then for any $\alpha < \omega_1$ A is $\mathbf{\Pi}^0_\alpha$ in X iff, for all compact zero-dimensional $P \subseteq X$, $P \cap A$ is $\mathbf{\Pi}^0_\alpha$ in P.*

Proof. Suppose $A \subseteq X \subseteq [0,1]^\omega$ where X is a $\mathbf{\Pi}^0_2$ subset of $[0,1]^\omega$ and A is a Borel set which is not $\mathbf{\Pi}^0_\alpha$. For $\alpha = 1$, if A is not closed, then there is a convergent sequence $x_n \to x$ with $x_n \in A$ all n and $x \notin A$. Just let $P = \{x_n : n < \omega\} \cup \{x\}$. So we can assume $\alpha \geqslant 2$. By Lemma 3 there exists a countable dense $D \subseteq [0,1]$ such that $A \cap ([0,1]-D)^\omega$ is not $\mathbf{\Pi}^0_\alpha$. Since $X \cap [(0,1]-D)^\omega$ is a zero-dimensional Polish space it is homeomorphic to a closed $X^* \subseteq \omega^\omega$ (see Kuratowski [**7**], p. 441). Let A^* be the image of $A \cap ([0,1]-D)^\omega$ under this homeomorphism. By Lemma 1 there exists a compact $P \subseteq \omega^\omega$ such that $A^* \cap P$ is not $\mathbf{\Pi}^0_\alpha$. Now just pull $P \cap X^*$ back to a compact subset of X in which A is not $\mathbf{\Pi}^0_\alpha$. ∎

Remark. By taking complements the theorem is also true for $\mathbf{\Sigma}^0_\alpha$.

Under a continuous map the preimage of a $\mathbf{\Pi}^0_\alpha$ set is a $\mathbf{\Pi}^0_\alpha$ set. Under certain conditions the preimage cannot be any simpler. The following theorem is due to Saint-Raymond [**14**] using a quite different proof.

THEOREM 5. *Suppose $f: X \to Y$ is a continuous onto map, where X and Y are compact metric spaces. Then, for any $\alpha < \omega_1$, if $A \in \mathbf{\Pi}^0_\alpha - \mathbf{\Delta}^0_\alpha$ then $f^{-1}(A) \in \mathbf{\Pi}^0_\alpha - \mathbf{\Delta}^0_\alpha$.*

Proof. By Theorem 4 there exists $P \subseteq Y$ a compact zero-dimensional space such that $A \cap P \notin \mathbf{\Delta}^0_\alpha$. Since we could replace X and Y by $f^{-1}(P)$ and P, we may assume without loss of generality that X and Y are compact zero-dimensional metric spaces and, in fact, subspaces of 2^ω. Furthermore we may assume that $X = Y = 2^\omega$. It is enough to see that we may assume neither X nor Y contain isolated points. Just replace X and Y by $X \times 2^\omega$ and $Y \times 2^\omega$ and define $\hat{f}: X \times 2^\omega \to Y \times 2^\omega$ by $\hat{f}(x,y) = (f(x), y)$. Then $\hat{f}$ is onto, $A \times 2^\omega$ is $\mathbf{\Pi}^0_\alpha - \mathbf{\Delta}^0_\alpha$ and $\hat{f}^{-1}(A \times 2^\omega) = f^{-1}(A) \times 2^\omega$ is $\mathbf{\Delta}^0_\alpha$.

From now on assume $f: 2^\omega \to 2^\omega$ is a continuous onto map, $A \in \mathbf{\Pi}^0_\alpha - \mathbf{\Delta}^0_\alpha$, and $f^{-1}(A) \in \mathbf{\Delta}^0_\alpha$.

Claim. For any continuous onto map $f: 2^\omega \to 2^\omega$, there exists $g: 2^\omega \to 2^\omega$ such that $f \circ g$ is the identity and, for every clopen C, $g^{-1}(C) \in \mathbf{\Delta}^0_\alpha$.

Proof. The lexicographical order on 2^ω is defined by $x <_l y$ iff there exists n such that $x \upharpoonright n = y \upharpoonright n$ and $x(n) < y(n)$. It is easy to see that every compact subset of 2^ω has a lexicographical least element. Define g by $g(y)$ is the lexicographical least element of $f^{-1}(y)$. For any $s \in 2^{<\omega}$ let $N_s = \{x \in 2^\omega : s \subseteq x\}$. Then

$$z \in g^{-1}(N_s) \quad \text{iff} \quad f^{-1}(z) \cap N_s \neq \varnothing$$

and

$$\text{for all } t <_l s \quad f^{-1}(z) \cap N_t = \varnothing.$$

By compactness $f^{-1}(z) \cap N_s \neq \varnothing$ iff, for all $n < \omega$, $f^{-1}(N_{z\restriction n}) \cap N_s \neq \varnothing$; and $f^{-1}(z) \cap N_t = \varnothing$ iff there exists $n < \omega$ such that $f^{-1}(N_{z\restriction n}) \cap N_t = \varnothing$. Since there are only finitely many t lexicographically less than s we see that $g^{-1}(N_s)$ is the intersection of an open set and a closed set. It follows that, for any clopen set C, $g^{-1}(C)$ is a $\mathbf{\Delta}^0_2$ set. |

For any $C \in \mathbf{\Pi}^0_\alpha$, $g^{-1}(C) \in \mathbf{\Pi}^0_{1+\alpha}$. Hence if $f^{-1}(A) \in \mathbf{\Delta}^0_\alpha$, then $A = g^{-1}(f^{-1}(A)) \in \mathbf{\Delta}^0_{1+\alpha}$. We have therefore proved the theorem for $\alpha \geqslant \omega$. We use Wadge theory to take care of the finite case.

Let
$$\mathscr{B} = \{B \subseteq 2^\omega : \exists h : 2^\omega \xrightarrow{\text{onto}} 2^\omega \wedge h^{-1}(B) \in \mathbf{\Delta}^0_\alpha\}.$$

By the claim we know that $\mathscr{B}$ is contained in the $\mathbf{\Delta}^0_{1+\alpha}$ sets. We derive a contradiction by showing that $\mathscr{B}$ is closed under countable unions and complements (complements are trivial).

Claim. $\mathscr{B}$ contains all $\mathbf{\Pi}^0_\alpha$ sets.

Proof. Let B be any $\mathbf{\Pi}^0_\alpha$ set. By Wadge's theorem there exists a continuous map $k: 2^\omega \to 2^\omega$ such that $k^{-1}(A) = B$. Let $Q = \{(x,y) : f(x) = k(y)\}$. Define $i(x,y) = y$ and $j(x,y) = x$.

$$\begin{tikzcd} 2^\omega \arrow[d, "f"'] & Q \arrow[l, "j"'] \arrow[d, "i"] \\ A \subseteq 2^\omega & 2^\omega \arrow[l, "k"'] \end{tikzcd}$$

This is known as the pull-back of the two maps. Since f is onto, i maps Q onto 2^ω. Also

$$i^{-1}(B) = j^{-1}(f^{-1}(A))$$

so $i^{-1}(B)$ is $\mathbf{\Delta}^0_\alpha$. If Q is not perfect replace Q by its perfect kernel, $\ker(Q)$ (i must map $\ker(Q)$ onto 2^ω since $Q - \ker(Q)$ is countable). Now take any homeomorphism of 2^ω onto Q and compose it with i to see that B is in $\mathscr{B}$. |

We use this claim to see that $\mathscr{B}$ is closed under countable intersection. Suppose $h_n: 2^\omega \to 2^\omega$ witness that B_n are in $\mathscr{B}$. Let

$$X = \{x \in (2^\omega)^\omega : \forall n, m \; h_n(x_n) = h_m(x_m)\}.$$

Define $g: X \to 2^\omega$ by $g(x) = f_1(x_1)$. Then for any $n < \omega$

$$g^{-1}(A_n) = \{x \in X : f_n(x_n) \in B_n\}$$

which is $\mathbf{\Delta}^0_\alpha$. As before we can assume $g: 2^\omega \to 2^\omega$ is onto. Since $g^{-1}(\bigcap_{n<\omega} B_n) \in \mathbf{\Pi}^0_\alpha$ there exists $h: 2^\omega \to 2^\omega$ such that

$$h^{-1}(g^{-1}(\textstyle\bigcap_{n<\omega} B_n)) \in \mathbf{\Delta}^0_\alpha$$

and we get that $\bigcap_{n<\omega} B_n$ is in $\mathscr{B}$. Thus $\mathscr{B}$ is a σ-algebra contradicting the fact that $\mathscr{B} \subseteq \mathbf{\Delta}^0_{1+\alpha}$ and proving Theorem 5. |

Theorem 5 has been generalized to the case where X and Y are arbitrary compact

Hausdorff spaces by Jayne and Rogers [**4**]. In this case we replace the Borel hierarchy by the Baire hierarchy. The Baire hierarchy on a space X is defined similarly to the Borel hierarchy except that we let $\mathbf{\Sigma}^0_1(X)$ be the open F_σ-sets and $\mathbf{\Pi}^0_1(X)$ the closed G_δ-sets. Of course, in a metric space the two hierarchies coincide. The following theorem is due to Rogers and Jayne [**4**].

THEOREM 6. *Suppose $f: X \to Y$ is a continuous onto map, X and Y are compact Hausdorff spaces and A is $\mathbf{\Pi}^0_\alpha(Y) - \mathbf{\Delta}^0_\alpha(Y)$. Then $f^{-1}(A)$ is $\mathbf{\Pi}^0_\alpha(X) - \mathbf{\Delta}^0_\alpha(X)$.*

Proof. We show how to reduce a counterexample to the second countable case, contradicting Theorem 5. Begin by embedding X and Y in a product of closed and bounded intervals as in the standard constructions of the Stone–Čech compactification (see Willard [**16**], section 19). More explicitly, let $C^*(X)$ be the family of all continuous, bounded, real-valued functions on X and for each $g \in C^*(X)$ let I_g be a compact interval containing the range of g. Let ΠI_g be the product over all $g \in C^*(X)$ and let $e: X \to \Pi I_g$ be the evaluation map, i.e. $[e(x)]_g = g(x)$. Define ΠI_h and $e^1: Y \to \Pi I_h$ similarly.

Define $F: \Pi I_g \to \Pi I_h$ by $[F(t)]_h = [t]_{f\circ h}$. The function F is continuous and $F \circ e = e^1 \circ f$. Thus if f is a counterexample to our theorem so is $F \restriction e(X)$ with range $e(Y)$. Now any Baire set relative to $e(X)$ is the intersection of a Baire set relative to ΠI_g with $e(X)$. Also for any Baire set B in ΠI_g there exists a countable set $\Sigma \subseteq C^*(X)$ such that Σ supports B, i.e. for any $x, y \in \Pi I_g$ if $x \restriction \Sigma = y \restriction \Sigma$ then $x \in B$ iff $y \in B$ (see Bockstein [**2**]). If $\pi: \Pi_{g\in C^*(X)} I_g \to \Pi_{g\in\Sigma} I_g$ is the projection map, then, since membership in B is determined by Σ, the Baire complexity of B is preserved by π.

Now suppose for contradiction that A is $\mathbf{\Pi}^0_\alpha(Y) - \mathbf{\Delta}^0_\alpha(Y)$ and $f^{-1}(A)$ is $\mathbf{\Delta}^0_\alpha(X)$. Let Σ^1 support a $\mathbf{\Pi}^0_\alpha(\Pi I_h)$ set with intersects $e(Y)$ in $e(A)$. Let Σ support a $\mathbf{\Sigma}^0_\alpha(\mathbf{\Pi} I_g)$ and a $\mathbf{\Pi}^0_\alpha(\Pi I_g)$ both of which intersect $e(X)$ in $e(f^{-1}(A))$, and also let Σ contain $f \circ h$ for each $h \in \Sigma^1$. Let π and π^1 be the corresponding projection maps for Σ and Σ^1 and define $F^1: \Pi_{g\in\Sigma} I_g \to \Pi_{h\in\Sigma^1} I_h$ just as F is defined, i.e. $[F(t)]_h = [t]_{f\circ h}$.

$$\begin{tikzcd}
\prod\limits_{g\in\Sigma} I_g \arrow[r, "F^1"] & \prod\limits_{h\in\Sigma^1} I_h \\
\prod\limits_{g\in C^*(X)} I_g \arrow[r, "F"] \arrow[u, "\pi"'] & \prod\limits_{h\in C^*(Y)} I_h \arrow[u, "\pi^1"'] \\
X \arrow[r, "f"] \arrow[u, "e"'] & Y \arrow[u, "e^1"']
\end{tikzcd}$$

The map F^1 restricted to $\pi(e(X))$ gives a counterexample to Theorem 5. ∎

The first draft of this paper was written before we discovered that Theorem 5 and Theorem 6 were already known. They were originally motivated by the following application to nonstandard analysis. Let $I = [0, 1] \subseteq R$ and let T be a $*$-finite set such that the standard part map, *st*, maps T onto I. Define the Borel hierarchy on T by declaring $\Sigma^0_0(T) = \Pi^0_0(T) = {}^*\mathscr{P}(T) =$ the internal power set of T. We wanted

THEOREM 7. *For $A \subset I$, $\alpha \geqslant 1$*

(*a*) $A \in \Pi^0_\alpha(I) \leftrightarrow st^{-1}(A) \in \Pi^0_\alpha(T)$

(*b*) $A \in \Sigma^0_\alpha(I) \leftrightarrow st^{-1}(A) \in \Sigma^0_\alpha(T)$.

Of course, a and b are equivalent and $\rightarrow$ is trivial by induction. For $\leftarrow$, let X be the Stone space of the Boolean algebra ${}^*\mathscr{P}(T)$, and consider the commutative diagram

$$\begin{tikzcd} T \arrow[r, "i", "1-1"'] \arrow[d, "st"', "\text{onto}"] & X \arrow[dl, "f", "\text{onto}"'] \\ I & \end{tikzcd}$$

Suppose $A \subset I$ and $st^{-1}(A) \in \Sigma_\alpha^0(T)$. Say $st^{-1}(A) = \phi(K_0, K_1, K_2, \ldots)$, where the K_i are internal and ϕ is some Σ_α^0 combination (think of ϕ as a Σ_α^0 propositional sentence of $\mathscr{L}_{\omega_1\omega}$). In X, let $C = \phi(N_{K_0}, N_{K_1}, \ldots)$, where N_K is the clopen set corresponding to K. So $C \in \Sigma_\alpha^0(X)$.

Note that $C = f^{-1}(A)$: if not, say $u \in f^{-1}(A)$ but $u \notin C$. By ω_1-saturation of the nonstandard model, there is a $t \in T$ such that $st(t) = f(u)$ and $t \in K_i \leftrightarrow u \in N_{K_i}$ $(i = 0, 1, \ldots)$. But then $t \notin \phi(K_0, K_1, \ldots) = st^{-1}(A)$, so $f(u) = st(t) \notin A$, a contradiction.

Now, Theorem 6 applies to give $A \in \Sigma_\alpha^0(I)$. ▌

For any family Γ of subsets of ω^ω we say that a set $A \subseteq \omega^\omega$ is compactly Γ iff for every compact $K \subseteq \omega^\omega$, $A \cap K \in \Gamma$. Lemma 1 says that any Borel A which is compactly $\mathbf{\Sigma}_\alpha^0$ is $\mathbf{\Sigma}_\alpha^0$. Rogers and Jayne asked whether analytic plus compactly Borel implies Borel. This was shown to be independent by Mathias, Ostaszewski, and Talagrand [**10**].

Fremlin (see [**12**], p. 483, problem 18) asked whether it is consistent that compactly analytic implies analytic. He notes that this is clearly false assuming CH but asks what about $MA + \neg CH$.

It is not necessary to consider arbitrary Polish spaces – this problem is really about ω^ω. Consider the following two facts:

(1) Every Polish space is the continuous, one-to-one, image of a closed subset of ω^ω.

(2) Every Polish space is either σ-compact or contains a closed subspace homeomorphic to ω^ω.

Thus by (2) if there is a compactly analytic non-analytic subset of ω^ω there is such a set in every non σ-compact Polish space. On the other hand if there is such a set in some Polish space it is easy by (1) to see that there is one in ω^ω. These remarks are due to Fremlin and Jayne; see also [**10**].

One advantage to working in ω^ω is that the compact subsets of ω^ω are easy to understand. For f and g elements of ω^ω let $f \leqslant g$ iff, for all $n < \omega$, $f(n) \leqslant g(n)$ and let $f \leqslant^* g$ iff, for all but finitely many n, $f(n) \leqslant g(n)$. It is not hard to show that for any compact subset K of ω^ω there is an $f \in \omega^\omega$ such that $K \subseteq \{g \in \omega^\omega : g \leqslant f\}$. Also any σ-compact set is contained in a set of the form $\{g \in \omega^\omega : g \leqslant^* f\}$ (a set which is itself σ-compact). Since the family of analytic sets is closed under countable union, we see that a set $A \subseteq \omega^\omega$ is compactly analytic iff, for every $f \in \omega^\omega$, $A \cap \{g \in \omega^\omega : g \leqslant^* f\}$ is analytic.

The next theorem shows that $MA + \neg CH$ is not sufficient to answer Fremlin's question.

THEOREM 8. *Suppose* $MA + (\omega_1^V = \omega_1^L) + (2^\omega = \omega_2)$. *Then there exists* $A \subseteq \omega^\omega$ *which is not projective such that* A *is compactly analytic.*

Proof. Let $A = \{f_\alpha : \alpha < \omega_2\} \subseteq \omega^\omega$ be well-ordered by $\leqslant^*$ and such that for every $g \in \omega^\omega$ there exists α such that $g \leqslant^* f_\alpha$. It is well known (see Rudin [**13**]) that such a set exists. Since any subset of A of cardinality ω_2 has the same property, we may assume A is not projective (since there are only $2^\omega = \omega_2$ projective sets). $MA + (\omega_1^V = \omega_1^L)$ implies that every subset of ω^ω of cardinality ω_1 is $\mathbf{\Pi}_1^1$ (Martin and Solovay [**9**]). Thus $\omega^\omega - A$ is compactly $\mathbf{\Sigma}_1^1$. ∎

Next we are going to show that it is consistent that compactly $\mathbf{\Sigma}_2^1$ implies $\mathbf{\Sigma}_2^1$. The model we use is obtained by iterating the eventually dominating order with finite support. This model is due to Hechler [**3**].

$\mathbb{P}$ is the eventually dominating order, i.e.

$$\mathbb{P} = \{(n,f) : n < \omega, f \in \omega^\omega\}$$

and

$$(n,f) \leqslant (m,g) \quad \text{iff} \quad n \geqslant m, \quad f \restriction m = g \restriction m, \quad \text{and} \quad \forall k \geqslant m, \quad f(k) \geqslant g(k).$$

Let $\mathbb{P}_\alpha$ be the usual partial order obtained by iterating $\mathbb{P}$ α times, using finite support. We will need some lemmas from Miller [**11**]. For the convenience of the reader we will make this account self-contained. Our first lemma is an easy exercise.

LEMMA 9. *Suppose $M \subseteq N$ are countable models of ZFC, $\mathbb{Q}_M \in M$ and $\mathbb{Q}_N \in N$ are partial orders, and $\mathbb{Q}_M$ is a suborder of $\mathbb{Q}_N$. Then (A) and (B) are equivalent. (A) For any G which is $\mathbb{Q}_N$-generic over N, $G \cap \mathbb{Q}_M$* is *$\mathbb{Q}_M$-generic over M. (B) For any $A \in M$, if $M \models$ '$A \subseteq \mathbb{P}_M$ is a maximal antichain', then $N \models$ '$A \subseteq \mathbb{P}_N$ is a maximal antichain'.*

Proof. $(A) \Rightarrow (B)$.

Suppose not and $N \models$ '$\forall q \in A$ p and q are incompatible'. Let G be $\mathbb{Q}_N$-generic over N with $p \in G$ and let $D = \{r \in \mathbb{Q}_M : \exists q \in A\ r \leqslant q\}$. But now $M \models$ 'D is dense in $\mathbb{Q}_M$' and $G \cap \mathbb{P}_M \cap D = \varnothing$.

$(B) \Rightarrow (A)$.

Since the notion of incompatibility must be absolute, $G \cap \mathbb{P}_M$ is a $\mathbb{P}_M$-filter. Given any $D \subseteq \mathbb{P}_M$ dense in M, let $M \models$ '$A \subseteq D$ is a maximal antichain'. But then

$$D^1 = \{q \in \mathbb{Q}_N : \exists p \in A\ q \leqslant p\}$$

is dense in $\mathbb{Q}_N$. So, if G is $\mathbb{P}_N$-generic over N, then $G \cap D^1 \neq \varnothing$; therefore $G \cap A \neq \varnothing$ and thus $G \cap \mathbb{P}_M \cap D \neq \varnothing$. ∎

Remark. Clearly we need only a fraction of ZFC to be true in M and N to prove this lemma.

LEMMA 10. *Suppose $M \subseteq N$ are models of ZFC and G is $\mathbb{P}^N$-generic over N, then $G \cap \mathbb{P}^M$* is *$\mathbb{P}^M$-generic over M. ($\mathbb{P}^N$ and $\mathbb{P}^M$ are the relativization to N and M of the eventually dominating partial order* $\mathbb{P}$.)

Proof. It is enough to see that if $M \models$ '$A \subseteq \mathbb{P}$ is a maximal antichain', then $N \models$ '$A \subseteq \mathbb{P}$ is a maximal antichain'. But the statement '$A \subseteq \mathbb{P}$ is a maximal antichain' is easily seen to be $\mathbf{\Pi}_1^1$ and hence absolute. ∎

This lemma is also true for Cohen forcing and random real forcing, but it can fail, for example, for Laver forcing or Sacks' real forcing. For example, suppose that M is a model of $V = L$ and N models that ω_1^L is countable. In this situation there is a perfect set P in N of reals each of which is Cohen over M. Thus there is a real which is Sacks generic over N but Cohen generic over M!

The next two lemmas are quite general, and they are true for many finite support iterations satisfying Lemma 10.

We now give a very explicit definition of $\mathbb{P}_\alpha$ by induction on α. $\mathbb{P}_1$ is just $\mathbb{P}$. Now suppose $\mathbb{P}_\alpha$ has been defined. We say that τ is a $\mathbb{P}_\alpha$ term for an element of ω^ω iff $\tau = \langle D_n, f_n : n < \omega\rangle$ where, for each $n < \omega, D_n \subseteq \mathbb{P}_\alpha$ is a maximal antichain and $f_n : D_n \to \omega$. (The realization of τ given G a $\mathbb{P}_\alpha$-generic filter is defined by $\tau^G(n) = m$ iff for the unique $p \in D_n \cap G, f_n(p) = m$.) Then $p \in \mathbb{P}_{\alpha+1}$ iff $p \restriction \alpha \in \mathbb{P}_\alpha$ and either $p(\alpha) = 1$ (the trivial condition) or $p(\alpha) = \langle n, \tau\rangle$ where $n < \omega$ and τ is a $\mathbb{P}_\alpha$ term for an element of ω^ω. We define $p \leqslant q$ iff $p \restriction \alpha \leqslant q \restriction \alpha$ and $p \restriction \alpha \Vdash$ '$p(\alpha) \leqslant q(\alpha)$'. For limit ordinals λ, $\mathbb{P}_\lambda$ is just the usual finite support limit, i.e. $p \in \mathbb{P}_\lambda$ iff the domain of p is λ, for all $\alpha < \lambda$

$$p \restriction \alpha \in \mathbb{P}_\alpha,$$

and for all but finitely many α $p(\alpha)$ is the trivial condition. The finite set of α on which p is non trivial is called the support of p. For p and q in $\mathbb{P}_\lambda$ if $\beta < \lambda$ contains the support of p and q, then $p \leqslant q$ iff $p \restriction \beta \leqslant q \restriction \beta$.

LEMMA 11. *Suppose $M \subseteq N$ are countable models of ZFC and $\beta \in M$. Then $\mathbb{P}^M_\beta = \mathbb{P}^N_\beta \cap M$ and $\mathbb{P}^M_\beta$ is a suborder of $\mathbb{P}^N_\beta$. Also for any G $\mathbb{P}^M_\beta$-generic over N, $G \cap \mathbb{P}^M_\beta$ is $\mathbb{P}^M_\beta$-generic over M.*

Proof. Both statements are proved simultaneously by induction on β. Suppose $\beta = \alpha + 1$. Since $\mathbb{P}_{\alpha+1}$ is defined explicitly in terms of antichains in $\mathbb{P}_\alpha$, we see that by Lemma 9 and the induction hypothesis that $\mathbb{P}^M_{\alpha+1} = \mathbb{P}^N_{\alpha+1} \cap M$. $\mathbb{P}^M_{\alpha+1}$ is a suborder of $\mathbb{P}^N_{\alpha+1}$ because for $p \in \mathbb{P}^M_\alpha, M \models$ '$p \models (n,\tau) \leqslant (m,\sigma)$' iff $N \models$ '$p \models (n,\tau) \leqslant (m,\sigma)$'. This follows from Π^0_1 absoluteness. Thus suppose, for contradiction, that M thinks that $p \models$ '$(n,\tau) \leqslant (m,\sigma)$' but N does not. Then for some G $\mathbb{P}^N_\alpha$-generic over N with $p \in G$ we have that

$$N[G] \models \text{‘}(n, \tau^G) \nleqslant (m, \sigma^G)\text{’}.$$

But $\hat{G} = G \cap \mathbb{P}^M_\alpha$ is $\mathbb{P}^M_\alpha$-generic over M so $M[\hat{G}] \models$ '$(n,\tau^G) \leqslant (m,\sigma^G)$'. This contradicts the fact that $(n,\tau^G) \leqslant (m,\sigma^G)$ is clearly a Π^0_1 sentence and hence absolute. A similar argument proves the contrapositive.

Now let us verify the second statement. Suppose G is $\mathbb{P}^N_{\alpha+1}$-generic over N, then $G = G_\alpha * G^\alpha$ where G_α is $\mathbb{P}^N_\alpha$-generic over N and G^α is $\mathbb{P}^{N[G_\alpha]}$-generic over $N[G_\alpha]$. By induction and Lemma 10 we get that $\hat{G}_\alpha = G_\alpha \cap \mathbb{P}^M_\alpha$ is $\mathbb{P}^M_\alpha$-generic over M and $G \cap \mathbb{P}^{M[\hat{G}_\alpha]}$ is $\mathbb{P}^{M[\hat{G}_\alpha]}$-generic over $M[\hat{G}_\alpha]$. Hence, by the product lemma, $G_\beta \cap \mathbb{P}^M_\beta$ is $\mathbb{P}^M_\beta$-generic over M. Now suppose α is a limit ordinal. In this case we know that, for every G $\mathbb{P}^N_\alpha$-generic over N and $\beta < \alpha$, $G_\beta \cap \mathbb{P}^M_\beta$ is $\mathbb{P}^M_\beta$-generic over M. We use condition (B) of Lemma 9. Suppose

$$M \models \text{‘}A \subseteq \mathbb{P}_\alpha \text{ is a maximal antichain’}.$$

To see that
$$N \models \text{‘}A \subseteq \mathbb{P}_\alpha \text{ is an antichain’}$$

note that for any two p and $q \in A$ there exists $\beta < \alpha$ such that the support of p and q is contained in β. Hence $M \models$ '$p \restriction \beta$ and $q \restriction \beta$ are incompatible in $\mathbb{P}_\beta$' and therefore by Lemma 9 and induction

$$N \models \text{‘}p \restriction \beta \quad\text{and}\quad q \restriction \beta \text{ are incompatible in } \mathbb{P}_\beta\text{’}.$$

To verify that A remains maximal in N, let $N \models$ '$p \in \mathbb{P}_\alpha$' and let $\beta < \alpha$ contain the support of p. Clearly

$$M \models \text{'}\forall p \in \mathbb{P}\, \exists q \in A \quad q \restriction \beta \quad \text{and} \quad p \quad \text{are compatible'}.$$

Hence by induction

$$N \models \text{'}\exists q \in A \quad q \restriction \beta \text{ and } p \restriction \beta \quad \text{are compatible'}.$$

But since the support of p is contained in β

$$N \models \text{'}\exists q \in A \quad p \quad \text{and} \quad q \quad \text{are compatible'}. \,\blacksquare$$

The next lemma says that if $\langle f_\alpha : \alpha < \gamma \rangle$ is an iteration sequence of eventually dominating reals over M, then, for any $X \subseteq \gamma$ in M, $\langle f_\alpha : \alpha \in X \rangle$ is an iteration of length-order type of X of eventually dominating reals over M.

To make things more explicit we define for each $p \in \mathbb{P}_\beta$ the super support of p as follows. For $p \in \mathbb{P}_{\alpha+1}$ if $p(\alpha) = (n, \tau)$ where $\tau = (D_n, f_n : n < \omega)$, then the super support of p is the union of the super support of $p \restriction \alpha$ and all the super support of $q \in \cup_{n<\omega} D_n$ and $\{\alpha\}$. If $p(\alpha)$ is the trivial condition, then the super support of P is just the super support of $p \restriction \alpha$. For $p \in \mathbb{P}_\lambda$ when λ is a limit, the super support of p is the super support of $p \restriction \beta$ where $\beta < \lambda$ is large enough to contain the support of p.

Since antichains are countable it is easy to prove that the super support of any condition is countable.

For any ordinal β and set X define $\mathbb{P}_\beta^X$ to be the set of $p \in \mathbb{P}_\beta$ such that the super support of p is contained in X and let $\mathbb{P}_\beta^X$ have the inherited ordering.

The next lemma is proved like Lemma 11.

LEMMA 12. *Suppose M is a countable transitive model of ZFC and X and α are elements of M. Then, for any G $\mathbb{P}_\alpha$-generic over M, $G \cap \mathbb{P}_\alpha^X$ is $\mathbb{P}_\alpha^X$-generic over M. Also, for any $p \in \mathbb{P}_\alpha^X$, n and m elements of ω, and σ and τ $\mathbb{P}_\alpha$-terms for elements of ω^ω with super support contained in X,*

$$p \models_{\mathbb{P}_\alpha} \text{'}(n, \tau) \leqslant (m, \sigma)\text{'} \quad \textit{iff} \quad p \Vdash_{\mathbb{P}_\alpha^X} \text{'}(n, \tau) \leqslant (m, \sigma)\text{'}.$$

Proof. This is proved by induction on α. The second conclusion follows from the first via the same $\mathbf{\Pi}_1^0$-absoluteness argument as was used in Lemma 11. It also allows us to see that ordering on $\mathbb{P}_{\alpha+1}^X$ is defined as the usual iteration and hence the same proofs as in Lemma 11 work. $\blacksquare$

LEMMA 13. *For any $X \subseteq \alpha$, if X has order type β, then $\mathbb{P}_\alpha^X$ is isomorphic to $\mathbb{P}_\beta$.*

Proof. Let $j : X \to \beta$ be an order isomorphism. We will do the successor step in the induction. Suppose j has already been defined on $\mathbb{P}_\gamma^X$. If $\gamma \notin X$, then $\mathbb{P}_{\gamma+1}^X = \mathbb{P}_\gamma^X \times \mathbb{1}$ ($\mathbb{1}$ is the trivial order). If $\gamma \in X$ first define j on $\mathbb{P}_\gamma$-terms τ for elements of ω^ω with super-support contained in X. Suppose $\tau = (D_n, f_n : n < \omega)$. Then let

$$j(\tau) = (j(D_n), f_n \circ j^{-1} : n < \omega).$$

Now define $j(p) = j(p \restriction \gamma)^\frown \langle n, j(\tau) \rangle$ where $p(\gamma) = (n, \tau)$.

As a corollary to Lemma 12 we see that, for any p and $q \in \mathbb{P}_{\gamma+1}^X$, $p \leqslant q$ iff $p \restriction \gamma \leqslant q \restriction \gamma$, and $p \restriction \gamma \Vdash_{\mathbb{P}_\gamma^X} \text{'}p(\gamma) \leqslant q(\gamma)\text{'}$. Thus j is an order isomorphism. $\blacksquare$

LEMMA 14. *Suppose M is a model of ZFC and G is $\mathbb{P}_\alpha$-generic over M. Then for any $x \in \omega^\omega \cap M[G]$ there exists $\beta < \omega_1^M$ and $\hat{G} \in M[G]$ such that $\hat{G}$ is $\mathbb{P}_\beta$-generic over M and $x \in M[\hat{G}]$.*

Proof. Use the countable chain condition to obtain a countable $X \subseteq \alpha$ such that $x \in M[G \cap \mathbb{P}_\alpha^X]$ and use the fact that $\mathbb{P}_\alpha^X$ is isomorphic to $\mathbb{P}_\beta$ where β is the order type of X. ∎

LEMMA 15. *Suppose $\phi(u, v)$ is a $\mathbf{\Sigma}_2^1$ formula with parameters from a model of set theory M, $p \in \mathbb{P}_\beta^M$ where $\beta < \omega_1^M$, and τ is a $\mathbb{P}_\beta^M$ term such that $p \Vdash$ '$\tau \in \omega^\omega$'. Then*

$$\{x \in \omega^\omega \cap M : p \Vdash \text{`}\phi(x, \tau)\text{'}\}$$

is a $\mathbf{\Sigma}_2^1$-set in M.

Proof. Work in M. Suppose N is a transitive model of some large fragment of ZFC, say ZFC_n, and N contains the parameters of ϕ, β, p, and τ. Then if $N \Vdash$ '$p \Vdash \phi(x, \tau)$', then (in M) $p \Vdash$ '$\phi(x, \tau)$'. Otherwise for some $G \cap \mathbb{P}_\beta^M$-generic over M, containing p, $M[G] \models$ '$\neg\phi(x, \tau)$'. But since $G \cap \mathbb{P}_\beta^N$ is $\mathbb{P}_\beta^N$-generic over N, $N[G \cap \mathbb{P}_\beta^N] \models$ '$\phi(x, \tau)$'. This contradicts the fact that $\mathbf{\Sigma}_2^1$ sentences are upward absolute.

Thus $p \Vdash$ '$\phi(x, \tau)$' iff there exists a countable transitive model N of ZFC_n containing the parameters of ϕ, β, p, and τ and $N \models$ '$p \Vdash$ "$\phi(x, \tau)$" '. This shows that the set in question is $\mathbf{\Sigma}_1$ over (HC, ϵ) and hence $\mathbf{\Sigma}_2^1$ in ω^ω (see Jech [**6**], p. 527). ∎

Remark. $\mathbf{\Sigma}_2^1$ cannot be improved to $\mathbf{\Pi}_1^1$ or $\mathbf{\Sigma}_1^1$ in Lemma 15. To see this let $WF \subseteq \omega^\omega$ be the set of reals which code well-founded subtrees of $\omega^{<\omega}$. Then WF is $\mathbf{\Pi}_1^1$ but not $\mathbf{\Sigma}_1^1$. Now suppose $f \in \omega^\omega$ eventually dominates every element of $\omega^\omega \cap M$. Then for any tree $T \subseteq \omega^{<\omega}$ contained in M, T has a branch iff T has a branch eventually beneath f. But to tell if a finite branching tree has a branch is $\mathbf{\Delta}_1^1$. Consequently there is a Borel set B with code f such that $B \cap M = WF \cap M$. This argument shows that the operation A in M 'becomes' a Borel operation in $M[f]$. Thus there are many $\mathbf{\Delta}_2^1$-sets in M which become Borel in $M[f]$.

LEMMA 16. *Suppose M is a countable transitive model of ZFC, $\alpha \in M$, and G $\mathbb{P}_\alpha$-generic over M. Also suppose that for some set of reals $A \in M[G]$, $M[G] \models$ 'A is a $\mathbf{\Sigma}_2^1$-set and $A \cap M \in M$'. Then $M \models$ '$A \cap M$ is a $\mathbf{\Sigma}_2^1$-set'.* ∎

Proof. By Lemma 14 the $\mathbf{\Sigma}_2^1$-code for A is in some $M[\hat{G}_\beta]$ for β countable. Thus by $\mathbf{\Sigma}_2^1$ absoluteness we may assume that α is countable. But now Lemma 15 implies that $A \cap M$ is $\mathbf{\Sigma}_2^1$ in M. ∎

THEOREM 17. Con (ZFC) *implies* Con $(ZFC + \textit{every compactly } \mathbf{\Sigma}_2^1\textit{-set is } \mathbf{\Sigma}_2^1)$.

Proof. Let M be a countable transitive model of $ZFC + CH$ and let G be $\mathbb{P}_{\omega_2}$-generic over M. Then in $M[G]$ every compactly $\mathbf{\Sigma}_2^1$-set is $\mathbf{\Sigma}_2^1$. To prove this let $A \subseteq \omega^\omega$ be a compactly $\mathbf{\Sigma}_2^1$-set in $M[G]$. Suppose for contradiction that $M[G] \models$ 'A is not $\mathbf{\Sigma}_2^1$'. Using a Lowenheim–Skolem type argument, CH in M, and the c.c.c. of $\mathbb{P}_{\omega_2}$, find $\alpha < \omega_2$ such that $A \cap M[G_\alpha] \in M[G_\alpha]$ and $M[G_\alpha] \models$ '$A \cap M[G_\alpha]$ is not a $\mathbf{\Sigma}_2^1$-set'. Now think of $M[G_\alpha]$ as the ground model and apply Lemma 16 to the $\mathbf{\Sigma}_2^1$-set $A \cap \{g \in \omega^\omega : g \leqslant^* f_{\alpha+1}\}$ where $f_{\alpha+1}$ is the $\alpha + 1$ eventually dominating real. This contradiction proves the result. ∎

To get this result for compactly $\mathbf{\Sigma}_1^1$ we found it necessary to have $\mathbf{\Delta}_2^1$-determinancy in the model $M[G]$. It seems unlikely that such strong hypotheses are necessary.

THEOREM 18. *Suppose every well ordering of reals in $L[\mathbb{R}]$ is countable and $\mathbf{\Delta}^1_2$-determinancy is true. Then there is a transitive model of ZFC + every compactly $\mathbf{\Sigma}^1_1$-set is $\mathbf{\Sigma}^1_1$.*

Proof. We use the technique of the Kunen–Moschovakis theorem for showing the consistency of $ZFC + \mathbf{\Delta}^1_n$-determinacy $+ 2^\omega = \omega_2$. (See Becher[1] theorem 20, also lemmas 18 and 19.) It is clear that by this technique we can obtain $M[G_{\omega_2}]$ such that in $M[G_{\omega_2}]$ every compactly $\mathbf{\Sigma}^1_2$-set is $\mathbf{\Sigma}^1_2$ and $\mathbf{\Delta}^1_2$-determinacy holds. But obviously every compactly $\mathbf{\Sigma}^1_1$-set must be $\mathbf{\Delta}^1_2$ and therefore by Lemma 1 and $\mathbf{\Delta}^1_2$-determinacy it must be $\mathbf{\Sigma}^1_1$. ∎

If one is willing to assume projective determinacy (PD) and every well-ordered set of reals in $L[\mathbb{R}]$ is countable, then one can easily get a transitive model of ZFC in which for every $n < \omega$ every compactly $\mathbf{\Sigma}^1_n$-set is $\mathbf{\Sigma}^1_n$. One works over the model $L[\tilde{T}]$ where $\tilde{T} = \langle T^1, T^3, T^5, \ldots \rangle$ (see Becher[1], p. 73). All models between $L[\tilde{T}]$ and V are $\mathbf{\Sigma}^1_n$-correct, so we have absoluteness.

Problem. Suppose every $A \subseteq \omega^\omega$ of cardinality ω_1 is eventually dominated. Does $\mathbf{\Delta}^1_2$ + compactly $\mathbf{\Sigma}^1_1$ imply $\mathbf{\Sigma}^1_1$?

Problem. Does (compactly $\mathbf{\Delta}^1_1 = \mathbf{\Delta}^1_1$) imply (compactly $\mathbf{\Sigma}^1_1 = \mathbf{\Sigma}^1_1$)?

REFERENCES

[1] H. BECHER, Partially playful universes. In *Cabal Seminar 76–77*, Lecture Notes in Math. vol. 689 (Springer-Verlag, 1978).

[2] M. BOCKSTEIN. Un théorème de séparabilité pour les produits topologiques. *Fund. Math.* **35** (1948), 242–246.

[3] S. H. HECHLER. On the existence of certain cofinal subsets of ω^ω. In *Proc. of Symp. in Pure Math.*, vol. 13, part 2, AMS (1974), 155–173.

[4] J. E. JAYNE and C. A. ROGERS. K-analytic sets. In *Analytic Sets* (Academic Press, 1980), pp. 1–175.

[5] J. E. JAYNE and C. A. ROGERS. The invariance of the absolute Borel classes. *Convex Analysis and Optimization*, Research Notes in Math. no. 57 (Pitman, 1982), 119–151.

[6] T. JECH. *Set Theory* (Academic Press, 1978).

[7] K. KURATOWSKI. *Topology*, vol. 1 (Academic Press, 1966).

[8] A. LOUVEAU. A separation theorem for Σ^1_1 sets. *Trans. Amer. Math. Soc.* **260** (1980), 363–378.

[9] D. A. MARTIN and R. M. SOLOVAY. Internal Cohen extensions. *Ann. Math. Logic* **2** (1970), 143–178.

[10] A. R. D. MATHIAS, A. J. OSTASZEWSKI and M. TALAGRAND. On the existence of an analytic set meeting each compact set in a Borel set. *Math. Proc. Cambridge Philos. Soc.* **84** (1978), 5–10.

[11] A. MILLER, Some properties of measure and category. *Trans. Amer. Math. Soc.* **266** (1981), 93–114. Corrections. *Trans. Amer. Math. Soc.* **271** (1982), 347–348.

[12] C. A. ROGERS *et al. Analytic Sets* (Academic Press, 1980).

[13] M. E. RUDIN. Martin's axiom. In *Handbook of Mathematical Logic* (North Holland, 1977), pp. 491–501.

[14] J. SAINT-RAYMOND. Fonctions Boréliennes sur un quotient. *Bull. Soc. Math.* 2ᵉ série, **100** (1976), 141–147.

[15] R. VAN WESEP. Wadge degrees and descriptive set theory. In *Cabal Seminar 76–77*, Lecture Notes in Math. vol. 689 (Springer-Verlag, 1978).

[16] S. WILLARD. *General Topology* (Addison-Wesley, 1970).