$$p_i \cdot f_j(t) - 1 = \pm\delta_{ij}\frac{h_i}{d_i} = \pm\delta_{ij}m.$$

All these incidence relations between vertices and facets of $S(t)$ are contained in the matrix equation

$$\begin{bmatrix} 1 & p_0 \\ \vdots & \vdots \\ 1 & p_m \end{bmatrix} \begin{bmatrix} 0 & -1 & \cdots & -1 \\ 1 & & & \\ \vdots & f_1(t) & \cdots & f_m(t) \\ 1 & & & \end{bmatrix}$$

$$(12) \qquad = \begin{bmatrix} h_0 m^{1/2} & & & 0 \\ & \pm m & & \\ & & \ddots & \\ 0 & & & \pm m \end{bmatrix}$$

It is well known that the determinant of the first matrix is $m!V(t)$. That of the second is $-Z^*_{m+1}(t)$, see §4. So, with (10) and (6), the determinant relation associated to (12) reads

$$(m-1)!\gamma_0(t)h_0Z^*_{m+1}(t) = \pm h_0 m^{m+1/2}$$

or

$$(13) \qquad \gamma_0(t) = |m^{m+1/2}/(m-1)!Z^*_{m+1}(t)|.$$

Combining this with (7) and (8), we get

$$\operatorname{vol} C_{m-1} = m^{-1/2}\left|\lim_{t\to 1-}\gamma_0(t)\right| = |m^m/(m-1)!Z^*_{m+1}(1)| = m^{m-3}/(m-1)!,$$

as asserted.

References

1. Ph.J. Davis, Circulant Matrices, Wiley, New York, 1979.
2. E. T. Wong, Polygons, circulant matrices, and Moore-Penrose inverses, this MONTHLY, 88(1981) 509–515.
3. B. Grunbaum, Convex Polytopes, Wiley, London, 1967.

Images of Monotone Functions

6218 [1978, 500; 1982, 134]. *Proposed by M. J. Pelling, Balliol College, Oxford, England.*

Let S be a subset of the real line R having cardinality of the continuum. Is there always a monotonic f: $R \to R$ such that $m^*f(S) > 0$ where m^* is outer Lebesgue measure?

Solution by A. W. Miller, University of Texas. This was partially solved by Fred Galvin (see this MONTHLY, 89 (1982) 134–135). He showed that the continuum hypothesis (actually SBCT which is much weaker) implies that there is a subset $S \subseteq R$ of cardinality the continuum such that every monotonic image of S has measure zero.

A positive answer to Pelling's problem is also consistent with the ZFC (Zermelo-Fraenkel set theory with the axiom of choice). A set of reals X has universal measure zero iff for every countably additive, nonatomic measure μ on the reals, $\mu(X) = 0$.

It is an unpublished result of J. Baumgartner and R. Laver that if ZFC is consistent, then so is ZFC plus every set of reals of cardinality the continuum that fails to have universal measure zero. This result was announced in Laver, R. "On the consistency of Borel's conjecture," Acta Math., 137 (1976) 151–169. A different proof of it will also appear in Miller, A., "Mapping a set of reals onto the reals," to appear in the Journal of Symbolic Logic (1983). This result combined with the following construction of Szpilrajn-Marczewski completely settles Pelling's problem.

THEOREM (Szpilrajn-Marczewski). *Suppose* $S \subseteq R$ *and* μ *is a countable additive, nonatomic measure such that* $\mu(S) > 0$. *Then there exists a monotonic function* $f: R \to R$ *such that* $m^*(f(s)) > 0$.

Proof. We may assume that μ does not vanish on any interval since we could replace μ by $\frac{1}{2}(\mu + \lambda)$ where λ is Lebesgue measure. Now define f by

$$f(x) = \mu([0, x]).$$

Since μ vanishes on no intervals and is atomless, we see that f is strictly increasing and continuous and thus a homeomorphism. Define

$$\nu(B) = \mu(f^{-1}(B))$$

for any Borel set B. We are done if we show that ν is Lebesgue measure. Suppose $I = [a, b]$ is any interval. Then

$$f^{-1}(I) = \{y: \mu([0, y]) \in I\} = [c, d]$$

where $\mu([0, c]) = a$ and $\mu([0, d]) = b$. But then

$$\nu(I) = \mu(f^{-1}(I)) = \mu([c, d]) = b - a.$$

Since ν agrees with Lebesgue measure on the intervals and the intervals generate the Borel sets, we have that ν is Lebesgue measure. □

Stochastic Matrices

6366 [1981, 711]. *Proposed by Emeric Deutsch, Polytechnic Institute of New York.*

Let A be a row stochastic matrix such that $\|A\| = 1$, $\|\ \|$ being the operator norm induced by the Euclidean vector norm. Show that A is doubly stochastic.

Solution by Enzo R. Gentile, Universidad de Buenos Aires, Argentina. Let $A = (a_{ij})$ be a real $n \times n$ matrix. We shall prove that

$$\|A\| = \max_{x \neq 0} \frac{|Ax|}{|x|} = 1 \quad \text{and} \quad \sum_{i,j} a_{ij} = n$$

imply that A is doubly stochastic. Since $\|A\| = \|A^T\| = 1$ where A^T is the transpose of A, it will be enough to prove that the sum

$$\begin{aligned} S &= \sum_j \Big(1 - \sum_i a_{ij}\Big)^2 \\ &= \sum_j \Big(1 + \Big(\sum_i a_{ij}\Big)^2 - 2\Big(\sum_i a_{ij}\Big)\Big) \\ &= -n + \sum_j \Big(\sum_i a_{ij}\Big)^2 \end{aligned}$$

is 0.