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Nyikos's Joke

In 1988 Peter Nyikos asked me if there were any Q-sets in Laver's model. After I solved it (Theorem 16), I asked Peter why he wanted to know. He said he was just joking, "no Q-points / no Q-sets", referring to my paper [6].

The file "nyikos.jok" [7] dated Nov 1, 1998 was part of a letter or email I sent to Jaime Ihoda (Haim Judah). At the time Judah was moving from Berkeley to Tel Aviv and we were writing the joint paper Judah-Miller-Shelah [3]. What follows may be some version of the proof I had in 1998, but I haven't been able to locate any other notes on the proof. At the time I was more interested in Theorem 12. I put this result for one Laver real (for any Borel $B \subseteq \omega^\omega$ there is a Laver tree p with $[p] \subseteq B$ or $[p] \cap B = \emptyset$) in the first version of our paper [4] (last section). A referee for the Journal of Symbolic Logic noted that it is an immediate consequence of Galvin-Prikry Theorem that Borel subsets of $[\omega]^\omega$ are Ramsey. Mathias trees are special Laver trees, this can be seen via the map from $[\omega]^\omega$ to ω^ω taking a set $X \subseteq \omega$ to the increasing enumeration of X.[1] It was rejected and we resubmitted it to the Archive for Mathematical Logic with the last section deleted.

Paul Larson recently asked me (May 2015) if I knew how to prove there are no Q-sets in the iterated perfect set model. Theorem 12 and 16 are true for iterated Sacks forcing, as well as, many other tree forcings, e.g. Superperfect forcing, Mathias forcing, Silver forcing, or mixed versions. Theorem 12 could probably be generalized to the analogues of Silver's Theorem (analytic sets are Ramsay) and Ellentuck's topological characterization of completely Ramsay sets [1].

I don't know if Theorem 12 for Laver, Superperfect, and Sacks forcing follows from Mathias forcing by a simple argument as in the case of one-step forcing. For the no Q-set result (Theorem 16), there are probably simpler arguments, perhaps based on homogeneity or definability.

Let $\mathbb{P}$ be Laver forcing ([6]). So $p \in \mathbb{P}$ iff $p \subseteq \omega^{<\omega}$ is a nonempty subtree such that there exists $s \in p$ (called the root of p) which is comparable to all $t \in p$ and has the property that every $t \in p$ with $s \subseteq t$ splits, i.e. there are

[1] Similarly Superperfect trees are Laver trees, so Theorem 12 (for one-step) for Laver trees implies it for Superperfect trees.

infinitely many $n < \omega$ with $t\hat{}\langle n\rangle \in p$. We use $t\hat{}\langle n\rangle$ to denote the sequence $(t_0, t_1, \ldots, t_k, n)$ where $t = (t_0, t_1, \ldots, t_k)$. Define $p \leq q$ iff $p \subseteq q$.

The splitting nodes of p can be put naturally into one-to-one correspondence with $\omega^{<\omega}$ preserving inclusion as well as the lexicographical order. Effectively list $\omega^{<\omega}$ so that for any $s \in \omega^{<\omega}$ and $n < \omega$:

- s is listed before $s\hat{}\langle n\rangle$ and
- $s\hat{}\langle n\rangle$ is listed before $s\hat{}\langle n+1\rangle$.

We let $\mathrm{split}_n(p)$ be the first $n+1$ split nodes of p in the induced order. We define $p \leq_n q$ iff $p \leq q$ and $\mathrm{split}_n(p) = \mathrm{split}_n(q)$.

For $p \in \mathbb{P}$ and $s \in p$ let $p_s = \{t \in p \ : \ s \subseteq t \text{ or } t \subseteq s\}$ and let

$$[p] = \{f \in \omega^\omega \ : \ \forall n < \omega \ f \restriction n \in p\}.$$

Lemma 1 *(See Judah-Shelah [2]) For any $p \in \mathbb{P}$ and $D \subseteq \mathbb{P}$ dense and downward closed there exists $q \leq_0 p$ such that for every $f \in [q]$ there is $k < \omega$ such that $q_{f\restriction k} \in D$.*

Proof:

Let

$$R_0 = \{s \in p \ : \ \exists q \leq_0 p_s \ \ q \in D\}$$

and put

$$R_\alpha = \{s \in p \ : \ \exists^\infty n \ \ s\hat{}\langle n\rangle \in \bigcup_{\beta<\alpha} R_\beta\}.$$

Let $\mathrm{rank}(s)$ be the least α with $s \in R_\alpha$ if there is one, otherwise $\mathrm{rank}(s) = \infty$.

Claim. If r is the root of p, then $\mathrm{rank}(r) \neq \infty$.

If $\mathrm{rank}(s) = \infty$ then there are infinitely many n such that $s\hat{}\langle n\rangle \in p$ and $\mathrm{rank}(s\hat{}\langle n\rangle) = \infty$. But then we could build $q \leq_0 p$ such that all s in q have rank ∞. This contradicts the density of D.

Now we can prove the lemma by induction on the rank of the root node. Let L be the nodes immediately below the root of p and having smaller rank than the root. For each $s \in L$ there is $q^s \leq_0 p_s$ such that for every $f \in [q^s]$ there is k with $q^s_{f\restriction k} \in D$ and so $q = \bigcup_{s\in L} q^s$ satisfies the lemma.
QED

Throughout we suppose that κ is any sufficiently large regular cardinal. H_κ is the set of all sets whose transitive closure has cardinality less than κ.

Lemma 2 *Suppose $M \preceq H_\kappa$ is countable, $p \in \mathbb{P} \cap M$. Then there exists $q \leq_0 p$ such that every $f \in [q]$ is Laver generic over M, in fact, for all downward closed dense $D \subseteq \mathbb{P}$ in M and $f \in [q]$ there exist k with $q_{f \restriction k} \in D$.*

Proof:

This is an easy fusion argument. Let D_n for $n < \omega$ list all dense downward closed subsets of $\mathbb{P}$ in M. Put $p_0 = p$ and build a sequence $p_{n+1} \leq_n p_n$ of conditions in $\mathbb{P} \cap M$ such that for every n and $f \in [p_{n+1}]$ there is k with $(p_{n+1})_{f \restriction k} \in D_n$. Let q be the fusion.
QED

Definition 3 *We define the countable support iteration of Laver forcing $\mathbb{P}_\alpha$ for $\alpha \leq \omega_2$:*

1. *$\alpha = 1$. $\mathbb{P}_1 = \mathbb{P}$ (Laver forcing).*
2. *α limit. $p \in \mathbb{P}_\alpha$ iff $p \restriction \beta \in \mathbb{P}_\beta$ for all $\beta < \alpha$ and its support* $\mathrm{supp}(p) =^{def} \{\beta : p(\beta) \neq 1\}$ *is countable.*
3. *$\alpha = \beta + 1$. $p \in \mathbb{P}_{\beta+1}$ iff $p \restriction \beta \in \mathbb{P}_\beta$, $p \restriction \beta \Vdash p(\beta) \in \mathring{\mathbb{P}}$, and $p(\beta)$ is a nice $\mathbb{P}_\beta$-name for a nonempty subtree of $\omega^{<\omega}$, which means:*
   (a) *$p(\beta) \subseteq \{\langle q, \check{s}\rangle : s \in \omega^{<\omega}$ and $q \in \mathbb{P}_\beta\}$*
   (b) *if $\langle q, \check{s}\rangle \in p(\beta)$, $t \subseteq s$, then $\langle q, \check{t}\rangle \in p(\beta)$*
   (c) *$\langle 1, \check{\langle\rangle}\rangle \in p(\beta)$*

$p \leq q$ iff $\forall \beta < \alpha \;\; p \restriction \beta \Vdash p(\beta) \leq q(\beta)$

Definition 4 *For $p \in \mathbb{P}_\alpha$, and we define $[p] \subseteq (\omega^\omega)^\alpha$ by induction on $\alpha \leq \omega_2$.*

$\alpha = 1$.

$$[p] = \{f \in \omega^\omega \; : \; \forall n < \omega \;\; f \restriction n \in p\}.$$

α limit.

$$[p] = \{f \in (\omega^\omega)^\alpha \; : \; \forall \beta < \alpha \;\; f \restriction \beta \in [p \restriction \beta].$$

$\alpha = \beta + 1$.

$$[p] = \{f \in (\omega^\omega)^{\beta+1} \; : \; f \restriction \beta \in [p \restriction \beta] \text{ and } f(\beta) \in [p(\beta)^{G_{f\restriction\beta}}]\}.$$

where as usual $p(\beta)^{G_{f\restriction\beta}} = \{s \; : \; \exists q \in G_{f\restriction\beta} \;\; \langle q, \check{s}\rangle \in p(\beta)\}$.
In all cases for $f \in (\omega^\omega)^\alpha$ we define $G_f = \{p \in \mathbb{P}_\alpha \; : \; f \in [p]\}$.

In general, $[p]$ could be the empty set since without a little genericity $[p(\beta)^{G_{f\restriction\beta}}]$ may be empty. Note that by using nice names, it will always be a nonempty subtree of $\omega^{<\omega}$.

Definition 5 *$\sigma : \alpha \to \omega$ is finitely supported iff*

$$\mathrm{supp}(\sigma) =^{def} \{\beta \ : \ \sigma(\beta) \neq 0\}$$

is finite.

Definition 6 *For $q \in \mathbb{P}_\alpha$, $f \in [q]$, and $\sigma : \alpha \to \omega$ finitely supported, we say that $q_{f,\sigma} \in \mathbb{P}_\alpha$ is defined iff $\mathrm{supp}(\sigma) \subseteq \mathrm{supp}(q)$ and for all $\beta < \alpha \quad (q_{f,\sigma}) \restriction \beta$ is defined and if $n = \sigma(\beta)$ and $s = f(\beta) \restriction n$ then*

$$(q_{f,\sigma}) \restriction \beta \Vdash \check{s} \in q(\beta).$$

We then put $q_{f,\sigma}(\beta) = q(\beta)_s =^{def} \{(p, \hat{t}) \in q(\beta) \ : t \subseteq s \text{ or } s \subseteq t\}$.

Since $\langle 1, \check{\langle\rangle} \rangle \in q(\beta)$ if $\sigma(\beta) = 0$, then $(q_{f,\sigma}) \restriction \beta \Vdash \check{s} \in q(\beta)$ is trivially true.

Definition 7 *$p \in \mathbb{P}_\alpha$ is determined is defined by induction on α.*

$\alpha = 1$. Every $p \in \mathbb{P}$ is determined.

α limit. $p \in \mathbb{P}_\alpha$ is determined iff $p \restriction \beta$ is determined for every $\beta < \alpha$.

$\alpha = \beta + 1$. $p \in \mathbb{P}_\alpha$ is determined iff $p \restriction \beta$ is determined and for every $f \in [p \restriction \beta]$ for every $n < \omega$ there is a finitely supported $\sigma : \beta \to \omega$ such that $(p \restriction \beta)_{f,\sigma}$ is defined and for some $T_n \subseteq \omega^{<\omega}$

$$(p \restriction \beta)_{f,\sigma} \Vdash \mathrm{split}_n(p(\beta)) = \check{T}_n.$$

Note that if p is determined, then for any $f \in [p]$ and $\beta < \alpha$ that $p(\beta)^{G_{f\restriction\beta}}$ is a Laver tree.

Lemma 8 *If $p \in \mathbb{P}_\alpha$ is determined, then $[p] \subseteq (\omega^\omega)^\alpha$ is closed.*

Proof:

Suppose $f \notin [p]$. Choose $\beta < \alpha$ minimal so that $f \restriction \beta \in [p \restriction \beta]$ but $f \restriction (\beta+1) \notin [p \restriction (\beta+1)]$. There must be some $n, N < \omega$ and $T_n \subseteq \omega^{<\omega}$ such that $\mathrm{split}_n(p(\beta)^{G_f}) = T_n$ and $s = f(\beta) \restriction N$ is a node which cannot be on any Laver tree whose n-split nodes are T_n. Let $\sigma : \beta \to \omega$ be finitely supported so that $(p \restriction \beta)_{f,\sigma}$ is defined and forces "$\mathrm{split}_n(p(\beta)) = \check{T}_n$". It follows that

$$\{g \in (\omega^\omega)^\alpha \ : \ g(\beta) \restriction N = f(\beta) \restriction N \text{ and } \forall \gamma < \beta \ \ g \restriction \sigma(\gamma) = f \restriction \sigma(\gamma)\}$$

is a basic open set containing f and disjoint from $[p]$.
QED

Lemma 9 *If $p \in \mathbb{P}_\alpha$ is determined, $f \in [p]$, and $\sigma : \alpha \to \omega$ is finitely supported, then there exists finitely supported $\rho : \alpha \to \omega$ with $\rho(\beta) \geq \sigma(\beta)$ all β and $p_{f,\rho}$ is defined.*

Proof:

By induction on the maximum of the support of σ. If σ is identically zero, then $\rho = \sigma$ works. Otherwise, let $\beta < \alpha$ be the maximum such that $\sigma(\beta) \neq 0$. By inductive hypothesis there is $\sigma_1 : \beta \to \omega$ which bounds $\sigma \upharpoonright \beta$ and $(p \upharpoonright \beta)_{f\upharpoonright\beta,\sigma_1}$ is defined. By the definition of determined there is $\sigma_2 : \beta \to \omega$ such that $(p \upharpoonright \beta)_{f\upharpoonright\beta,\sigma_2}$ is defined and forces $f(\beta) \upharpoonright \sigma(\beta) \in p(\beta)$. (It is enough to decide $\text{split}_n(p)$ for $n = \sigma(\beta)$.) But note that it is easy to show that for $\rho \upharpoonright \beta$ the maximum of σ_1 and σ_2 that $(p \upharpoonright \beta)_{f\upharpoonright\beta,\rho\upharpoonright\beta}$ is defined. So now extend $\rho \upharpoonright \beta$ to α by letting $\rho(\beta) = \sigma(\beta)$.

QED

Definition 10 *For $\alpha \leq \omega_2$, $n < \omega$, $F \in [\alpha]^{<\omega}$ and $p, q \in \mathbb{P}_\alpha$*
$q \leq_n^F p$ iff $p \leq q$ and for all $\beta \in F$ $\quad q \upharpoonright \beta \Vdash q(\beta) \leq_n p(\beta)$.

Lemma 11 *Suppose $\alpha \leq \omega_2$, $n < \omega$, $F \in [\alpha]^{<\omega}$ and $p \in \mathbb{P}_\alpha$.*

[det] There is a determined $q \leq_n^F p$.

[one] If $D \subseteq \mathbb{P}_\alpha$ dense and downward closed, then there is a determined $q \leq_n^F p$ such that for every $f \in [q]$ there is a finite partial $\sigma : \alpha \to \omega$ such that $q_{f,\sigma}$ is defined and $q_{f,\sigma} \in D$.

[ctlb] If $\mathcal{D}$ is a countable collection of dense and downward closed subsets of $\mathbb{P}_\alpha$, then there is a determined $q \leq_n^F p$ such that for every $f \in [q]$ and $D \in \mathcal{D}$ there is a finite partial $\sigma : \alpha \to \omega$ such that $q_{f,\sigma}$ is defined and $q_{f,\sigma} \in D$.

[gen] For any countable $M \preceq H_\kappa$ with p and α in M there is a determined $q \leq_n^F p$ such that for every $f \in [q]$, G_f^M is $\mathbb{P}_\alpha$-generic over M where

$$G_f^M =^{def} \{r \in \mathbb{P}_\alpha^M \ : \ \exists \sigma \in \Sigma \ \ q_{f,\sigma} \leq r\}$$

and Σ is the set of finite partial maps $\sigma : \alpha \to \omega$ such that $q_{f,\sigma}$ is defined.

Proof:

Induction on α followed by the size of F. The case $\alpha = 1$ is done by Lemmas 1 and 2.

[det]:

To simplify notation assume $n = 0$. Put $F_0 = F$ and $q_0 = p$ and construct a fusion sequence $q_n \in \mathbb{P}_\alpha$, $F_n \in [\alpha]^{<\omega}$, and $\beta_n < \alpha$, for $n < \omega$ so that the following is true:

- $q_{n+1} \leq_n^{F_n} q_n$,
- $\bigcup_n F_n = \bigcup_n \mathrm{supp}(q_n)$, $F_n \subseteq F_{n+1}$,
- $\beta_n \in F_n$,
- for every $\beta \in \cup_n F_n$ there are infinitely many n with $\beta_n = \beta$,
- $q_n \restriction \beta_n$ is determined, and
- for every $f \in [q_n \restriction \beta_n]$ there is a finitely supported $\sigma : \beta_n \to \omega$ such that $(q_n)_{f,\sigma} \restriction \beta_n$ is defined and for some $T_n \subseteq \omega^{<\omega}$

$$(q_n)_{f,\sigma} \restriction \beta_n \Vdash \mathrm{split}_n(q_n(\beta_n)) = \check{T}_n.$$

Let q be the fusion. We prove that q is determined. Fix $\beta < \alpha$ and assume we have shown that $q \restriction \beta$ is determined and let $f \in [q \restriction \beta]$. Fix n so that $\beta_n = \beta$. By construction there is $T_n \subseteq \omega^{<\omega}$ finitely supported $\sigma_n : \beta \to \omega$ so that

$$(q_n)_{f,\sigma_n} \restriction \beta \Vdash \mathrm{split}_n(q_n(\beta)) = \check{T}_n.$$

Since $\beta = \beta_n \in F_n$ and $q \leq_n^{F_n} q_n$

$$q \restriction \beta \Vdash \mathrm{split}_n(q(\beta)) = \mathrm{split}_n(q_n(\beta)).$$

By Lemma 9 there exists $\rho_n : \beta \to \omega$ dominating σ_n so that $(q)_{f,\rho_n} \restriction \beta$ is determined. Since $(q)_{f,\rho_n} \restriction \beta \leq (q_n)_{f,\sigma_n} \restriction \beta$

$$(q)_{f,\rho_n} \restriction \beta \Vdash \mathrm{split}_n(q(\beta)) = \check{T}_n.$$

Since there are infinitely many n such that $\beta = \beta_n$, hence $q \restriction (\beta + 1)$ is determined.

[one]:

Case 1. F is empty. First extend to be in D, then extend to a determined condition. For any f take σ to be the constant zero function.

Case 2. $0 \in F$. Let $F = F_0 \cup \{0\} \subseteq \alpha$ where $0 \notin F_0$. By the maximality principle there is a name $\mathring{q}$ so that

$$p(0) \Vdash \mathring{q} \leq_n^{F_0} p \restriction [1,\alpha) \text{ is determined and } \forall f \in [\mathring{q}]\ \exists \mathring{\sigma} : \alpha \to \omega\ \ \mathring{q}_{f,\sigma}^{def} \in \mathring{E}$$

where in $V[G]$

$$E = \{(r \restriction [1,\alpha))^G \ \ : \ \ r(0) \in G \text{ and } r \in D\}.$$

Put all relevant sets in a countable $M \preceq H_\kappa$. By Lemma 2 there exists $q(0) \leq_n p(0)$ such that G_f is $\mathbb{P}$-generic over M for every $f \in [q(0)]$. Consider the concatenation of $q(0)$ with $\mathring{q}$. More precisely, for $\beta > 0$ define

$$q(\beta) = \{\langle r, \check{s}\rangle \ : \ r \leq q \restriction \beta \text{ and } r \Vdash \check{s} \in \mathring{q}(\beta)\}.$$

Given $h \in [q(0]$ since G_h is $\mathbb{P}$-generic over M we have that

$$M[G_h] \models q^{G_h} \in \mathbb{P}_{[1,\alpha)} \text{ is determined.}$$

Also in $M[G_h]$ for any $f \in [q^{G_h}]$ there is a finitely supported $\sigma : [1,\alpha) \to \omega$ such that $q_{f,\sigma}^{G_h}$ is defined and an element of E. It follows that there is an $N < \omega$ such that

$$(q(0), q \restriction [1,\alpha))_{h\hat{\ }f, N\hat{\ }\sigma} \in D.$$

Thus the conclusion of [one] holds for all $f \in [q]$ such that $f(0) \in [q(0)]$ and $f \restriction [1,\alpha) \in M[G_{f(0)}]$. To show that it holds for all $f \restriction [1,\alpha) \in V$ note that $[q \restriction [1,\alpha)^{G_{f(0)}}]$ is a closed set (Lemma 8) and so by $\mathbf{\Pi^1_1}$ absoluteness [one] holds for all f in V.

Case 3. $0 < \gamma = \min(F) \subseteq \alpha$. This is similar to case 2. Let

$$p \restriction \gamma \Vdash \mathring{q} \leq_n^{F} p \restriction [\gamma,\alpha) \text{ is determined and } \forall f \in [\mathring{q}]\ \exists \mathring{\sigma} : [\gamma,\alpha) \to \omega\ \ \mathring{q}_{f,\sigma}^{def} \in \mathring{E}$$

where

$$E = \{(r \restriction [\gamma,\alpha))^G \ \ : \ \ r \restriction \gamma \in G \text{ and } r \in D\}.$$

Taking M as before, by inductive hypotheses we can find determined $q \restriction \gamma \leq p \leq \gamma$ such that G_h is $\mathbb{P}_\gamma$-generic over M for every $h \in [q]$. Combining $q \restriction \gamma$ and $\mathring{q} \restriction [\gamma,\alpha)$ gives us the required condition q.

[ctble]:

This will follow from [one] by the usual fusion arguments: We may assume without loss that $\mathcal{D}$ includes the countably many dense sets need in the proof of [det] to show that the fusion is determined. Let D_n for $n < \omega$ list $\mathcal{D}$. To simplify notation assume $n = 0$. Put $F_0 = F$ and $p_0 = p$. Let F_n be an increasing sequence of finite subsets of α constructed inductively so that

$$\bigcup_n F_n = \bigcup_n \text{supp}(p_n),$$

where the p_n are determined conditions with $p_n \in M$, $p_{n+1} \leq_n^{F_n} p_n$, and for every $f \in [p_{n+1}]$ there is a finitely supported σ such that $(p_{n+1})_{f,\sigma}$ is defined and an element of D_n. Let q be the fusion.

Given $f \in [q]$ and n since $f \in [p_{n+1}]$ there is a finitely supported $\sigma : \alpha \to \omega$ such that $(p_{n+1})_{f,\sigma}$ is defined and an element of D_n. By Lemma 9 there is a $\rho : \alpha \to \omega$ bounding σ such that $q_{f,\rho}$ is defined. But $q_{f,\rho} \leq (p_{n+1})_{f,\sigma}$ and since D_n is downward closed we are done.

[gen]:

This is immediate from [ctlbe] just taking $\mathcal{D}$ to be all downward closed subsets of $\mathbb{P}_\alpha$ which are in M.

QED

I don't know if using Shelah's notion of q-generic condition (as in his definition of proper forcing) would give an easier proof of Lemma 11.

Theorem 12 *For any $\alpha < \omega_1$, Borel set $B \subseteq (\omega^\omega)^\alpha$, and $p \in \mathbb{P}_\alpha$, there exists a determined condition $q \leq p$ such that either $[q] \subseteq B$ or $[q] \cap B = \emptyset$.*

Proof:

Choose countable $M \preceq H_\kappa$ containing $p, \mathbb{P}_\alpha$, and a Borel code for B. Extend p so that we may assume that $p \Vdash \mathring{f} \in B$ or $p \Vdash \mathring{f} \notin B$. Where f is the generic Laver sequence. Assume the $p \Vdash \mathring{f} \in B$. By Lemma 11 we have a determined $q \leq p$ such that all $f \in [q]$ are generic over M. Hence

$$M[G_f] \models f \in B$$

By absoluteness of Borel predicates $[q] \subseteq B$. Similarly, if $p \Vdash \mathring{f} \notin B$, then $[q] \cap B = \emptyset$.

QED

Definition 13 *$p \in \mathbb{P}_\alpha$ is canonically determined (c.d.) iff it is determined and for every $\beta < \alpha$ and $s \in \omega^{<\omega}$ if $\langle q, \hat{s}\rangle \in p(\beta)$, then $q = (p \upharpoonright \beta)_{f,\sigma}$ for some $f \in [p \upharpoonright \beta]$ and finitely supported $\sigma : \beta \to \omega$ such that $q = (p \upharpoonright \beta)_{f,\sigma}$ is defined.*

Lemma 14 *If $p \in \mathbb{P}_\alpha$ is determined, then there exists $p' \in \mathbb{P}_\alpha$ canonically determined with the same support, $p \equiv p'$, and $[p] = [p']$.*

Proof:

Note that if q is canonically determined, then $q_{f,\sigma}$ is canonically determined whenever it is defined. Define $p'(\beta)$ by induction. Let $p(0) = p'(0)$. Define

$$\Gamma_\beta = \{(p' \upharpoonright \beta)_{f,\sigma} \text{ defined } : f \in [p \upharpoonright \beta] \text{ and } \sigma : \beta \to \omega \text{ finitely supported}\}$$

$$p'(\beta) = \{\langle q, \hat{s}\rangle \ : \ q \in \Gamma_\beta \text{ and } q \Vdash \hat{s} \in p(\beta)\}.$$

Assume $p \upharpoonright \beta \equiv p' \upharpoonright \beta$. Then for any G $\mathbb{P}_\beta$-generic over V, by determinateness $p(\beta)^G = (p'(\beta))^G$ and so $p \upharpoonright (\beta+1) \equiv p' \upharpoonright (\beta+1)$.
QED

Lemma 15 *Let $\Sigma \in [\alpha]^\omega$ and define $\mathbb{P}^{c.d.}_\Sigma$ to be the canonically determined conditions of $\mathbb{P}_\alpha$ with support Σ. Let β be the order type of Σ and $\mathbb{P}^{c.d.}_\beta$ the canonically determined conditions of $\mathbb{P}_\beta$. Then there is natural isomorphism $j : \mathbb{P}^{c.d.}_\Sigma \to \mathbb{P}^{c.d.}_\beta$*

Proof:

Let $j : \Sigma \to \beta$ be the order isomorphism. For $\gamma \in \Sigma$ define

$$j(p)(\gamma)(j(\gamma)) = \{\langle j(q), \hat{s}\rangle \ : \ \langle q, \hat{s}\rangle \in q(\gamma)\}.$$

QED

I think that $\mathbb{P}_\beta$ is isomorphic to the set of conditions in $\mathbb{P}_\alpha$ which have hereditary support Σ, i.e. $\text{supp}(q) \subseteq \Sigma$ and if $\langle p, \hat{s}\rangle \in q(\beta)$, then p is hereditarily of support Σ.

Theorem 16 *In Laver's model for the Borel conjecture [6] there is no uncountable Q-set.*

Proof:

If an uncountable Q-set does occur by the usual arguments due to Laver ([6] page 164) we may assume that this Q-set X is in the ground model V. The continuum hypothesis is true in V, hence by a transfinite argument of length ω_1 and Theorem 12 we may construct $Y \subseteq X$ such that for every $\alpha < \omega_1$, canonically determined $q \in \mathbb{P}_\alpha$, and Borel set $B \subseteq [q] \times 2^\omega$ coded in V there is a $u \in X$ and a canonically determined $r \leq q$ such that for all $f \in [r]$

$$u \in Y \text{ iff } (f, u) \notin B.$$

Claim. In $V[G]$ where G is $\mathbb{P}_{\omega_2}$-generic over V, the set Y is not relatively Borel in X.

Proof:

Suppose for contradiction that it is $\mathbf{\Sigma}^{\mathbf{0}}_\beta$. Let $U \subseteq 2^\omega \times 2^\omega$ be a universal $\mathbf{\Sigma}^{\mathbf{0}}_\beta$ set coded in the ground model V. Let τ be name for an element of 2^ω and $p \in G$ a condition such that

$$p \Vdash \forall u \in X \ \ (u \in Y \text{ iff } (\tau, u) \in U).$$

Put p, X, Y, τ into a countable $M \preceq H_\kappa$. Using Lemma 11 and 14 let $q \leq p$ be canonically determined with $\text{supp}(q) = \Sigma = M \cap \omega_2$. Let $j : \mathbb{P}^{c.d.}_\Sigma \to \mathbb{P}^{c.d.}_\beta$ be the isomorphism from Lemma 15. Let $r = j(q)$.

Define $B \subseteq [r] \times 2^\omega$ by

$$(f, u) \in B \text{ iff } (\tau^{G_{j(f)}}, u) \in U.$$

This map is continuous and so B is Borel. By our construction there is $u_0 \in X$ and a canonically determined $r_0 \leq r$ in $\mathbb{P}_\alpha$ so that for every $h \in [r_0]$

$$u_0 \in Y \text{ iff } (h, u_0) \notin B.$$

Let $j(r_0) = q_0 \leq q$,

Now we get a contradiction. Suppose G is $\mathbb{P}_{\omega_2}$-generic over V with $q_0 \in G$. Let f_G be the ω_2-sequence of Laver reals. Let $h \in [r_0]$ be the Laver sequence with $h(j(\beta)) = f_G(\beta)$. By the choice of u_0 and $[r_0]$ we have that

$$u_0 \in Y \text{ iff } (h, u_0) \notin B.$$

By the definition of B

$$(h, u_0) \in B \text{ iff } (\tau^G, u_0) \in U$$

so

$$u_0 \in Y \text{ iff } (\tau^G, u_0) \notin U$$

which contradicts

$$p \Vdash u_0 \in Y \text{ iff } (\tau, u_0) \in U.$$

QED

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