Compact subsets of the Baire space

Arnold W. Miller
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Results in this note were obtained in 1994 and reported on at a meeting on Real Analysis in Łodz, Poland, July 1994.

Let ω^ω be the Baire space, infinite sequences of natural numbers with the product topology. In this topology a set $K \subseteq \omega^\omega$ is compact iff there exists a finite branching tree $T \subseteq \omega^{<\omega}$ such that

$$K = [T] =^{def} \{x \in \omega^\omega : \forall n \in \omega \;\; x \upharpoonright n \in T\}.$$

Theorem 1 *If there exists a countable standard model of ZFC, then there exists M, a countable standard model of ZFC, $N \supseteq M$, a generic extension of M, and $T \in N$ a finite branching subtree of $\omega^{<\omega}$ with the properties that*

(a) $\forall f \in [T] \cap N \;\; \exists g \in M \cap \omega^\omega \;\; \forall^\infty n \; f(n) < g(n)$ *and*

(b) $\forall g \in M \cap \omega^\omega \;\; \exists f \in [T] \cap N \;\; \exists^\infty n \; g(n) < f(n)$.

I don't know how to do this over an arbitrary ground model M, for example a model of $V = L$.

I don't know if we can have (a) and the stronger condition:

$(b)'$ $\forall g \in M \cap \omega^\omega \; \exists f \in [T] \cap N \;\; \forall^\infty n \; g(n) < f(n)$.

This Theorem is related to Michael's problem [4] of whether there must be a Lindelöf space X such that $X \times \omega^\omega$ is not Lindelöf and M.E.Rudin's characterization of that problem [5]. See also Alster [1] and Lawrence [3].

Proof of the Theorem.

Let $\mathbb{P}$ be the natural forcing for producing a finite splitting tree using finite conditions. Namely, $p \in \mathbb{P}$ iff $p \subseteq \omega^{<\omega}$ is a finite subtree and $p \leq q$ iff $p \supseteq q$ is an end extension of q. End extension means if $s \in p \setminus q$ then $s \supseteq t$ for some $t \in q$ which is terminal in q, i.e., has no extensions in q. This order is countable and hence forcing equivalent to adding a single Cohen real. The union of $\mathbb{P}$-generic filter is a tree $T \subseteq \omega^{<\omega}$ which determines a compact set $K \subseteq \omega^\omega$ by $K = [T]$.

Given $X \subseteq \omega^\omega$ define $\mathbb{H}_X$ to be a version of Hechler forcing restricted to X:

$$\mathbb{H}_X = \{(s,F) \ : \ s \in \omega^{<\omega} \text{ and } F \in [X]^{<\omega}\}$$

and $(s,F) \leq (t,H)$ iff $F \supseteq H$, $s \supseteq t$, and

$$\forall f \in H \ \ \forall n \ \ [\ |t| \leq n < |s| \ \rightarrow \ f(n) \leq s(n)\ ].$$

Forcing with this determines a $g \in \omega^\omega$ such that

$$\forall f \in X \ \ \forall^\infty n \ \ f(n) \leq g(n).$$

For V a countable transitive model of set theory let $K = [T]$ be $\mathbb{P}$-generic over V and g_α $\mathbb{H}_{X_\alpha}$-generic over $V[K]\langle g_\beta \ : \ \beta < \alpha\rangle$ where $X_\alpha = H_\alpha \cup K_\alpha$;

$$H_\alpha = \omega^\omega \cap V\langle g_\beta : \beta < \alpha\rangle \quad \text{and} \quad K_\alpha = K \cap V[K]\langle g_\beta \ : \ \beta < \alpha\rangle.$$

We use finite supports and so this is a ccc.

Our models are $M = V[\langle g_\alpha \ : \ \alpha < \omega_1\rangle]$ and $N = M[K]$.

Condition (a): In N, $K = \cup_{\alpha<\omega_1} K_\alpha$ and g_α eventually dominates each element of K_α.

Condition (b): It suffices to show that for all $\alpha < \omega_1$ there exists $f \in [T]$ such that $\exists^\infty n \ g_\alpha(n) < f(n)$. This is sufficient because $\{g_\alpha \ : \ \alpha < \omega_1\}$ is a dominating family in M.

Claim. For any $\alpha < \omega_1$ and $s \in T$ there exists $t \supseteq s$ with $t \in T$ such that $t(n) > g_\alpha(n)$ for some n with $|s| < n < |t|$.

Assuming the Claim it is easy to produce $f \in [T]$ which is infinitely often larger than g_α. The rest of the proof is to prove the Claim.

We now describe the posets.

1. For each $\alpha \leq \omega_1$ define $\mathbb{P}_\alpha$:
   (a) $(p, (s_i, K_i, H_i)_{i<\alpha}) \in \mathbb{P}_\alpha$ iff
   (b) $p \in \mathbb{P}$ and
   (c) $s_i = K_i = H_i = \emptyset$ for all but finitely many i,

(d) $s_i \in \omega^{<\omega}$, K_i is finite set of $\mathbb{P}_i$-names for elements of ω^ω and for all $\tau \in K_i$

$$(p, (s_j, K_j, H_j)_{j<i}) \Vdash_{\mathbb{P}_i} \tau \in [T]$$

(e) H_i is a finite set of $\mathbb{H}_i$-names for elements of ω^ω.

(f) Let $N = max\{s(n) \ : s \in p \text{ and } n < s|\}$ and let $f \in \omega^\omega$ be the constant function N. Then $\check{f} \in H_i$ for every $i < \alpha$ with K_i nonempty.

2. $\mathbb{H}_\alpha = \{(p, (s_i, H_i, K_i) : i < \alpha) \in \mathbb{P}_\alpha \ : \ p = \emptyset \text{ and } \forall i < \alpha \ \ K_i = \emptyset\}$

3. τ is $\mathbb{H}_i$-name for an element of ω^ω iff $\tau \subseteq \mathbb{H}_i \times \{\check{\langle n, m \rangle} : n, m < \omega\}$ and

$$1 \Vdash_{\mathbb{P}_i} \forall n \ \exists! m \ \langle n, m \rangle \in \tau.$$

4. Define the ordering on $\mathbb{P}_\alpha$ by: $(p', (s'_i, K'_i, H'_i)_{i<\alpha}) \leq (p, (s_i, K_i, H_i)_{i<\alpha})$ iff

(a) $p' \leq p$ and for each $i < \alpha$

(b) $H_i \subseteq H'_i$ and $K_i \subseteq K'_i$

(c) $\forall \tau \in H_i \cup K_i \ \ \forall n \ \ |s_i| \leq n < |s'_i| \ \ \rightarrow$

$$(p, (s_j, K_j, H_j)_{j<i}) \Vdash_{\mathbb{P}_i} s'(n) \geq \tau(n).$$

There are two possible orderings on $\mathbb{H}_\alpha$. They only differ on the forcing relation in the conclusion of clause 4c, i.e., use $\Vdash_{\mathbb{H}_i}$ instead of $\Vdash_{\mathbb{P}_i}$. It will be necessary to show that these are in fact the same. It is the reason behind the innocuous condition 1f. We will need to drop p but retain some information about p in the H_i's.

Note that any condition not satisfying condition 1f can be extended to one that does simply by putting the required $\check{f}$ into H_i. Define

$$H_i^+ = \begin{cases} H_i \cup \{\check{f}\} & \text{if } K_i \neq \emptyset \\ H_i & \text{otherwise} \end{cases}$$

Then $(p, (s_i, K_i, H_i^+)_{i<\alpha}) \leq (p, (s_i, K_i, H_i)_{i<\alpha})$. So conditions satisfying (1a-1f) are dense in those satisfying (1a-1e).

Lemma 2 *(a) Suppose that* $\tau \subseteq \mathbb{H}_\alpha \times \{\langle n, \check{m} \rangle : n, m < \omega\}$ *and* $\theta(x)$ *is a Borel predicate with parameters from* V*. For any* $(p, (s_i, K_i, H_i)_{i<\alpha}) \in \mathbb{P}_\alpha$

$$(\emptyset, (s_i, \emptyset, H_i)_{i<\alpha}) \Vdash_{\mathbb{H}_\alpha} \theta(\tau) \text{ iff } (p, (s_i, K_i, H_i)_{i<\alpha}) \Vdash_{\mathbb{P}_\alpha} \theta(\tau).$$

(b) Every $D \subseteq \mathbb{H}_\alpha$ *in* V *which is dense in* $\mathbb{H}_\alpha$ *is predense in* $\mathbb{P}_\alpha$*.*
(c) For every G *which is* $\mathbb{P}_\alpha$*-generic over* V *if* $G' = G \cap \mathbb{H}_\alpha$*, then* G' *is* $\mathbb{H}_\alpha$*-generic over* V*.*

Proof
This is by induction on α. $\mathbb{H}_0 = \{\emptyset\}$ is the trivial partial order and so τ evaluates to a ground model real. Hence the trivial condition in $\mathbb{P}_0 = \mathbb{P} * \{\emptyset\}$ decides $\theta(\tau)$. Suppose that the Lemma is true for all $i < \alpha$. This implies that the forcing in clause 4c coincides for the posets $\mathbb{H}_i$ and $\mathbb{P}_i$. Also the forcing in the notion of $\mathbb{H}_i$-name (3) for an element of ω^ω.

Proof of (b). Suppose $D \subseteq \mathbb{H}_\alpha$ is dense in $\mathbb{H}_\alpha$. We claim that it is predense in $\mathbb{P}_\alpha$. Given an arbitrary $(p, (s_i, K_i, H_i)_{i<\alpha}$.

Let $(\emptyset, (s'_i, \emptyset, H'_i)_{i<\alpha}) \leq (\emptyset, (s_i, \emptyset, H_i)_{i<\alpha})$ be any element of D. We need to show that for some $p^+ \leq p$ that

$$(p^+, (s'_i, K_i, H'_i)_{i<\alpha} \leq (p, (s_i, K_i, H^+_i)_{i<\alpha}$$

The only problem is clause 4c for the case that $\tau \in K_i$.

To choose p^+ let $n_1 = \max\{|s'_i| : i < \alpha\}$ and let p^+ be obtained by adding $n_1 + 1$ or more zeros to each terminal node of p. Note that the $f \in H_i$ given by clause 1f dominates all $s \in p^+$. For any $i < \alpha$ and $\tau \in K_i$ since $f \in H_i$ we know that for any n with $|s_i| \leq n < |s'_i|$ that $s'_i(n) \geq f(n) \geq r(n)$ for any $r \in p^+$ and since $\tau \upharpoonright n+1$ is forced to be in p^+ (since it is forced that $\tau \in [T]$), we have verified clause 4c.

Proof of (c). Since the suborder is the same, condition (b) implies that if G $\mathbb{P}_\alpha$-generic over V, then $G \cap \mathbb{H}_\alpha$ is $\mathbb{H}_\alpha$-generic over V.

Proof of (a). Suppose that $(\emptyset(s_i, \emptyset, H_i)_{i<\alpha}) \Vdash_{\mathbb{H}_\alpha} \theta(\tau)$. Then given any G $\mathbb{P}_\alpha$-generic over V with $(p, (s_i, K_i, H_i)_{i<\alpha}) \in G$ we have that

$$(\emptyset, (s_i, \emptyset, H_i)_{i<\alpha}) \in G' =^{def} G \cap \mathbb{H}_\alpha.$$

By the definition of forcing we have that $V[G'] \models \theta(\tau^{G'})$. But $\tau^{G'} = \tau^G$ and by Borel absoluteness $V[G] \models \theta(\tau^G)$. Consequently by the definition of forcing

$$(p, (s_i, K_i, H_i)_{i<\alpha}) \Vdash_{\mathbb{P}_\alpha} \theta(\tau)$$

On the other hand, if it is not the case that $(\emptyset, (s_i, \emptyset, H_i)_{i<\alpha}) \Vdash_{\mathbb{H}_\alpha} \theta(\tau)$, then there exists $(\emptyset, (s'_i, \emptyset, H'_i)_{i<\alpha}) \leq (\emptyset, (s_i, \emptyset, H_i)_{i<\alpha})$ such that

$$(\emptyset, (s'_i, \emptyset, H'_i)_{i<\alpha}) \Vdash_{\mathbb{H}_\alpha} \neg\theta(\tau)$$

as we saw in the proof of (b) we may construct $p^+ \leq p$ such that

$$(p^+, (s'_i, K_i, H'_i)_{i<\alpha}) \leq (p, (s_i, K_i, H_i)_{i<\alpha})$$

$$(p^+, (s'_i, K_i, H'_i)_{i<\alpha}) \Vdash_{\mathbb{P}_\alpha} \neg\theta(\tau)$$

and therefor it is not the case that $(p, (s_i, K_i, H_i)_{i<\alpha}) \Vdash_{\mathbb{P}_\alpha} \theta(\tau)$.
QED

Our next lemma finds a node t in T below s which has nothing to do with the currently mentioned elements of K. This will allow us to extend t without having to worry about extensions of the s_i dominating these elements of K.

Lemma 3 *For any β, $(p, (s_i, K_i, H_i)_{i<\beta}) \in \mathbb{P}_\beta$ and $s \in p$ there exists*

$$(p', (s'_i, K'_i, H'_i)_{i<\beta}) \leq (p, (s_i, K_i, H_i)_{i<\beta})$$

and $t \in p'$ with $s \subseteq t$ with the property that for every $j < \beta$ and every $\tau \in K'_j$ that

$$(p', (s'_i, K'_i, H'_i)_{i<j}) \Vdash t \not\subseteq \tau$$

Proof
The proof is by induction on β. For β limit it is trivial. For the successor case $\beta + 1$ suppose we are given $(p, (s_i, K_i, H_i)_{i\leq\beta})$ and $s \in p$. Let $n = |K_\beta|$. Extend p so as to have at least $n + 1$ nodes $t_1, t_2, \ldots, t_{n+1}$ below s and having the same length, say $|t_i| = k > |s|$. Extend $(p, (s_i, K_i, H_i)_{i<\beta})$ to $(\hat{p}, (\hat{s}_i, \hat{K}_i, \hat{H}_i)_{i<\beta})$ which decides $\tau \restriction k$ for each $\tau \in K_\beta$. Some $\hat{t} = t_i$ is not ruled out. Apply the induction hypothesis to $(\hat{p}, (\hat{s}_i, \hat{K}_i, \hat{H}_i)_{i<\beta})$ and $\hat{t}$.
QED

We are now ready to prove the Claim. Let $T_s = \{t \in T \ : \ s \subseteq t\}$ and suppose for contradiction that for some $(p, (s_i, K_i, H_i)_{i\leq\alpha}) \in \mathbb{P}_{\alpha+1}$ with $s \in p$ that

$$(p, (s_i, K_i, H_i)_{i\leq\alpha}) \Vdash \forall t \in T_s \ \forall n \ \ (|s| < n < |t| \ \rightarrow \ t(n) < g_\alpha(n)).$$

Without loss by Lemma 3 we may assume that there exists $t \in p$ a terminal node of p extending s with the property that all $\tau \in \bigcup_{i \leq \alpha} K_i$ are being forced incompatible with t. Extend t by concatenating zeros to it so that $|t| = n_0 > |s_\alpha|$. By Lemma 2 we can find

$$(\emptyset, (s'_i, \emptyset, H'_i)_{i<\alpha}) \leq (\emptyset, (s_i, \emptyset, H_i)_{i<\alpha})$$

and $N < \omega$ such that for each $\tau \in H_\alpha$ and for each n with $|s_\alpha| \leq n \leq n_0$

$$(\emptyset, (s'_i, \emptyset, H'_i)_{i<\alpha}) \Vdash_{\mathbb{H}_\alpha} \tau(n) < N.$$

In addition we may assume that $N > s(n)$ for all $s \in p$ and $n < |s|$. Now we define p^+ as follows. We extend t by adding $N+1$ to it's end, i.e., $t' \supseteq t$ with $t'(n_0) = N+1$. For all other terminal nodes of p we extend by adding zeros until they are longer than any of the lengths of the s'_i. We define s'_α to be the extension of s_α of length $n_0 + 1$ gotten by adding the constant sequence N.

Then

$$(p^+, (s'_i, K_i, H'_i)_{i \leq \alpha}) \leq (p, (s_i, K_i, H_i)_{i \leq \alpha})$$

because none of the elements of any K_i go thru the node t and all the other nodes are extended by zeros. But this is a contradiction, $t'(n_0) = N+1$ and this condition forces that $g_\alpha(n_0) = N$ since $s'_\alpha(n_0) = N$.

This proves the Claim and therefore Theorem 1.

QED

By general forcing facts N is a ccc generic extension of M. In fact, N is a generic extension of M using a ccc suborder of $\mathbb{P}_{\omega_1}$, see Solovay [6] p.22 definition of Σ. For a proof using complete Boolean algebras see Grigorieff [2]. However this factor forcing cannot be countable in M even though K is added by a poset countable in V.

Proposition 4 *Suppose M is a countable transitive model of ZFC, G $\mathbb{P}$-generic over M where $M \models \mathbb{P}$ is countable. Then M and $N = M[G]$ fail to satisfy Theorem 1.*

Proof

Suppose not. In N let $T \subseteq \omega^{<\omega}$ be a finitely branching tree satisfying:

(a) $\forall f \in [T] \cap N \;\; \exists g \in M \cap \omega^\omega \;\; \forall^\infty n \; f(n) < g(n)$ and

(b) $\forall g \in M \cap \omega^\omega \;\; \exists f \in [T] \cap N \;\; \exists^\infty n \; g(n) < f(n)$.

Define

$$B = \{s \in T \; : \; \exists g \in M \cap \omega^\omega \;\; \forall f \in [T_s] \quad f \leq^* g\}$$

to be the M-bounded nodes of T. Without loss of generality we may assume that B is empty. To see this replace T by $T_0 = T \setminus B$. This will be an M-unbounded tree for which the corresponding B_0 is empty. This is true because given any $H \subseteq M \cap \omega^\omega$ such that $N \models H$ is countable, there exists $g \in M \cap \omega^\omega$ with $h \leq^* g$ for all $h \in H$.

Working in M suppose for some p_0 and name T

$$p_0 \Vdash T \subseteq \omega^{<\omega} \text{ is finitely branching tree and } B = \emptyset.$$

We will prove that (a) fails. Suppose $p \leq p_0$ and $p \Vdash s \in T$, then let

$$T(p) = \{t \in \omega^{<\omega} \; : \; s \subseteq t \text{ and } \exists q \leq p \;\; q \Vdash t \in T\}.$$

Note that $T(p)$ is a subtree of ω^ω. Clearly it cannot be finite branching because then s is M-bounded. Hence for some k the set $\{t(k) : t \in T(p)\}$ is infinite. Using this observation we can build a name τ for an element of $[T]$ which is infinitely often larger than any ground model real, so (a) fails. Construct (p_t, t_s) for $s \in \omega^{<\omega}$ such that

1. $p_{\langle\rangle} = p_0$ and $t_{\langle\rangle} = \langle\rangle$.
2. p_{si} for $i < \omega$ is a maximal antichain beneath p_s.
3. $p_s \Vdash t_s \in T$.
4. $t_{si} \supseteq t_s$ and for some k the set $\{t_{si}(k) : i < \omega\}$ is infinite.
5. For any $q \in \mathbb{P}$ with $q \leq p_0$ there exists s such that $p_s \leq q$.

For any generic filter G containing p_0 there will be a unique $f \in \omega^\omega$ $p_{f\restriction n} \in G$ for all n. Let τ be a name for $\cup_{n<\omega} t_{f\restriction n}$. We claim

$$p_0 \Vdash \tau \in [T] \text{ and } \forall g \in M \; \exists^\infty \; k \;\; g(k) < \tau(k).$$

Suppose for contradiction that $q \leq p_0$ and $g \in M$ satisfy

$$q \Vdash \forall k \;\; \tau(k) \leq g(k).$$

Then some for some s we have $p_s \leq q$. By condition 4 we may find k and i so that $t_{si}(k) > g(k)$. But then $p_{si} \Vdash \tau(k) > g(k)$ which is a contradiction. QED

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Arnold W. Miller
miller@math.wisc.edu
http://www.math.wisc.edu/∼miller
University of Wisconsin-Madison
Department of Mathematics, Van Vleck Hall
480 Lincoln Drive
Madison, Wisconsin 53706-1388