A MAD Q-set

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Abstract

A MAD (maximal almost disjoint) family is an infinite subset $\mathcal{A}$ of the infinite subsets of $\omega = \{0, 1, 2, \ldots\}$ such that any two elements of $\mathcal{A}$ intersect in a finite set and every infinite subset of ω meets some element of $\mathcal{A}$ in an infinite set. A Q-set is an uncountable set of reals such that every subset is a relative G_δ set. It is shown that it is relatively consistent with ZFC that there exists a MAD family which is also a Q-set in the topology in inherits a subset of $P(\omega) = 2^\omega$.

In this paper we answer a question of Hrušák by showing that it is consistent that there exists a maximal almost disjoint family $\mathcal{A} \subseteq [\omega]^\omega$ which is also a Q-set. The reference Hrušák [6] contains some related problems. A topological space is a Q-set iff every subset is a G_δ-set. His reason for asking this question was because in a certain argument involving a topological space $\Psi(\mathcal{A})$ built from a MAD family it would have been helpful to assume that a MAD family cannot be a Q-set. Szeptycki [11] contains some results on van Douwen's Ψ and also on Q-sets.

Our construction is similar to that in Fleissner and Miller [3] where a Q-set is obtained which is concentrated on the rationals. In Judah and Shelah [7] it is shown consistent to have a Q-set while at the same time $\mathfrak{b} = \mathfrak{d} = \omega_1$. Their Q-set forcing has the Sack's property. Their forcing is also used in Nowik and Weiss [10] to construct a Q-set with certain properties and also Gruenhage and Koszmider [4] to construct a topological space with certain properties. In our model as in [3] we have that $\mathfrak{d} = \mathfrak{c} = \omega_2$ and $\mathfrak{b} = \omega_1$.

In Dow [2] and Brendle [1] a type of Q-set forcing is used which preserves towers (so $\mathfrak{p} = \omega_1$) which generalizes Hechler dominating real forcing, and $\mathfrak{b} = \mathfrak{d} = \mathfrak{c}$.

Theorem 1 *It is relative consistent with ZFC, that there exists a MAD family $\mathcal{A} \subseteq [\omega]^\omega$ which is also a Q-set.*

Proof

We begin by forcing a generic MAD family and then we iterate our Q-set forcing to make the generic MAD into a Q-set. The difficulty is to ensure the family stays maximal.

Define: Let $\mathbb{P}$ be the usual poset for forcing a MAD family: $(p, q) \in \mathbb{P}$ iff

1. $p : F \to 2^N$ for some finite $F \subseteq \omega_1$ and $N < \omega$ (write $F = \mathrm{dom}(p)$ and $N = N_p$)

2. q a partial function from a subset of $[F]^2$ into N

3. if $q(\alpha, \beta) = n$, then for every i with $n \leq i < N$ either $p(\alpha)(i) = 0$ or $p(\beta)(i) = 0$.

The uniformity N of lengths in condition (2) is not strictly necessary but it will be convenient and would occur on a dense set anyway.

Define:
$(p_1, q_1) \leq (p_2, q_2)$ iff

1. $\mathrm{dom}(p_1) \supseteq \mathrm{dom}(p_2)$,

2. $p_1(\alpha) \supseteq p_2(\alpha)$ for all $\alpha \in dom(p_2)$, and

3. $q_1 \supseteq q_2$.

Intuitively, we are describing a family $\{a_\alpha \subseteq \omega : \alpha < \omega_1\}$ as follows:

1. $p(\alpha) = s$ means $(i \in a_\alpha$ iff $s(i) = 1)$ for $i < |s|$

2. $q(\alpha, \beta) = n$ promises that $a_\alpha \cap a_\beta \subseteq n$.

Note that (p_1, q_1) and (p_2, q_2) are compatible iff there exists $p_3 \leq p_1, p_2$ such that $(p_3, q_1 \cup q_2)$ is in $\mathbb{P}$.

This forcing is due to Hechler [5]. For G $\mathbb{P}$-generic over M define

$$x_\alpha^G = \bigcup \{p(\alpha) : \exists q \ (p, q) \in G\}$$

Let $X = \{x_\alpha^G : \alpha < \omega_1\}$ and let $\mathcal{A} = \{a_\alpha \subseteq \omega : \alpha < \omega_1\}$ where each x_α is the characteristic function of a_α, i.e.

$$a_\alpha = \{n : x_\alpha(n) = 1\}.$$

The following lemma is due to Hechler.

Lemma 2 $\mathbb{P}$ *is ccc. If G is $\mathbb{P}$-generic over M, then in $M[G]$ the set $\mathcal{A}$ is a maximal almost disjoint family of infinite subsets of ω.*

We will in a sense need to reprove this lemma since we will show that after our new version of Q-set forcing our generic family still remains a maximal almost disjoint family. The idea of the argument is that given a name τ for some infinite subset of ω, we find an α which is not involved with deciding $n \in \tau$ for any n. Then we get a contradiction by swapping the value of $x_\alpha(n) = 0$ to $x'_\alpha(n) = 1$ while still forcing $n \in \tau$. In the usual Q-set forcing while the condition forcing $n \in \tau$ doesn't directly talk about x_α, it may decide that $[s] \subseteq U_n$ where the other condition says $x_\alpha \notin U_n$. These conditions may become inconsistent when we change to x'_α because it might be that $s \subseteq x'_\alpha$ even though s is not a subset of x_α.

A new Q-set forcing

The following is to motivate our definition of $\mathbb{P} * \widetilde{\mathbb{Q}}$. It would be the definition of the new Q-set forcing in the model $M[G]$ where G is $\mathbb{P}$-generic.

Define. For $x \in 2^\omega, s \in 2^{<\omega}, k < \omega$

$$\mathrm{swap}(x, s, k) = \{y \in 2^\omega : s \subseteq y, |\{i \geq |s| : y(i) \neq x(i)\}| \leq k\}.$$

Note that $\mathrm{swap}(x, s, k)$ is a countable closed subset of $[s]$. It contains x if $s \subseteq x$. Also $\mathrm{swap}(x, \langle\rangle, 0) = \{x\}$.

Suppose we are given $X \subseteq 2^\omega$ such that for all $x \neq y \in X$ there are infinitely many n with $x(n) \neq y(n)$. For $Y \subseteq X$ define $\mathbb{Q}(X, Y)$ as follows:
$r \in \mathbb{Q}(X, Y)$ iff r is a finite subset of

$$\{(n, s) : n < \omega, s \in 2^{<\omega}\} \cup \{(n, (x, t, k)) : x \in Y, t \in 2^{<\omega}, \; n, k < \omega\}$$

subject to the condition:

$$\text{if } (n, s) \in r \text{ and } (n, (x, t, k)) \in r, \text{ then } [s] \cap \mathrm{swap}(x, t, k) = \emptyset.$$

The ordering is by inclusion $r_1 \leq r_2$ iff $r_1 \supseteq r_2$. The meaning of these conditions is

1. (n, s) means "$[s] \subseteq U_n$"

2. $(n, (x, t, k))$ means "$\mathrm{swap}(x, t, k) \cap U_n = \emptyset$"

Now suppose G is $\mathbb{Q}(X, Y)$-generic over a model N. Define

$$U_n^G = \bigcup\{[s] : \exists r \in G \; (n, s) \in r\}$$

An easy genericity argument shows that

$$X \cap \bigcap_{n < \omega} U_n^G = X \setminus Y$$

To see this suppose $y \in Y$ and r is any condition. Let n be sufficiently large so as to not appear in r at all. Then let $r' = r \cup \{(n, (y, \langle\rangle, 0))\}$ and note that

$$r' | {\vdash} y \notin U_n.$$

On the otherhand let $y \in X \setminus Y$, r be any condition, and $n < \omega$ be arbitrary. Since y is infinitely often different from any element of X mentioned in r (they must come from Y), we can find $l < \omega$ so that

$$[y \upharpoonright l] \cap \operatorname{swap}(x, s, k) = \emptyset$$

for any $(n, (x, s, k)) \in r$. Now we let $r' = r \cup \{(n, y \upharpoonright l)\}$, then

$$r' | {\vdash} y \in U_n.$$

Next we describe the ordering $\mathbb{P} * \widetilde{\mathbb{Q}}$ which is a basic building block of our iteration. If G is $\mathbb{P}$-generic over M then $\widetilde{\mathbb{Q}}^G$ is essentially the same as $\mathbb{Q}(X, X)$.

Define.
$((p, q), r) \in \mathbb{P} * \widetilde{\mathbb{Q}}$ iff

1. $(p, q) \in \mathbb{P}$

2. r is a finite subset of the union of

$$\{(n, t) : n < N_p, \; t \in 2^{<N_p}\}$$

and

$$\{(n, (\alpha, s, k)) : \alpha \in \operatorname{dom}(p), s \in 2^{<N_p}, \; n, k < N_p\}$$

3. if $(n, (\alpha, s, k)) \in r$ and $(n, t) \in r$, then either s and t are incomparable or $s \subseteq t$ and

$$|\{i : |s| \leq i < |t|, \; t(i) \neq p(\alpha)(i)\}| > k.$$

4

Condition (3) guarantees that for any $x \in 2^\omega$ such that $x \supseteq p(\alpha)$ that

$$\text{swap}(x, s, k) \cap [t] = \emptyset.$$

The ordering is given by

$((p_1, q_1), r_1) \leq ((p_2, q_2), r_2)$ iff $(p_1, q_1) \leq (p_2, q_2)$ and $r_1 \supseteq r_2$.

Note that $((p_1, q_1), r_1)$ and $((p_2, q_2), r_2)$ are compatible iff there exists $p_3 \leq p_1, p_2$ such that $((p_3, q_1 \cup q_2), r_1 \cup r_2)$ is a condition.

The ω_2 iteration.

Our iteration can be described as a suborder of the product

$$\mathbb{P} \times \sum_{\alpha < \omega_2} \mathbb{E}$$

where $\mathbb{E}$ is the set of all finite subsets of

$$\{(n, t) : n < \omega, t \in 2^{<\omega}\} \cup \{(n, (\alpha, s, k)) : \alpha \in \omega_1, s \in 2^{<\omega}, n, k < \omega\}$$

and $\sum_{\alpha < \omega_2} \mathbb{E}$ is the set of all $r : \omega_2 \to \mathbb{E}$ such that $r(\delta)$ is trivial (i.e., the empty set) for all but finitely many δ.

By induction on $\beta \leq \omega_2$ define

$$\mathbb{P}_\beta \subseteq \mathbb{P} \times \sum_{\alpha < \beta} \mathbb{E}$$

as follows:

Define. $\mathbb{P}_0 = \mathbb{P}$, and suppose that we have defined $\mathbb{P}_\beta$ and we are also given a $\mathbb{P}_\beta$ name Y_β for a subset of ω_1, i.e.,

$$\Vdash_\beta Y_\beta \subseteq \omega_1$$

Define. $((p, q), r) \in \mathbb{P}_{\beta+1}$ iff

1. $((p, q), r \restriction \beta) \in \mathbb{P}_\beta$,

2. $((p, q), r(\beta)) \in \mathbb{P} * \widetilde{\mathbb{Q}}$

3. $((p, q), r \restriction \beta) \Vdash_\beta \alpha \in Y_\beta$
   whenever $(n, (\alpha, s, k)) \in r(\beta)$ for some n, s, k, α.

5

For limit ordinals $\lambda \leq \omega_2$ we define $((p, q), r) \in \mathbb{P}_\lambda$ iff for all $\beta < \lambda$ we have $((p, q), r \upharpoonright \beta) \in \mathbb{P}_\beta$ and for all but finitely many $\beta < \lambda$ we have that $r(\beta)$ is the trivial condition (i.e. empty set).

Since the iteration of ccc forcing is ccc all of these forcings are ccc. To see this directly we can argue as follows: Standard arguments using Δ systems show that $\mathbb{P}_\beta$ has precalibre ω_1, i.e., any ω_1 sequence of conditions contain an ω_1 subsequence which is centered. Start with $((p_\alpha, q_\alpha), r_\alpha) \in \mathbb{P}_\beta$ for $\alpha < \omega_1$. We can find an uncountable $\Sigma \subseteq \omega_1$ and finite sets F and H and $N < \omega$ so that

1. $N_\alpha = N$ for all $\alpha \in \Sigma$,

2. $\mathrm{dom}(p_\alpha) \cap \mathrm{dom}(p_\beta) = F$ for $\alpha \neq \beta \in \Sigma$,

3. $\mathrm{dom}(r_\alpha) \cap \mathrm{dom}(r_\beta) = H$ for $\alpha \neq \beta \in \Sigma$,

4. $p_\alpha \upharpoonright F$ are all the same for $\alpha \in \Sigma$,

5. $q_\alpha \upharpoonright [F]^2$ are all the same for $\alpha \in \Sigma$, and

6. $r_\alpha \upharpoonright H$ are all the same with respect to $\{(n, s) : n < \omega, s \in 2^{<\omega}\}$ for $\alpha \in \Sigma$.

Then any two (or even finite subset) of them are compatible.

Assuming that the ground model satisfies the GCH by the usual book keeping argument we can arrange things so that for any $Y \subseteq \omega_1$ which appears in $M[G_{\omega_2}]$ there will be a name for it in the list Y_α for some $\alpha < \omega_2$. The simplest way to do this is to take

$$\{\langle Z_\beta^\alpha : \beta < \omega_1 \rangle : \alpha < \omega_2\}$$

which lists all ω_1 sequences of countable subsets of $\mathbb{P} \times \sum_{\alpha < \omega_2} \mathbb{E}$ with ω_2 repetitions and then define

$$Y_\alpha = \{\langle p, \check{\beta} \rangle : \beta < \omega_1, p \in Z_\beta^\alpha \cap \mathbb{P}_\alpha\}$$

If we define

$$x_\alpha = \bigcup \{s \in 2^{<\omega} : \exists ((p, q), r) \in G, \ \ s = p(\alpha)\} \text{ and } X = \{x_\alpha : \alpha < \omega_1\}$$

6

then X will be the characteristic functions of an almost disjoint family $\mathcal{A} = \{a_\alpha : \alpha < \omega_1\}$. Furthermore if we define the open sets

$$U_n^\beta = \cup\{[s] : \exists((p,q),r) \in G \ (n,s) \in r(\beta)\}$$

then by the usual genericity argument

$$\bigcap_{n<\omega} U_n^\beta \cap X = \{x_\alpha : \alpha \notin Y_\beta^G\}$$

and so X will be a Q-set.

The nontrivial part of our argument is to prove that $\mathcal{A}$ remains a maximal almost disjoint family. So let τ be a name for a counterexample, i.e., suppose

$$((p_0,q_0),r_0)|\vdash\tau \in [\omega]^\omega \text{ and } \forall\alpha < \omega_1\ \tau\cap a_\alpha \text{ is finite }.$$

Let $\Sigma \subseteq \mathbb{P}_{\omega_2}$ be a countable set of conditions extending $((p_0,q_0),r_0)$ such that for any $n \in \omega$ Σ contains a maximal antichain beneath $((p_0,q_0),r_0)$ which decides $n \in \tau$. Let $\alpha_0 < \omega_1$ be any ordinal not mentioned in any condition from Σ. We show $a_{\alpha_0} \cap \tau$ is infinite.

Suppose for contradiction that we have $((p_1,q_1),r_1) \leq ((p_0,q_0),r_0)$, and $N_1 < \omega$ such that

$$((p_1,q_1),r_1)|\vdash\tau \cap a_{\alpha_0} \subseteq N_1$$

Without loss of generality we may assume that $N_1 = N_{p_1}$. By tacking on strings of zeros to the conditions in p_1 we may assume that every integer occurring in r is bounded by $N_1 - 2$ (and not just as required by N_1). Let

$$F = \{\beta : \{\alpha_0, \beta\} \in \text{dom}(q_1)\}$$

Define $r' \supseteq r_1$ as follows:

$$r'(\delta) = r_1(\delta) \cup \{(n,(\alpha_0,t',k+1)) : (n,(\alpha_0,t,k)) \in r_1(\delta), t' \in A_{\delta,n}, t' \supseteq t\}$$

for each δ where

$$A_{\delta,n} = \{t' \in 2^{N_1-1} : t' \text{ is incomparable with all } s \text{ such that } (n,s) \in r_1(\delta)\}$$

Note that $((p_1,q_1),r')$ is a valid condition because α_0 is forced into Y_δ and t' incomparable with all s which might be a problem. Let G be a generic filter containing $((p_1,q_1),r')$. Since τ^G is almost disjoint from each a_β^G and

infinite, there exists some $n_0 \in \tau^G$ with $n_0 > N_1$ and $n_0 \notin a_\beta^G$ for all $\beta \in F$. Let $((p_2, q_2), r_2) \in \Sigma \cap G$ be so that

$$((p_2, q_2), r_2) | \vdash n_0 \in \tau.$$

Since it is from Σ it does not mention α_0.

Let $((p^*, q^*), r^*) \in G$ be stronger than both $((p_1, q_1), r')$ and $(p_2, q_2), r_2)$ and such that $N^* > n_0$. Note that $((p^*, q_1 \cup q_2), r' \cup r_2)$ is a valid condition. Any γ that needs to be forced into some Y_β is already forced in by either $((p_1, q_1), r' \upharpoonright \beta)$ or $((p_2, q_2), r_2 \upharpoonright \beta)$.

If $p^*(\alpha_0)(n_0) = 1$ then we already have a contradiction and there is nothing to prove. So assume not, and define p' to be exactly the same as p^* except $p'(\alpha_0)(n_0) = 1$.

Claim. $((p', q_1 \cup q_2), r \cup r_2)$ is a valid condition, extending both $((p_1, q_1), r_1)$ and $((p_2, q_2), r_2)$.

Proof: Note that we have dropped the extra conditions from r', these were put there just to prove this Claim. The fact that p' extends both p_1 and p_2 uses that $n_0 > N_{p_1} = N_1$ and α_0 is not in the domain of p_2. Similarly since q_2 does not mention α_0, so if $\{\alpha_0, \beta\} \in \mathrm{dom}(q_1 \cup q_2)$, then $\beta \in F$ and we know that $p^*(\beta)(n) = 0$ for each $\beta \in F$. So making $p'(\alpha_0)(n) = 1$ does not violate any promises of disjointedness made in $q_1 \cup q_2$. So we have that $(p', q_1 \cup q_2) \in \mathbb{P}$.

Now fix δ and we must check that

$$((p', q_1 \cup q_2), r_1(\delta) \cup r_2(\delta)) \in \mathbb{P} * \widetilde{\mathbb{Q}}$$

We need to check condition (3)

(3) if $(n, (\alpha, s, k)), (n, t) \in r_1(\delta) \cup r_2(\delta)$ then either s and t are incomparable or $s \subseteq t$ and

$$|\{i : |s| \leq i < |t|, \ t(i) \neq p'(\alpha)(i)\}| > k$$

Suppose it fails. It can only fail if the $\alpha = \alpha_0$ and since r_2 does not mention α_0 it must be that $(n, (\alpha_0, s, k)) \in r_1(\delta)$ and $(n, t) \in r_2(\delta)$. Also it must be that s and t are comparable with $s \subseteq t$ but

$$|\{i : |s| \leq i < |t|, \ t(i) \neq p'(\alpha_0)(i)\}| \leq k$$

Note also that $|t| > n_0 > N_1$ because otherwise

$$\{i : |s| \leq i < |t|, \ t(i) \neq p'(\alpha_0)(i)\} = \{i : |s| \leq i < |t|, \ t(i) \neq p^*(\alpha_0)(i)\}$$

8

But then
$$|\{i : |s| \le i < |t|, \ t(i) \ne p^*(\alpha_0)(i)\}| > k$$
because $((p^*, q_1 \cup q_2), r_1(\delta) \cup r_2(\delta)) \in \mathbb{P} * \widetilde{\mathbb{Q}}$.

Now let $t' = t \upharpoonright (N_1 - 1)$.

Case 1. t' is comparable with some s' such that $(n, s') \in r_1(\delta)$.

Recall that every integer occurring in $r_1(\delta)$ is bounded by $N_1 - 1$. So it must be that $s' \subseteq t'$ but intuitively this is easy because $r_1(\delta)$ is already asserting $[s'] \subseteq U_n^\delta$ and this implies $[t] \subseteq U_n^\delta$. More formally, $s' \subseteq t'$ and therefore s' and s are both initial strings of t' and so comparable, but then we know:
$$|\{i : |s| \le i < |s'| < N_1, s'(i) \ne p_1(\alpha_0)(i)\}| > k$$

But this is still true for p' since we have not changed it below N_1.

Case 2. $t' \in A_{\delta,n}$ and so we added $(\alpha_0, t', k+1)$ to $r'(\delta)$.

But remember $((p^*, q_1 \cup q_2), r' \cup r_2)$ is a valid condition, which means that
$$|\{i : N_1 \le i < |t|, t(i) \ne p^*(\alpha_0)(i)\}| > k + 1$$

but $k < N_1 - 2$ and $p^*(\alpha_0)$ agrees with $p'(\alpha_0)$ except at exactly one coordinate so
$$|\{i : |s| < N \le i < |t|, t(i) \ne p'(\alpha_0)(i)\}| > k$$

This proves that $((p', q_1 \cup q_2), r_1(\delta) \cup r_2(\delta)) \in \mathbb{P} * \widetilde{\mathbb{Q}}$ for every δ.

Finally we must show that

$$((p', q_1 \cup q_2), (r_1 \cup r_2) \upharpoonright \beta)|\vdash_\beta \gamma \in Y_\beta$$

whenever $(n, (\gamma, s, k)) \in (r_1 \cup r_2)(\beta)$ for some n, s, k. But by induction

$$((p', q_1 \cup q_2), (r_1 \cup r_2) \upharpoonright \beta)$$

extends both $((p_1, q_1), r_1 \upharpoonright \beta)$ and $((p_2, q_2), r_2 \upharpoonright \beta)$, one of which does the required forcing.

This proves the Claim. The theorem now follows from the contradiction that

$$((p_1, q_1), r_1)|\vdash \tau \cap a_{\alpha_0} \subseteq N_1$$
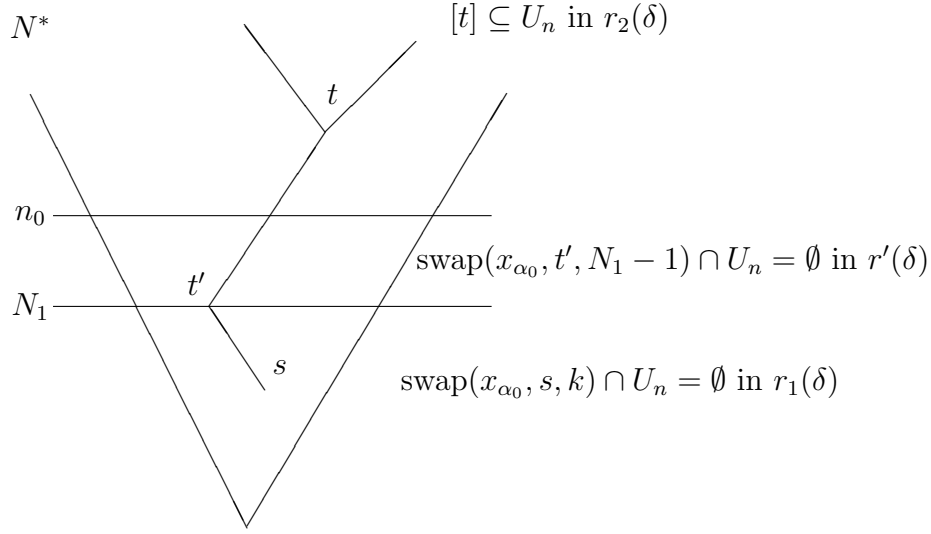$$((p_2, q_2), r_2)|\vdash n_0 \in \tau$$

9

Figure 1: The swap

where $n_0 > N_1$ and

$$((p', q_1 \cup q_2), r_1 \cup r_2)|\vdash n_0 \in a_{\alpha_0}.$$

QED

Remark. The usual Q-set forcing kills the maximality of an almost disjoint family X. To see this suppose $\{x_n : n < \omega\} \subseteq X$ and conditions are finite consistent sets of sentences of the form: "$[s] \subseteq U_n$" or "$x \notin U_n$" where $x \in X \setminus \{x_n : n < \omega\}$. So when we force we get a G_δ set so that

$$\cap_{n<\omega}U_n \cap X = \{x_n : n < \omega\}.$$

In the generic extension we can find $\{k_n : n < \omega\}$ increasing so that

$$k_{n+1} \notin \cup_{i<n}x_i \text{ and } \{y \subseteq \omega : k_{n+1} \in y\} \subseteq \bigcap_{i<n} U_i.$$

Why? Given p find $k_{n+1} > k_n$ not in any x_i for $i < n$ or in any x mentioned in p and put

$$p' = p \cup \{[s_1] \subseteq U_i : i < n, s \in 2^{k_{n+1}-1}\}$$

10

But then $\{k_n : n < \omega\}$ is almost disjoint from all elements of X.

Remark. Since there are perfect almost disjoint families, eg.,

$$\mathcal{A} = \{\{x \upharpoonright n : n < \omega\} : x \in 2^\omega\} \subset P(2^{<\omega})$$

there are always MAD families of arbitrarily large Borel order. Obviously a Q-set cannot have cardinality continuum, however a σ-set can.

Define. $X \subseteq 2^\omega$ is a σ-set iff for every Borel set $B \subseteq 2^\omega$ there exists a G_δ set G such that $B \cap X = G \cap X$.

A Sierpinski set is an example of σ-set (Poprougenko, see Miller [9]).

Theorem 3 *It is consistent with any cardinal arithmetic that there exists a MAD σ-set of size the continuum.*

Proof
This is an easy modification of the argument of the main theorem. Taking any countable transitive model M first force a generic MAD of size continuum, then do a finite support iteration of length continuum to make it into a σ-set.
QED

Remark. H.Woodin, see Larson [8], has shown that if there exists a measurable Woodin cardinal κ, and V and $V[G]$ are both models of CH where $V[G]$ is a generic extension using a partial order of size less than κ, then V and $V[G]$ model exactly the same Σ_1^2 sentences. The existence of a MAD σ-set is a Σ_1^2 sentence. It follows that

CH + there exists a measurable Woodin cardinal implies there is a MAD σ-set.

It is virtually certain that MAD σ-sets have nothing to do with large cardinals, so we have the conjecture:

Conjecture 4 *CH implies there exists a MAD σ-set.*

Theorem 5 *The generic MAD set $\mathcal{A} = \{a_\alpha : \alpha < \omega\}$ is concentrated on $\{a_n : n < \omega\}$, i.e., every open set containing $\{a_n : n < \omega\}$ contains all but countably many elements of $\mathcal{A}$.*

Proof

Let M be a countable standard model of ZFC and G be $\mathbb{P}$-generic over M. Working in M suppose

$$|\vdash \{a_n : n < \omega\} \subseteq U \text{ an open set.}$$

Let $\Sigma \subseteq \mathbb{P}$ be countable so that for every $s \in 2^{<\omega}$ there exist a maximal antichain in Σ which decides "$[s] \subseteq U$".

Claim. $|\vdash a_\alpha \in U$ for any α larger than any mentioned in Σ.

Proof

Suppose not and let $(p,q)|\vdash a_\alpha \notin U$. Choose some n so that n is not in the domain of p. Let $p' = p \cup \{(n,s)\}$ where $(\alpha, s) \in p$ and let

$$q' = q \cup \bigcup \{(\{n,\beta\},k) : (\{\alpha,\beta\},k) \in q\}$$

so $(p',q') \leq (p,q)$ and it says the same things about a_n and a_α. There exists $(\hat{p},\hat{q}) \in \Sigma$ compatible with (p',q') such that $N_{\hat{p}} > N_p$ and

$$(\hat{p},\hat{q})|\vdash [x_n \upharpoonright N_{\hat{p}}] \subseteq U$$

Let (p^*, q^*) extend both (p',q') and $(\hat{p},\hat{q})$. Change p^* to r with same domain but $r(\alpha) = p^*(n)$ and other coordinate all the same. But then $(r, q' \cup \hat{q})$ is a common extension of both (p',q') and $(\hat{p},\hat{q})$. And this is a contradiction.

This proves the Claim and Theorem.

QED

Theorem 6 *CH implies exists a MAD family which is concentrated on the finite subsets of ω and is a λ-set (i.e., every countable subset is a relative G_δ.*

Proof

It is easy to construct a MAD family $\{a_\alpha : \alpha < \omega_1\}$ so that if $f_\alpha : \omega \to a_\alpha$ is the strictly increasing enumeration of α, then for every $\alpha < \beta$ we have that $f_\alpha <^* f_\beta$ and for every $g \in \omega^\omega$ there exists $\alpha < \omega_1$ such that $g \leq^* f_\alpha$, i.e., they form a scale. Rothberger (see Miller [9]) showed that any well-ordered subset of $(\omega^\omega, \leq^*)$ is a λ-set and that any ω_1-ordered unbounded set is concentrated on the rationals.

QED

The same large cardinal results lead to the following conjecture:

12

Conjecture 7 *CH implies there exists a MAD family which is concentrated on a countable subset of itself.*

Paul Szeptycki pointed out that the Q-set forcing using in Theorem 1 can be used to prove the following:

Theorem 8 *It is relatively consistent that there exists a Q-set $X \subseteq [\omega]^\omega$ satisfying the property that for every $a \in [\omega]^\omega$ for all but countably many $x \in X$ we have that $|x \cap a| = |x \setminus a| = \omega$, ie X is a strong splitting family.*

Proof
We replace $\mathbb{P}$ by the Cohen real partial order, i.e., just drop the q's from the (p, q). We use the same $\mathbb{P} * \widetilde{\mathbb{Q}}$. Note that in the basic argument for p' we could have flipped $p'(\alpha_0)(n_0) = 1 - p^*(\alpha_0)(n_0)$ and α_0 could be any $\alpha < \omega_1$ not mentioned in Σ.
QED

Allan Dow asked if it is possible to have $X = \{x_\alpha \in 2^\omega : \alpha < \omega_1\}$ and $Y = \{y_\alpha \in 2^\omega : \alpha < \omega_1\}$ such that $x_\alpha =^* y_\alpha$ for every $\alpha < \omega_1$, and X a Q-set and Y not a Q-set. The answer is yes, in fact, we force something stronger:

Theorem 9 *It is relatively to consistent to have $X = \{x_\alpha \in 2^\omega : \alpha < \omega_1\}$ and $\{y_n \in 2^\omega : n < \omega\}$ such that $x_n =^* y_n$ for every $n < \omega$, and X a Q-set concentrated on $\{y_n : n < \omega\}$ (hence $Y = \{y_n : n < \omega\} \cup \{x_\alpha : \omega \leq \alpha < \omega_1\}$ is not a Q-set).*

Proof
This is a variant of the forcing used in Fleissner-Miller [3]. In that forcing we start by adding an ω_1 batch of Cohen reals $X = \{x_\alpha \in 2^\omega : \alpha < \omega_1\}$. The Q-set forcing is modified to always allow statements of the form $e \notin U_n^\alpha$ where e is any eventually zero element of 2^ω. With this modification the Q-set X is shown to be concentrated on the eventually zero elements of 2^ω.

We modify this slightly as follows. Let y_n be defined by $y_n(m) = x_n(m)$ except $y_n(n) = 1 - x_n(n)$, i.e. we change only one coordinate. Now in the inductive construction of $\mathbb{P}_\alpha$ we use the y_n's instead of the eventually zero reals, i.e. if $x \notin U_m^\alpha \in p(\alpha)$, then either $x = y_n$ for some n or $p \upharpoonright \alpha | \vdash x \in Y_\alpha$.

The rest of the argument is the same as in [3].
QED

I am not sure how to do Theorem 9 with a MAD family. This would be interesting because it would show that the Ψ space does not determine whether or not you have a Q-set.

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