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Theorem 1 *Suppose X is a Luzin set and Y a Sierpinski set. Then $X \times Y$ is $S_1(\Gamma, \mathcal{O})$.*

Lemma 2 *Assume $X \subseteq 2^\omega$ is a dense Luzin set and $Y \subseteq 2^\omega$ is measure dense Sierpinski set (i.e. Y meets every positive measure Borel set). Suppose $(\mathcal{U}_n : n < \omega)$ are families of open sets in $2^\omega \times 2^\omega$ which γ-cover $X \times Y$. Then there exists $(V_n \in \mathcal{U}_n : n < \omega)$ and countable $X_0 \subseteq X$ and $Y_0 \subseteq Y$ such that:*

$$(X \setminus X_0) \times (Y \setminus Y_0) \subseteq \bigcup_n V_n.$$

Proof
Let $\{x_n \in X : n < \omega\}$ be dense in 2^ω. For each n let $\mathcal{U}_n = \{U_{n,m} : m < \omega\}$ and define:

$$U_{n,m}^{x_n} = \{y \in 2^\omega : (x_n, y) \in U_{n,m}\}.$$

Since

$$Y \subseteq \bigcup_N \bigcap_{m>N} U_{n,m}^{x_n}$$

we can choose k_n so that:

$$\mu(U_{n,k_n}^{x_n}) > 1 - \frac{1}{2^n}.$$

Let $C_n \subseteq U_{n,k_n}^{x_n}$ be compact with $\mu(C_n) > 1 - \frac{1}{2^n}$. Since Y is Sierpinski there exists a countable $Y_0 \subseteq Y$ such that:

$$Y \setminus Y_0 \subseteq \bigcup_N \bigcap_{n>N} C_n.$$

Choose $f : \omega \to \omega$ so that for every n:

$$[x_n \upharpoonright f(n)] \times C_n \subseteq U_{n,k_n}.$$

Since X is Luzin there is a countable $X_0 \subseteq X$ such that

$$\forall x \in X \setminus X_0 \quad \exists^\infty n \;\; x \in [x_n \upharpoonright f(n)].$$

It follows that

$$(X \setminus X_0) \times (Y \setminus Y_0) \subseteq \bigcup_n U_{n,k_n}.$$

QED

If $X \subseteq 2^\omega$ is any Luzin set, then there exists a countable $X_0 \subseteq X$ such that $X \setminus X_0$ is homeomorphic to a dense Luzin set. Given any $Y \subseteq 2^\omega$ Sierpinski we may find disjoint closed sets C_n such that $\bigcup_n C_n$ covers Y and for each n either $C_n \cap Y$ is countable or C_n has positive measure and $Y \cap C_n$ is relatively measure dense in C_n. Since the countable union of $S_1(\Gamma, \mathcal{O})$ sets is $S_1(\Gamma, \mathcal{O})$ it is enough to prove the Theorem for dense Luzin sets and measure dense Sierpinski sets.

Given γ-covers $\mathcal{U}_n$ of $X \times Y$ first split ω into infinitely many infinite pieces $(P_n : n < \omega)$. Then apply the Lemma to find $(V_n \in \mathcal{U}_n : n \in P_0)$ and countable X_0, Y_0 such that:

$$(X \setminus X_0) \times (Y \setminus Y_0) \subseteq \bigcup_{n \in P_0} V_n.$$

Using the remaining P_k to cover each of the countably many sets:

$$\{x\} \times Y \text{ and } X \times \{y\}$$

for $x \in X_0$ and $y \in Y_0$.
QED