# Long Borel Hierarchies

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Abstract

We show that there is a model of ZF in which the Borel hierarchy on the reals has length $\omega_2$. This implies that $\omega_1$ has countable cofinality, so the axiom of choice fails very badly in our model. A similar argument produces models of ZF in which the Borel hierarchy has exactly $\lambda+1$ levels for any given limit ordinal $\lambda$ less than $\omega_2$. We also show that assuming a large cardinal hypothesis there are models of ZF in which the Borel hierarchy is arbitrarily long.

## Contents



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# 1 Introduction

In this paper we do not assume the axiom of choice, not even in the form of choice functions for countable families. Define the classical Borel families, $\mathbf{\Pi}^0_\alpha$ and $\mathbf{\Sigma}^0_\alpha$, of subsets of $2^\omega$ for any ordinal $\alpha$ as usual:

1. $\mathbf{\Sigma}^0_0 = \mathbf{\Pi}^0_0 =$clopen subsets of $2^\omega$,
2. $\mathbf{\Pi}^0_{<\alpha} = \bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta$, $\qquad \mathbf{\Sigma}^0_{<\alpha} = \bigcup_{\beta<\alpha} \mathbf{\Sigma}^0_\beta$,
3. $\mathbf{\Sigma}^0_\alpha = \{\bigcup_{n<\omega} A_n \;:\; (A_n : n < \omega) \in (\mathbf{\Pi}^0_{<\alpha})^\omega\}$, and
4. $\mathbf{\Pi}^0_\alpha = \{\bigcap_{n<\omega} A_n \;:\; (A_n : n < \omega) \in (\mathbf{\Sigma}^0_{<\alpha})^\omega\}$.

It follows from these definitions that:

**Proposition 1** *Without using the Axiom of Choice:*

1. $\mathbf{\Pi}^0_\alpha \cup \mathbf{\Sigma}^0_\alpha \subseteq \mathbf{\Pi}^0_\beta \cap \mathbf{\Sigma}^0_\beta$ *for all* $\alpha < \beta$.
2. $\mathbf{\Pi}^0_{<\lambda} = \mathbf{\Sigma}^0_{<\lambda}$ *for limit ordinal* $\lambda$.
3. $\mathbf{\Pi}^0_\alpha = \{2^\omega \setminus X \;:\; X \in \mathbf{\Sigma}^0_\alpha\}$ *for all* $\alpha$.

Proof
The first is immediate from the definitions and the second follows from the first. The last follows from De Morgan's Laws.
QED

The family of Borel subsets of $2^\omega$ is the smallest family of sets containing the clopen sets and closed under countable unions and countable intersections. Equivalently, Borel= $\mathbf{\Sigma}^0_{<\infty} = \mathbf{\Pi}^0_{<\infty}$ where

$$\mathbf{\Sigma}^0_{<\infty} = \bigcup\{\mathbf{\Sigma}^0_\alpha \;:\; \alpha \text{ an ordinal }\} \text{ and } \mathbf{\Pi}^0_{<\infty} = \bigcup\{\mathbf{\Pi}^0_\alpha \;:\; \alpha \text{ an ordinal }\}.$$

The length of the Borel hierarchy is the least $\alpha$ such that every Borel set is $\mathbf{\Sigma}^0_{<\alpha}$. It cannot be $\infty$ since then there would be a map from the power set of $2^\omega$ onto the class of all ordinals.

It is a classical Theorem of Lebesgue 1905 [8] (see Kechris [7] p. 168) that assuming the axiom of choice for countable families, the length of the Borel hierarchy is $\omega_1$. To see that it has height at least $\omega_1$, he shows that $\mathbf{\Sigma}^0_\alpha \neq \mathbf{\Pi}^0_\alpha$ for all $\alpha$ with $1 \leq \alpha < \omega_1$ by constructing universal sets in each Borel class. This requires choosing codes for Borel sets. In the absence of the axiom of choice this may fail. Feferman and Lévy 1963 (see Cohen [3] p.143, Jech [6] p.142) showed that it is relatively consistent with ZF that $2^\omega$ is the countable union of countable sets. This implies that every subset of $2^\omega$ is a countable union of countable sets. Since singletons are closed this means that every subset of $2^\omega$ is the countable union of countable unions of closed sets, i.e., $F_{\sigma\sigma}$. We show in Theorem 2.1 that it is always the case that $\mathbf{\Sigma}^0_3 \neq \mathbf{\Pi}^0_3$. So in the Feferman-Lévy model since $F_{\sigma\sigma} \subseteq \mathbf{\Sigma}^0_4$ we have that $\mathcal{P}(2^\omega) = \mathbf{\Sigma}^0_4 = \mathbf{\Pi}^0_4$ and therefore there are exactly four levels for the Borel hierarchy which is the least possible.

The second place that the axiom of choice is used in the Lebesgue proof is to prove that the Borel hierarchy has length at most $\omega_1$. The regularity of $\omega_1$ implies that $\mathbf{\Sigma}^0_{<\omega_1}$ is closed under countable unions and countable intersections and hence it contains all Borel sets.

Since there is a map from $2^\omega$ onto $\omega_1$, it follows that in any ZF model in which $2^\omega$ is the countable union of countable sets, $\omega_1$ has cofinality $\omega$. In fact, in the Feferman-Lévy model, $\omega_1 = \aleph^L_\omega$.

Péter Komjáth asked if it is possible for the Borel hierarchy to have length greater than $\omega_1$ in some model of ZF. We show that it can be. This is the main result of our paper.

**Theorem 1.1** *There is a symmetric submodel $\mathcal{N}$ of a generic extension of the Feferman-Lévy model $V$ in which the Borel hierarchy on $2^\omega$ has length $\omega_2$, i.e., $\omega_2$ is the least $\alpha$ such that $\mathbf{\Sigma}^0_{<\alpha}$ is the family of all Borel sets.*

Komjáth asks if the Borel hierarchy can have length greater than $\omega_2$. This would require a model in which both $\omega_1$ and $\omega_2$ have cofinality $\omega$. In Gitik 1980 [5] a model of ZF is produced (assuming the consistency of ZFC plus unboundedly many strongly compact cardinals) in which every $\aleph_\alpha$ has cofinality $\omega$. Schindler [10] shows that the consistency strength of two successor singular cardinals is at least a Woodin cardinal (approximately).

We prove:

**Theorem 1.2** *Suppose $V$ is a countable transitive model of ZF in which every $\aleph_\alpha$ has countable cofinality. Then for every ordinal $\lambda$ in $V$, there is symmetric submodel $\mathcal{N}$ of a generic extension of $V$ with the same $\aleph_\alpha$'s as $V$ and the length of the Borel hierarchy in $\mathcal{N}$ is greater than $\lambda$.*

The argument is actually easier than Theorem 1.1, since we can separate the levels of the hierarchy by cardinality.

Note that we do not compute an upper bound on the length of the Borel hierarchy but only prove that it can be arbitrarily long. We leave the exact determination of the length to some interested graduate student who needs a dissertation topic.

The arguments we use are really about closure under countable unions and not about complementation. As we will see, complementation does not help to generate the Borel hierarchy in our models.

Specker 1957 [12] following Church 1927 [2] defines[2] the classes $\mathcal{G}_\alpha$ for $\alpha$ an ordinal as follows:

1. $\mathcal{G}_0$ is the class of countable sets,
2. $\mathcal{G}_{<\alpha} = \bigcup_{\beta<\alpha} \mathcal{G}_\beta$, and
3. $\mathcal{G}_\alpha = \{\bigcup_{n<\omega} X_n \;:\; (X_n : n < \omega) \in (\mathcal{G}_{<\alpha})^\omega\}$.

Gitik proves that in his model every set is in $\mathcal{G}_{<\infty}$, i.e., $V = \mathcal{G}_{<\infty}$. Löwe [9] calls ZF+$V = \mathcal{G}_{<\infty}$ the theory ZFG and discusses some of its philosophical properties.

**Proposition 2** *(Specker [12])*

1. *$\omega_2$ is not the countable union of countable sets. ($\aleph_2 \notin \mathcal{G}_1$)*
2. *In fact, more generally $\aleph_\alpha \notin \mathcal{G}_{<\alpha}$ for any ordinal $\alpha$.*
3. *Similarly $\mathcal{P}(\aleph_\alpha) \notin \mathcal{G}_\alpha$ for any ordinal $\alpha$.*
4. *If every $\aleph_\alpha$ has cofinality $\omega$, then $\aleph_\alpha \in \mathcal{G}_\alpha \setminus \mathcal{G}_{<\alpha}$ for every ordinal $\alpha$.*

Using Proposition 2 it is easy to give a simple example of a $\sigma$-algebra with what might seem like an impossibly long hierarchy:

[2] Actually Specker defines $\mathcal{G}_\alpha$ to be what we would write as $\mathcal{G}_\alpha \setminus \mathcal{G}_{<\alpha}$. We find our definition easier to work with.

**Proposition 3** *Suppose* $\mathrm{cof}(\aleph_\alpha) = \omega$ *for every* $\alpha \leq \omega_2$*. Let* $\mathcal{C}_0$ *be the set of countable or co-countable subsets of* $\aleph_{\omega_2}$*. If* $\mathcal{C}$ *is the* $\sigma$*-algebra generated by* $\mathcal{C}_0$*, then* $\mathcal{C} = \mathcal{P}(\aleph_{\omega_2})$ *and it takes exactly* $\omega_2 + 1$ *steps to generate* $\mathcal{C}$ *from* $\mathcal{C}_0$ *using countable unions and countable intersections.*

Proof
Note that $\aleph_{\omega_2} \in \mathcal{G}_{\omega_2} \subseteq \mathcal{C}$. Since the $\mathcal{G}$'s are closed under taking subsets, we have that every subset of $\aleph_{\omega_2}$ is in $\mathcal{C}$. Define

$$\mathcal{C}_\alpha = \{X \subseteq \aleph_{\omega_2} \;:\; |X| \leq \aleph_\alpha \text{ or } |\aleph_{\omega_2} \setminus X| \leq \aleph_\alpha\}.$$

As usual $\mathcal{C}_{<\alpha} = \bigcup_{\beta<\alpha} C_\beta$. The following facts are easy to show:

1. $X \in \mathcal{C}_\alpha$ iff $\aleph_{\omega_2} \setminus X \in \mathcal{C}_\alpha$.
2. If $\langle X_n : n < \omega\rangle \in (\mathcal{C}_{<\alpha})^\omega$, then $\bigcup_{n<\omega} X_n \in \mathcal{C}_\alpha$ and $\bigcap_{n<\omega} X_n \in \mathcal{C}_\alpha$.
3. If $X \in \mathcal{C}_\alpha$, then there exists $\langle X_n : n < \omega\rangle \in (\mathcal{C}_{<\alpha})^\omega$ such that either $X = \bigcup_{n<\omega} X_n$ or $X = \bigcap_{n<\omega} X_n$.
4. If $A \subseteq \aleph_{\omega_2}$ has the property that $|A| = |\aleph_{\omega_2} \setminus A| = \aleph_{\omega_2}$, then $A \notin \mathcal{C}_{<\omega_2}$.

This shows that the hierarchy has exactly $\omega_2 + 1$ levels.
QED

Apter and Gitik [1] give a partial solution to a (presumably still open) problem of Specker:

Is it possible to have a model of ZF in which $\mathcal{P}(\aleph_\alpha) \in \mathcal{G}_{\alpha+1}$ for every ordinal $\alpha$?

In the Feferman-Lévy model $\mathcal{P}(\omega) \in \mathcal{G}_1 \backslash \mathcal{G}_0$. Gitik shows that in his model that $\mathcal{P}(\omega) \in \mathcal{G}_2 \backslash \mathcal{G}_1$. There is a variation of the Feferman-Lévy model where it is also true that $\mathcal{P}(\omega) \in \mathcal{G}_2 \backslash \mathcal{G}_1$.

In the following Theorem $\mathcal{N}_\alpha$ will be a ZF submodel of the model $\mathcal{N}$ from Theorem 1.1.

**Theorem 1.3** *For* $2 \leq \alpha < \omega_2^V$ *in the model* $\mathcal{N}_\alpha$*,* $\mathcal{P}(\omega) \in (\mathcal{G}_\alpha \backslash \mathcal{G}_{<\alpha})$*. It follows that*

$$\text{Borel} = \mathcal{P}(2^\omega) = \mathcal{G}_\alpha \cap \mathcal{P}(2^\omega).$$

*If* $\alpha$ *is a limit ordinal, then the Borel hierarchy in* $\mathcal{N}_\alpha$ *has exactly* $\alpha+1$ *levels.*

We will make use of several hierarchies determined by closing under countable unions. For example, $\mathcal{A}_\alpha$ (Definition 5.1), is the hierarchy generated by the finite subsets of $2^\omega$. It is the same as the $\mathcal{G}_\alpha$ hierarchy restricted to the subsets of $2^\omega$ but off by one for finite $\alpha$. For technicals reasons it is easier to use than the $\mathcal{G}_\alpha$. The $\mathcal{M}_\alpha$ hierarchy (Definition 5.3) is generated by starting with the nowhere dense sets. In ZFC this would be just the meager sets and would stop at the first step. The $\mathcal{B}_\alpha$ sets (Definition 5.6) are the analog of the sets with the property of Baire, i.e., almost open mod $\mathcal{M}_\alpha$. Note that

$$\mathcal{A}_\alpha \subseteq \mathcal{M}_\alpha \subseteq \mathcal{B}_\alpha.$$

## 2 $\mathbf{\Pi}^0_3 \neq \mathbf{\Sigma}^0_3$

The Hausdorff terminology for the Borel hierarchy is defined as follows: $F$ is the family of closed sets, $G$ is the family of open sets, $F_\sigma$ is the family of sets which can written as the countable union of closed sets, $G_\delta$ is the family of sets which can written as the countable intersection of open sets, $F_{\sigma\delta}$ is the family of sets which can written as the countable intersection of $F_\sigma$ sets, etc.

**Theorem 2.1** *Without using the axiom of choice*

$$F_{\sigma\delta} \neq G_{\delta\sigma}$$

*equivalently* $\mathbf{\Pi}^0_3 \neq \mathbf{\Sigma}^0_3$.

Proof
Let $\mathbb{Q}$ be the set of $x \in 2^\omega$ which are eventually zero. Define $P = \mathbb{Q}^\omega \subseteq (2^\omega)^\omega$. We can identify $(2^\omega)^\omega$ with $2^\omega$ via a recursive pairing function on $\omega^2$. It is easy to check that $P$ is a $F_{\sigma\delta}$-set. We show that $P$ cannot be $G_{\delta\sigma}$.

**Claim.** *Suppose $G \subseteq (2^\omega)^\omega$ is a $G_\delta$ set and $(q_i \in \mathbb{Q} : i < n)$ has the property that*

$$G \subseteq \prod_{i<n}\{q_i\} \times \prod_{n\leq k<\omega} \mathbb{Q}.$$

*Then there exists $m > n$ and $(q_i \in \mathbb{Q} : n \leq i < m)$ such that*

$$G \cap \left(\prod_{i<m}\{q_i\} \times \prod_{m\leq k<\omega} \mathbb{Q}\right) = \emptyset.$$

To prove the Claim assume for simplicity that $n = 0$. So $G \subseteq P$. $G$ is not dense else we could effectively construct $x \in G$ with the property that $x_n \notin \mathbb{Q}$ for every $n$. To see this write $G$ as a descending sequence of dense open sets $U_n$ and construct sequences $(s_n^m \in 2^{<\omega} : m < N_n)$ with

1. $N_n < N_{n+1} < \omega$,
2. $s_m^n \subseteq s_m^{n+1}$ for $m < N_n$,
3. $\{x \in (2^\omega)^\omega \;:\; \forall i < N_n \;\; s_i^n \subseteq x_i\} \subseteq U_n$, and
4. $s_m^{n+1}(k) = 1$ for some $k > |s_m^n|$ and for all $m < N_n$.

By taking the union of the $s_m^n$'s we get $x \in G$ such that $x_n \notin \mathbb{Q}$ for all $n$.

Since $G$ is not dense it is easy to find the required $q_i$'s. This proves the Claim.

Now we prove the theorem. Suppose for contradiction $P = \bigcup_{n<\omega} G_n$ where each $G_n$ is a $G_\delta$. Construct $(q_i \in \mathbb{Q} : i < N_n)$ so that:

$$G_n \cap \left( \prod_{i<N_n} \{q_i\} \times \prod_{N_n \leq k < \omega} \mathbb{Q} \right) = \emptyset$$

by applying the Claim to the $G_\delta$ set

$$G_n \cap \left( \prod_{i<N_{n-1}} \{q_i\} \times \prod_{N_{n-1} \leq k < \omega} 2^\omega \right).$$

But then $(q_i : i < \omega) \in P \setminus \bigcup_{n<\omega} G_n$ which is a contradiction.
QED

We don't know if Theorem 2.1 is new. The set $P$ used in the proof is the same as $P_3$ in Kechris [7] p. 179. The proof Kechris gives to show $P_3$ is properly $\mathbf{\Pi}_3^0$ is to show that the universal $\mathbf{\Pi}_3^0$ set Wadge reduces to $P_3$. But universal sets don't exist in our context.

# 3 Forcing and the Feferman-Lévy Model $V$

**Theorem 3.1** *Suppose $M$ is a countable transitive model of ZF and $\mathbb{P}$ is a partial order in $M$. For any $G$ which is $\mathbb{P}$-generic over $M$ we have that $M[G]$ is a model of ZF with $M \subseteq M[G]$. Furthermore, if $M$ satisfies AC, then $M[G]$ satisfies AC.*

**Theorem 3.2** *Suppose $M$ is a countable transitive model of ZF, $\mathbb{P}$ is a partial order in $M$, and $\mathcal{F} \in M$ is a normal filter of groups of automorphisms of $\mathbb{P}$. For any $G$ which is $\mathbb{P}$-generic over $M$ the symmetric model $\mathcal{N}$ is a model of ZF with $M \subseteq \mathcal{N} \subseteq M[G]$.*

Theorem 3.1 is proved in Shoenfield [11] and Kunen [4] Chapter VII. Both of these author's assume that $M$ is a model of ZFC. The proof does not use it. We have been unable to find a reference stating this well-known fact.[3]

We cannot find a reference for Theorem 3.2 although we think it must be known. In Jech's book on the Axiom of Choice [6] the boolean algebraic version of forcing is used and he assumes that the ground model satisfies AC. At the suggestion of one of the referees we include the proof of Theorem 3.2 here.

## 3.1 Elementary forcing facts

Let $M$ be a countable transitive model of ZF. Let $\mathbb{P}$ be a partial order in $M$. Define

1. $G$ is a $\mathbb{P}$-filter iff
   (a) $G \subseteq \mathbb{P}$
   (b) $p \leq q$ and $p \in G$ implies $q \in G$
   (c) $p, q \in G$ implies there exists $r \in G$ with $r \leq p$ and $r \leq q$.
2. $D \subseteq \mathbb{P}$ is dense iff for every $p \in \mathbb{P}$ there exists $q \leq p$ with $q \in D$.
3. $G$ is $\mathbb{P}$-generic over $M$ iff $G$ is a $\mathbb{P}$-filter and $G \cap D \neq \emptyset$ for every $D \in M$ dense in $\mathbb{P}$.

[3]Kunen says that he didn't mention it because his book is not about the Axiom of Choice. When I was a graduate student in the early 1970's and puzzled over whether the ground model must satisfy AC, Jack Silver told me,"But forcing has nothing to do with the Axiom of Choice."

4. The $\mathbb{P}$-names are defined inductively on rank. $\tau$ is a $\mathbb{P}$-name iff each element of $\tau$ is of the form $(p,\sigma)$ where $p \in \mathbb{P}$ and $\sigma$ is a $\mathbb{P}$-name.

5. Given a $\mathbb{P}$-filter $G$ and $\mathbb{P}$-name $\tau$, the realization[4] of $\tau$ given $G$ is defined inductively by
$$\tau^G = \{\sigma^G \ : \ \exists p \in G \ (p,\sigma) \in \tau\}.$$

6. If $G$ is $\mathbb{P}$-generic over $M$, then
$$M[G] = \{\tau^G \ : \ \tau \text{ is a } \mathbb{P}\text{-name in } M\}.$$

7. Forcing is defined by: $p \Vdash \theta(\vec{\tau})$ iff

   for every $G$ $\mathbb{P}$-generic over $M$ if $p \in G$ then $M[G] \models \theta(\vec{\tau}^G)$. [5]

It is shown that if $M$ is a countable transitive model of $ZF$ then $M[G]$ is a countable transitive model of ZF with $M \subseteq M[G]$. If in addition $M$ satisfies AC, then $M[G]$ also satisfies AC.

This is proved using the two key properties of forcing:

1. (definability) For any formula $\theta(x_1,\ldots,x_n)$,
$$p \Vdash_{\mathbb{P}} \theta(\tau_1,\ldots,\tau_n)$$
   is definable in $M$ by a formula of the form $\psi(p,\mathbb{P},\tau_1,\ldots,\tau_n)$.

2. (truth) If $M[G] \models \theta(\vec{\tau}^G)$, then
$$\exists p \in G \ \ p \Vdash \theta(\vec{\tau}).$$

If $\pi$ is an automorphism of $\mathbb{P}$ in $M$, then $\pi$ extends to the $\mathbb{P}$-names by induction on rank:
$$\pi(\tau) = \{(\pi(p),\pi(\sigma)) \ : \ (p,\sigma) \in \tau\}.$$

A basic fact about such automorphisms is

[4]We prefer the notation $\tau^G$ to Kunen's $val(\tau,G)$.

[5]We may assume our forcing language includes a predicate $\mathring{M}$ denoting the ground model and that $M$ is a definable class of $M[G]$ .

**Lemma 3.3** *If $\pi$ is an automorphism of $\mathbb{P}$ in $M$, then for any formula $\theta$, $p \in \mathbb{P}$, and $\mathbb{P}$-names, $\tau_1, \ldots, \tau_n$*

$$p \Vdash \theta(\tau_1, \ldots, \tau_n) \text{ iff } \pi(p) \Vdash \theta(\pi(\tau_1), \ldots, \pi(\tau_n)).$$

Proof
First prove by induction on rank that

$$\tau^{\pi^{-1}(G)} = \pi(\tau)^G$$

and note that $M[G] = M[\pi^{-1}(G)]$.
The following are equivalent:

1. $p \Vdash \theta(\tau)$.
2. For all $G$ $\mathbb{P}$-generic over $M$ with $p \in G$ $M[G] \models \theta(\tau^G)$.
3. For all $G$ $\mathbb{P}$-generic over $M$ with $p \in \pi^{-1}(G)$ $M[\pi^{-1}(G)] \models \theta(\tau^{\pi^{-1}(G)})$.
4. For all $G$ $\mathbb{P}$-generic over $M$ with $\pi(p) \in G$ $M[G] \models \theta(\pi(\tau)^G)$.
5. $\pi(p) \Vdash \theta(\pi(\tau))$.

We have written the parameters $\tau_1, \ldots, \tau_n$ as $\tau$ to simplify the notation.
QED

## 3.2 The symmetric submodel

Suppose that $\mathcal{H}$ is a group of automorphisms of $\mathbb{P}$ in $M$. Then we can define in $M$:

1. For any $\mathbb{P}$-name $\tau$ the subgroup of $\mathcal{H}$:

   $$\text{Fix}(\tau) = \{\pi \in \mathcal{H} \;:\; \pi(\tau) = \tau\}.$$

2. $\mathcal{F}$ is a normal filter of subgroups of $\mathcal{H}$ iff
   (a) if $H \subseteq K \subseteq \mathcal{H}$ are subgroups and $H \in \mathcal{F}$, then $K \in \mathcal{F}$,
   (b) if $H, K \in \mathcal{F}$, then $H \cap K \in \mathcal{F}$, and
   (c) if $H \in \mathcal{F}$ and $\pi \in \mathcal{H}$, then $\pi H \pi^{-1} \in \mathcal{F}$.

3. $\tau$ is symmetric iff $\mathrm{Fix}(\tau) \in \mathcal{F}$.

4. $\tau$ is hereditarily symmetric iff $\tau$ is symmetric and $\sigma$ is hereditarily symmetric for every $(p, \sigma) \in \tau$.

**Remark**. Suppose $H = \mathrm{Fix}(\tau)$ and $\pi \in \mathcal{H}$, then $\pi H \pi^{-1} \subseteq \mathrm{Fix}(\pi(\tau))$. Hence if $\tau$ is an hereditarily symmetric name and $\pi \in \mathcal{H}$ then $\pi(\tau)$ is an hereditarily symmetric name.

For $G$ which is $\mathbb{P}$-generic over $M$ define the symmetric model:

$$\mathcal{N} = \{\tau^G \ : \ \tau \text{ is an hereditarily symmetric } \mathbb{P}\text{-name in } M \ \}.$$

The fact that $\mathcal{N}$ is transitive follows from the definition of hereditarily symmetric names. $M \subseteq \mathcal{N}$ because the canonical names

$$\check{x} = \{(1, \check{y}) \ : \ y \in x\}$$

are fixed by every automorphism of $\mathbb{P}$. $\mathcal{N} \subseteq M[G]$ is obvious.

Axioms of ZF are true in $\mathcal{N}$:

1. Pair. A name for the pair $\{\tau^G, \sigma^G\}$ is $\{(1, \tau), (1, \sigma)\}$ and

$$\mathrm{Fix}(\tau) \cap \mathrm{Fix}(\sigma) \subseteq \mathrm{Fix}(\{(1, \tau), (1, \sigma)\}).$$

It follows that if $\sigma$ and $\tau$ are hereditarily symmetric, then so is this name for their pair.

2. Union. Given $\mathring{x}$, let

$$\mathring{y} = \{(p, \sigma) \ : \ \exists (r, \rho) \in \mathring{x} \ \ \exists s \ \ (((s, \sigma) \in \rho) \wedge (p \leq s) \wedge (p \leq r))\}$$

Then

$$\Vdash \mathring{y} = \bigcup \mathring{x}$$

and $\mathrm{Fix}(\mathring{x}) \subseteq \mathrm{Fix}(\mathring{y})$. If $\mathring{x}$ is hereditarily symmetric, so is $\mathring{y}$.

3. Power Set. Given $\mathring{x}$ hereditarily symmetric, let

$$Q = \{\sigma \ : \ \exists p \in \mathbb{P} \ \ (p, \sigma) \in \mathring{x}\}.$$

Note that each element of $Q$ is hereditarily symmetric. Let

$$\mathring{y} = \{(p, \sigma) \ : \ \sigma \subseteq \mathbb{P} \times Q \text{ is symmetric and } p \Vdash \sigma \subseteq \mathring{x}\}.$$

Then $\mathring{y}$ is a hereditarily symmetric name for the power set of $\mathring{x}$ in $\mathcal{N}$. Note that the normality condition guarantees that if $\sigma$ is hereditarily symmetric then so is $\pi(\sigma)$ for every $\pi \in \mathcal{H}$. Also if

$$p \Vdash \sigma \subseteq \mathring{x}$$

and $\pi \in \text{Fix}(\mathring{x})$ then

$$\pi(p) \Vdash \pi(\sigma) \subseteq \mathring{x}.$$

So $\text{Fix}(\mathring{x}) \subseteq \text{Fix}(\mathring{y})$.

4. Comprehension. Given a formula $\theta(v, \vec{\tau})$ with hereditarily symmetric parameters and a hereditarily symmetric $\mathring{x}$ then defining $Q$ as before let

$$\mathring{y} = \{(p, \sigma) \in \mathbb{P} \times Q \ : \ p \Vdash \sigma \in \mathring{x} \quad \mathcal{N} \models \theta(\sigma, \vec{\tau})\}.$$

If $\pi$ fixes $\mathring{x}$ and each $\tau_i$ then $\pi(\mathring{y}) = \mathring{y}$.

5. Replacement. We may assume that $M$ is a definable class in $M[G]$ by adding a predicate $\mathring{M}$ if necessary. Since $M[G]$ models replacement and $\mathcal{N}$ is a definable class in $M[G]$ for any formula $\theta(x, y)$ and set $A \in \mathcal{N}$ there will be a set $B \in M$ of hereditarily symmetric names such that for every $a \in A$ if $\mathcal{N} \models \exists y \ \theta(a, y)$ then there exist $\tau \in B$ such that $\mathcal{N} \models \theta(a, \tau^G)$. Then:

$$C = \{(1, \pi(\tau)) : \tau \in B \text{ and } \pi \in \mathcal{H}\}$$

is hereditarily symmetric and $\{\tau^G : \tau \in B\} \subseteq C^G \in \mathcal{N}$.

This concludes the proof of Theorem 3.2.
QED

## 3.3 The Feferman-Lévy Model

Next we describe some of the basic properties of the Feferman-Lévy Model. The ground model satisfies $V = L$, let us call it $L$. In $L$ let $\mathbb{C}ol$ be the following version of the Lévy collapse of $\aleph_\omega$:

$$\mathbb{C}ol = \{p : F \to \aleph_\omega \ : \ F \in [\omega \times \omega]^{<\omega} \text{ and } \forall (n, m) \in F \ \ p(n, m) \in \aleph_n\}$$

ordered by inclusion.

For any $n < \omega$ let $\mathbb{C}ol_n = \{p : \text{dom}(p) \subseteq n \times \omega\}$ and for $G^{\text{col}}$ $\mathbb{C}ol$-generic over $L$ let $G_n^{\text{col}} = G^{\text{col}} \cap \mathbb{C}ol_n$.

The properties we will use of $V$ are summarized in the next Lemma.

**Lemma 3.4** *$L \subseteq V \subseteq L[G^{\text{col}}]$ and $G_n^{\text{col}} \in V$ for each $n$. In $V$, $\mathcal{P}(\omega)$ is the countable union of countable sets, in fact,*

$$\mathcal{P}(\omega) \cap V = \bigcup_{n<\omega} (L[G_n^{\text{col}}] \cap \mathcal{P}(\omega)).$$

*More generally, if $X \subseteq Y \in L$ and $X \in V$, then for some $n < \omega$ we have that $X \in L[G_n^{\text{col}}]$. It follows that $\omega_1^V = \aleph_\omega^L$ and $\omega_2^V = \aleph_{\omega+1}^L$ and is regular in $V$.*

# 4 The model $\mathcal{N}$ for Theorem 1.1

The model $\mathcal{N}$ will be a symmetric submodel of a generic extension of the Feferman-Lévy Model $V$. Working in $L$ we will construct a well-founded tree $T \subseteq (\aleph_{\omega+1})^{<\omega}$. First we make the following definitions:

1. For $s \in (\aleph_{\omega+1})^{<\omega}$ and $\delta < \aleph_{\omega+1}$:

   $s\hat{\ }\langle\delta\rangle$ is the finite sequence of length $|s| + 1$ which begins with $s$ and has one more element $\delta$.

2. For $s \in T$, we let:

   $$\text{Child}(s) = \{\delta : s\hat{\ }\langle\delta\rangle \in T\}.$$

3. For $s \in T$, we let:

   $$\text{rank}(s) = \sup\{\text{rank}(s\hat{\ }\langle\delta\rangle) + 1 : \delta \in \text{Child}(s)\}.$$

4. The terminal nodes or leaves of $T$ are:

   $$\text{Leaf}(T) = \{s \in T : \text{rank}(s) = 0\}.$$

Then $T$ should have the following properties:

1. $\text{Child}(\langle\rangle) = \aleph_{\omega+1}$ and $\text{rank}(\langle\alpha\rangle) = \alpha$ for each $\alpha < \aleph_{\omega+1}$.

2. If $\text{rank}(s) = \alpha + 1$ is a successor ordinal, then

$$\{\delta \ : \ s\hat{}\langle\delta\rangle \in T\} = \omega$$

   and $\text{rank}(s\hat{}\langle n\rangle) = \alpha$ for all $n < \omega$.

3. If $\text{rank}(s) = \lambda$ is a limit ordinal and $\text{cof}(\lambda) = \omega_n$, then

$$\text{Child}(s) = \omega_n$$

   and $\text{rank}(s\hat{}\langle\delta\rangle)$ for $\delta < \omega_n$ is strictly increasing and (necessarily) cofinal in $\lambda$.

It is easy to inductively construct such a $T$ in $L$. Note that in $V$ each $\omega_n^L$ is countable, so except for the root node $\langle\rangle$, $T$ is countably branching in $V$, i.e., $\text{Child}(s)$ is countable for every $s \in T$ except the root node.

Working in $V$ we make the following definitions:

1. Define $\mathbb{P}$ to be the set of finite partial functions $p : F \to 2^{<\omega}$ where $F \in [\text{Leaf}(T)]^{<\omega}$. $\mathbb{P}$ is ordered by $p \leq q$ iff $\text{dom}(p) \supseteq \text{dom}(q)$ and $p(s) \supseteq q(s)$ for every $s \in \text{dom}(q)$.[6]

2. For $\pi$ a permutation, define the support of $\pi$:

$$\text{supp}(\pi) = \{t \in \text{dom}(\pi) \ : \ \pi(t) \neq t\}.$$

3. Let $\mathcal{H}$ be the group of automorphisms of $\mathbb{P}$ which are induced by finite support permutations of $\text{Leaf}(T)$. That is, $\pi \in \mathcal{H}$ iff there exists a finite support permutation $\hat{\pi} : \text{Leaf}(T) \to \text{Leaf}(T)$ such that $\pi : \mathbb{P} \to \mathbb{P}$ is defined by

$$\text{dom}(\pi(p)) = \hat{\pi}(\text{dom}(p)) \text{ and } \pi(p)(s) = p(\hat{\pi}(s)).$$

4. For any $r \in T$ put $\text{Leaf}(r) = \{t \in \text{Leaf}(T) \ : \ r \subseteq t\}$. Note that $\text{Leaf}(s) = \{s\}$ for $s \in \text{Leaf}(T)$.

[6] $\mathbb{P}$ is forcing equivalent to Cohen real forcing, $\text{Fn}(\aleph_{\omega+1}, 2)$. Elements of the partial order $\text{Fn}(A, B)$ are of the form $p : X \to Y$ where $X$ is a finite subset of $A$ and $Y$ is a finite subset of $B$. It is ordered by $p \leq q$ iff $p$ is an extension of $q$. See Kunen [4] p. 211.

5. For any $s \in T \setminus \text{Leaf}(T)$ define:

$$H_s = \{\pi \in \mathcal{H} \;:\; \hat{\pi}(\text{Leaf}(s\hat{\ }\langle\delta\rangle)) = \text{Leaf}(s\hat{\ }\langle\delta\rangle) \text{ for all } \delta \in \text{Child}(s)\}.$$

6. For any $t \in \text{Leaf}(T)$ define $H_t = \{\pi \in \mathcal{H} \;:\; \hat{\pi}(t) = t\}$.

7. Let $\mathcal{F}$ be the filter of subgroups of $\mathcal{H}$ which are generated by the $H_s$, i.e., $H \in \mathcal{F}$ iff there is a finite $Q \subseteq T$ with

$$H_Q \subseteq H \subseteq \mathcal{H} \text{ where } H_Q =^{\text{def}} \bigcap\{H_s \;:\; s \in Q\}.$$

Note that we defined $H_t$ for $t \in \text{Leaf}(T)$ just for convenience of notation, since if $s\hat{\ }\langle n\rangle = t$, then $H_s \subseteq H_t$.

**Lemma 4.1** *The filter of subgroups $\mathcal{F}$ is normal, i.e., for any $\pi \in \mathcal{H}$ and $H \in \mathcal{F}$, we have that $\pi^{-1}H\pi \in \mathcal{F}$.*

Proof
Fix $\pi \in \mathcal{H}$ and $Q \subseteq T$ finite with $H_Q \subseteq H$. Let $R$ be a finite superset of $Q$ which contains the support of $\hat{\pi}$. We claim that $\pi H_R \pi^{-1} = H_R$. This follows from the fact that for any $\sigma \in H_R$ the support of $\hat{\sigma}$ is disjoint from the support of $\hat{\pi}$ and so $\pi\sigma\pi^{-1} = \sigma$.

It follows that:

$$\pi H_R \pi^{-1} = H_R \subseteq H_Q \text{ implies } H_R \subseteq \pi^{-1}H_Q\pi \subseteq \pi^{-1}H\pi$$

and hence $\pi^{-1}H\pi$ is in $\mathcal{F}$.
QED

Let $G$ be $\mathbb{P}$-generic over $V$ and let $\mathcal{N}$ be the symmetric model determined by $\mathcal{H}$ and $\mathcal{F}$.

**Lemma 4.2** $\omega_1^V = \omega_1^{\mathcal{N}}$, $\omega_2^V = \omega_2^{\mathcal{N}}$, *and* $\omega_2^{\mathcal{N}}$ *remains regular in* $\mathcal{N}$.

Proof
It is enough to verify that this is true for $V[G]$ in place of $\mathcal{N}$, since:

$$V \subseteq \mathcal{N} \subseteq V[G].$$

This would seem obvious since $\mathbb{P}$ is forcing equivalent to the poset of the finite partial functions, $\mathrm{Fn}(\kappa, 2)$, where $\kappa$ is $\omega_2^V = \aleph_{\omega+1}^L$. If $V$ were a model of the axiom of choice, then we would know that forcing with $\mathbb{P}$ cannot collapse cardinals.[7]

First we verify that $\omega_1^V = \aleph_\omega^L$ is not collapsed in $V[G]$. Working in $V$, suppose for contradiction there exists $p_0 \in \mathbb{P}$ and a name $\tau$ such that

$$p_0 \Vdash \tau : \omega \to \aleph_\omega^L \text{ is onto.}$$

Define:

$$A = \{(p, n, \beta) \in \mathbb{P} \times \omega \times \aleph_\omega^L \ : \ p \leq p_0 \text{ and } p \Vdash \tau(n) = \check{\beta}\}.$$

Note that for any $(p, n, \beta), (q, n, \gamma) \in A$ that if $\beta \neq \gamma$, then $p$ and $q$ are incompatible.

The set $A$ is a subset of a set in $L$, so it follows from Lemma 3.4 that there exist $k < \omega$ such that $A \in L[G_k^{\text{col}}]$. In $L[G_k^{\text{col}}]$, $\omega_1$ is $\aleph_{k+1}^L$. Since $L[G_k^{\text{col}}]$ is a model of the axiom of choice, the range of $A$, i.e., $\{\alpha \ : \ \exists p, n \ \ (p, n, \alpha) \in A\}$, cannot even cover $\aleph_{k+1}^L$.

Now suppose in $V$:

$$p_0 \Vdash \tau : \omega \to \aleph_{\omega+1}^L \text{ is cofinal.}$$

Define $A$ similarly and suppose $A \in L[G_k^{\text{col}}]$. Then since $\omega_2^V = \aleph_{\omega+1}^L = \aleph_{\omega+1}^{L[G_k^{\text{col}}]}$ it follows that the range of $A$ cannot be cofinal in $\omega_2^V = \aleph_{\omega+1}^L$. This shows that the cofinality of $\omega_2$ is $\omega_2$ in $V[G]$ and hence it is not collapsed and it remains regular.[8]
QED

[7]Can there be a model of ZF in which for some $\kappa$ forcing with $\mathrm{Fn}(\kappa, 2)$ collapses a cardinal? This question interests us because a yes answer would be another example of just how badly set theory can go wrong if the axiom of choice fails. Since we are more interested in a yes answer, we could ask more generally: Can forcing with $\mathrm{Fn}(X, 2)$ for some set $X$ ever make two sets $A$ and $B$ of different cardinality in the ground model become the same cardinality in the generic extension?

[8]An alternative proof for $\omega_2$ being regular in $\mathcal{N}$ is to note that it is $\omega_1$ in the model $L[G^{\text{col}}]$. Since $L[G^{\text{col}}]$ is a model of ZFC forcing with $\mathrm{Fn}(\kappa, 2)$ cannot collapse $\omega_1$. The proof of Theorem 1.2 has an alternative argument for showing that cardinals are not collapsed in $\mathcal{N}$.

# 5 In $\mathcal{N}$ the Borel hierarchy has length $\omega_2$

For each $t \in \text{Leaf}(T)$ let $x_t \in 2^\omega$ be the Cohen real attached to $t$ which is determined by $G$, i.e.,

$$x_t = \bigcup\{p(t) \ : \ t \in \text{dom}(p) \text{ and } p \in G\}.$$

For each $s \in T$ define:

$$A_s = \{x_t \ : \ t \in \text{Leaf}(s)\}.$$

So $A_{\langle\rangle}$ is the set of all the Cohen reals, $x_t$ for $t$ a leaf of $T$. Working in $\mathcal{N}$:

**Definition 5.1** *for each ordinal $\alpha$ define the family $\mathcal{A}_\alpha$ inductively as follows:*

1. *$\mathcal{A}_0$ is the set of finite subsets of $2^\omega$, i.e. $\mathcal{A}_0 = [2^\omega]^{<\omega}$,*
2. *$\mathcal{A}_{<\alpha} = \bigcup_{\beta<\alpha} \mathcal{A}_\beta$, and*
3. *$\mathcal{A}_\alpha = \{\bigcup_{n<\omega} X_n \ : \ (X_n : n < \omega) \in (\mathcal{A}_{<\alpha})^\omega\}$.*

It is not hard to check that

$$\mathcal{A}_{1+\alpha} = \mathcal{P}(2^\omega) \cap \mathcal{G}_\alpha.$$

(Note that a countable union of finite subsets of $2^\omega$ is countable, because $2^\omega$ can be linearly ordered, so $\mathcal{A}_1 = \mathcal{P}(2^\omega) \cap \mathcal{G}_1$.) For technical reasons (for example the statement of the next Lemma) it is easier to work with the hierarchy $\mathcal{A}_\alpha$ than $\mathcal{G}_\alpha$.

**Lemma 5.2** *For each $s \in T$ the set $A_s$ is in $\mathcal{N}$. For each $s \in T$ (except the root node) $A_s \in \mathcal{A}_\alpha$ where* $\text{rank}(s) = \alpha < \omega_2$.

Proof
If $s \in \text{Leaf}(T)$, then the name[9] of $x_s$:

$$\mathring{x}_s = \{(p, \langle \check{n}, \check{i}\rangle) \ : \ p \in \mathbb{P}, \ \ p(s) = \sigma, \text{ and } \sigma(n) = i\}$$

[9]For any $z$ in the ground model we use $\check{z}$ for its canonical name, see Kunen [4] p. 190. For arbitrary elements $z$ of the generic extension, we let $\mathring{z}$ stand for some $\mathbb{P}$-name of $z$. Hence for any $G$ generic $z = \mathring{z}^G$. For example, $\mathring{G} = \{\langle p, \check{p}\rangle \ : \ p \in \mathbb{P}\}$.

is fixed by all $\pi \in H_s$. For any $s \in T$ the set $A_s = \{x_t : t \in \text{Leaf}(s)\}$ has the name $\mathring{A}_s = \{(1, \mathring{x}_t) \ : \ t \in \text{Leaf}(s)\}$ which is fixed by $H_s$.

Fix $s \in T$ with $\text{rank}(s) = \alpha < \omega_2^{\mathcal{N}}$ and assume by induction that for every $\delta \in \text{Child}(s)$ that $A_{s\hat{\ }\langle\delta\rangle} \in \mathcal{A}_{<\alpha}$. Then $H_s$ fixes each $\mathring{A}_{s\hat{\ }\langle\delta\rangle}$ for $\delta \in \text{Child}(s)$ and so it fixes a name for the sequence $\langle A_{s\hat{\ }\langle\delta\rangle} \ : \ \delta \in \text{Child}(s)\rangle$. So this sequence is in $\mathcal{N}$. Since $\text{Child}(s)$ is countable in $V \subseteq \mathcal{N}$, we see that $A_s \in \mathcal{A}_\alpha$.
QED

The elements of $\mathcal{A}_\alpha$ are Borel sets, since finite sets are closed. Similarly in the model $\mathcal{N}$:

**Definition 5.3** *Define $\mathcal{M}_\alpha$ for each $\alpha$ by induction:*

1. *$\mathcal{M}_0$ is the family of nowhere dense subsets of $2^\omega$, i.e., sets whose closure has no interior,*
2. *$\mathcal{M}_{<\alpha} = \bigcup_{\beta<\alpha} \mathcal{M}_\beta$, and*
3. *$\mathcal{M}_\alpha = \{\bigcup_{n<\omega} X_n \ : \ (X_n : n < \omega) \in (\mathcal{M}_{<\alpha})^\omega\}$.*

Note that $\mathcal{A}_\alpha \subseteq \mathcal{M}_\alpha$ since finite sets are nowhere dense. The following Lemma is proved by induction on $\alpha$ and is also true for the $\mathcal{A}_\alpha$ and $\mathcal{G}_\alpha$.

**Lemma 5.4** *For any ordinal $\alpha$ the family $\mathcal{M}_\alpha$ is closed under finite unions and subsets, i.e., if $X, Y \in \mathcal{M}_\alpha$, then $X \cup Y \in \mathcal{M}_\alpha$ and if $X \subseteq Y \in \mathcal{M}_\alpha$, then $X \in \mathcal{M}_\alpha$.*

Proof
Left to reader.
QED

The usual clopen basis for $2^\omega$ consists of sets of the form:

$$[\sigma] = \{x \in 2^\omega \ : \ \sigma \subseteq x\}$$

for $\sigma \in 2^{<\omega}$. The following is the main lemma of the proof of Theorem 1.1.

**Lemma 5.5** *For each $s \in T$ not the root node and $\sigma \in 2^{<\omega}$*

$$(A_s \cap [\sigma]) \notin \mathcal{M}_{<\alpha}$$

*where $\alpha = \text{rank}(s)$.*

Proof
The proof is by induction on rank($s$). For $s \in \mathrm{Leaf}(T)$, i.e., rank($s$) $= 0$, there is nothing to prove. For rank($s$) $= 1$ it easy to see by genericity that $A_s$ is dense in $2^\omega$ and so $A_s \cap [\sigma]$ cannot be in $\mathcal{M}_0$, the nowhere dense sets.

Working in $V$, for contradiction, choose $\alpha > 1$ minimal so that for some $s \in T$ with rank($s$) $= \alpha$ there exists $p_0 \in \mathbb{P}$ and $\sigma \in 2^{<\omega}$ and $\beta < \alpha$ such that

$$p_0 \Vdash (\mathring{A}_s \cap [\sigma]) \in (\mathcal{M}_\beta)^{\mathcal{N}}.$$

Choose a hereditarily symmetric name $(\mathring{X}_n : n < \omega)$ such that

$$p_0 \Vdash \text{“}(\mathring{A}_s \cap [\sigma]) = \bigcup_{n<\omega} \mathring{X}_n \text{ where } \mathring{X}_n \in \mathcal{M}_{\beta_n} \text{ for some } \beta_n < \beta < \alpha.\text{”}$$

Choose a finite $Q \subseteq T$ such that $H_Q$ fixes $\langle \mathring{X}_n : n < \omega \rangle$ and $\mathrm{dom}(p_0) \subseteq Q$. Find an ordinal $\delta$ with

1. $\delta \in \mathrm{Child}(s)$,
2. $\mathrm{rank}(s\hat{\ }\langle\delta\rangle) \geq \beta$, and
3. $Q$ disjoint from $\{r \in T : s\hat{\ }\langle\delta\rangle \subseteq r\}$.

Choose an arbitrary $r \in \mathrm{Leaf}(s\hat{\ }\langle\delta\rangle)$. Since

$$p_0 \cup \{\langle r, \sigma\rangle\} \Vdash \mathring{x}_r \in \mathring{A}_s \cap [\sigma]$$

we can find an extension $p_1 \leq p_0 \cup \{\langle r, \sigma\rangle\}$ and an $n_0$ so that

$$p_1 \Vdash \mathring{x}_r \in \mathring{X}_{n_0} \cap [\sigma].$$

By extending $p_1$ even more, if necessary, we may assume that $p_1(r) = \tau \supseteq \sigma$ where $\tau \in 2^{<\omega}$ has the property that it is incompatible with $p_1(r')$ for every $r' \in \mathrm{dom}(p_1)$ different from $r$.

Claim. $p_1 \Vdash ([\tau] \cap \mathring{A}_{s\hat{\ }\langle\delta\rangle}) \subseteq \mathring{X}_{n_0}$.

Suppose not. Then there exists $p_2 \leq p_1$ and $r' \supseteq s\hat{\ }\langle\delta\rangle$ in $\mathrm{dom}(p_2)$ with $p_2(r') \supseteq \tau$ and

$$p_2 \Vdash \mathring{x}_{r'} \notin \mathring{X}_{n_0}.$$

Let $\pi \in \mathcal{H}$ be determined by the automorphism of Leaf$(T)$ which swaps $r'$ and $r$. Note that $r' \notin \text{dom}(p_1)$ since $\tau$ was incompatible with the range of $p_1$ except $p_1(r)$. It follows from this that $\pi(p_2) \cup p_1$ is a condition in $\mathbb{P}$ (in fact $\pi(p_2) \leq p_1$). By a general property of automorphisms and forcing we have that

$$\pi(p_2) \Vdash \pi(\mathring{x}_{r'}) \notin \pi(\mathring{X}_{n_0}).$$

Since $\pi \in H_Q$ we have that $\pi(\mathring{X}_{n_0}) = \mathring{X}_{n_0}$ and since $\hat{\pi}$ swaps $r'$ and $r$ we have that $\pi(\mathring{x}_{r'}) = \mathring{x}_r$ and so

$$\pi(p_2) \Vdash \mathring{x}_r \notin \mathring{X}_{n_0}.$$

But

$$p_1 \Vdash \mathring{x}_r \in \mathring{X}_{n_0}$$

which contradicts the fact that $\pi(p_2)$ and $p_1$ are compatible.

The Claim contradicts the minimal choice of $\alpha$ since $\beta_{n_0} < \alpha$ and $\mathcal{M}_{\beta_{n_0}}$ is closed under taking subsets. This proves the lemma.
QED

Working in $\mathcal{N}$:

**Definition 5.6** *For any ordinal $\alpha$ define $\mathcal{B}_\alpha$ to be all subsets of $2^\omega$ whose symmetric difference with an open set is in $\mathcal{M}_\alpha$, i.e.,*

$$\mathcal{B}_\alpha = \{X \subseteq 2^\omega \ : \ \exists U \subseteq 2^\omega \textit{ open such that } X \triangle U \in \mathcal{M}_\alpha\}.$$

**Lemma 5.7** *In $\mathcal{N}$*

$$\mathbf{\Sigma}^0_\alpha \cup \mathbf{\Pi}^0_\alpha \subseteq \mathcal{B}_\alpha$$

*for each ordinal $\alpha < \omega_2$.*

Proof
First we note that

(a) $\mathcal{B}_\alpha$ is closed under complementation. If $X \in \mathcal{B}_\alpha$, then $(2^\omega \setminus X) \in \mathcal{B}_\alpha$.

To see this, suppose that $X = U \triangle Y$ where $U$ is open and $Y \in \mathcal{M}_\alpha$. Let $Y' = \text{cl}(U) \setminus U$, then since $Y'$ is nowhere dense we have that $Y' \in \mathcal{M}_0$. Put $V = 2^\omega \setminus \text{cl}(U)$ and then we have that:

$$(2^\omega \setminus X) \triangle V \subseteq Y' \cup Y \in \mathcal{M}_\alpha.$$

Next we claim that

(b) If $\langle X_n : n < \omega\rangle \in (\mathcal{B}_{<\alpha})^\omega$, then $\bigcup_{n<\omega} X_n \in \mathcal{B}_\alpha$.

We need to see we can get the sequence of open sets required without using the axiom of choice.

It follows from Lemma 5.5 that no nonempty open set is in $\mathcal{M}_\alpha$ for $\alpha < \omega_2$. An open set $U \subseteq 2^\omega$ is regular iff it is equal to the interior of its closure, i.e., $U = \text{int}(\text{cl}(U))$. If $U \subseteq 2^\omega$ is an arbitrary open set, then $V = \text{int}(\text{cl}(U))$ is a regular open set containing $U$ such that $V \triangle U$ is nowhere dense and hence in $\mathcal{M}_0$. ($V \triangle U = V \setminus U \subseteq \text{cl}(U) \setminus U$)

It follows that for every $X \in \mathcal{B}_\alpha$ there exists a regular open set $U$ such that $X \triangle U \in \mathcal{M}_\alpha$.

Suppose $U$ and $V$ are regular open sets with $X \triangle U = A$ and $X \triangle V = B$ where $A, B \in \mathcal{M}_\alpha$. Then $U \triangle V = A \triangle B \subseteq A \cup B \in \mathcal{M}_\alpha$. Since $\mathcal{M}_\alpha$ contains no nontrivial open sets and $U$ and $V$ are regular, it must be that $U = V$.

Hence for any $X \in \mathcal{B}_\alpha$ there is a unique regular open set $U$ such that $X \triangle U \in \mathcal{M}_\alpha$. Hence given $\langle X_n : n < \omega\rangle \in (\mathcal{B}_{<\alpha})^\omega$, choose $U_n$ the unique regular open set such that $X_n \triangle U_n = Y_n \in \mathcal{M}_{<\alpha}$. Then

$$(\bigcup_{n<\omega} X_n) \triangle (\bigcup_{n<\omega} U_n) \subseteq \bigcup_{n<\omega} Y_n \in \mathcal{M}_\alpha.$$

From (a) and (b), induction and De Morgan's Laws we have that $\mathbf{\Pi}^0_\alpha$ and $\mathbf{\Sigma}^0_\alpha$ are subsets of $\mathcal{B}_\alpha$.
QED

Proof of Theorem 1.1:
Note that if $\text{rank}(s) = \alpha$ then $A_s \notin \mathcal{B}_{<\alpha}$. If it were, then $A_s = U \triangle Y$ where $U$ open and $Y \in \mathcal{M}_{<\alpha}$. If $U$ is the empty set, then this would contradict Lemma 5.5. But if $U$ is a nonempty set then $U \subseteq A_s \cup Y$. By Lemma 5.2 $A_s \in \mathcal{A}_\alpha \subseteq \mathcal{M}_\alpha$. But Lemma 5.5 implies that no nontrivial open set is in $\mathcal{M}_\alpha$.

It follows since each $A_s$ is Borel that the Borel hierarchy has length at least $\omega_2$. But since $\omega_2$ is a regular cardinal in $\mathcal{N}$ (Lemma 4.2) it must have length exactly $\omega_2$.
QED

**Remark.** Note that in $\mathcal{N}$ if $X$ is any topological space which contains a homeomorphic copy of $2^\omega$, then the Borel order of $X$ is $\omega_2$.

# 6 Proof of Theorem 1.2

Suppose $V$ is countable transitive model of ZF and $\lambda$ is a limit ordinal in $V$. Suppose that in $V$ we have $\mathrm{cof}(\aleph_\gamma) = \omega$ for all $\gamma < \lambda$. We will find a symmetric submodel $\mathcal{N}$ of a generic extension of $V$ with the same $\aleph_\alpha$'s as $V$ and the length of the Borel hierarchy in $\mathcal{N}$ is at least $\lambda$.

Throughout this section let $\kappa = \aleph_\lambda^V$.

Working in $V$ make the following definitions:

1. $\mathbb{P} = \{p : F \to 2^{<\omega} \;:\; F \in [\kappa]^{<\omega}\}$.
2. $\mathcal{H}$ is the group of automorphisms $\pi$ of $\mathbb{P}$ determined by a finite support permutations $\hat{\pi}$ of $\kappa$.
3. For any $q = (X_n : n < \omega)$ a partition of $\kappa$ let
$$H_q = \{\pi \in \mathcal{H} \;:\; \forall n \;\; \hat{\pi}(X_n) = X_n\}.$$
4. $\mathcal{F}$ is the filter of subgroups generated by the set of all such $H_q$.

It is easy to check that $\mathcal{F}$ is a normal filter. For any $G$ which is $\mathbb{P}$-generic over $V$ let $\mathcal{N}$ be the symmetric model determined by $\mathcal{F}$. Let $x_\alpha \in 2^\omega$ be the Cohen real attached to $\alpha$, i.e.,

$$x_\alpha = \bigcup\{p(\alpha) \;:\; p \in G\}.$$

For $X \subseteq \kappa$ define

$$A(X) = \{x_\alpha \;:\; \alpha \in X\}.$$

Note that each $x_\alpha$ is in $\mathcal{N}$ and for any $X \in V$ the set $A(X)$ is in $\mathcal{N}$ since $H_q$ fixes its name where $q = \{X, \kappa\backslash X\}$.

**Lemma 6.1** *For any $\alpha < \lambda$ and set $X \in ([\kappa]^{\aleph_\alpha})^V$*

$$\mathcal{N} \models A(X) \in \mathcal{G}_\alpha.$$

Proof

The proof is by induction on $\alpha$. In $V$ we have that $X = \bigcup_n X_n$ where each $|X_n| = \aleph_{\alpha_n}$ for some $\alpha_n < \alpha$ and the $X_n$ are pairwise disjoint. Take $q$ in $V$ to be the partition

$$q = \{X_n \;:\; n < \omega\} \cup \{\kappa\backslash X\}.$$

Note that $H_q$ fixes the name for the sequence $\langle A(X_n) \ : \ n < \omega\rangle$ and so by induction, $A(X) \in \mathcal{G}_\alpha$.
QED

**Lemma 6.2** *For any $\alpha < \lambda$ if $X \in ([\kappa]^{\aleph_\alpha})^V$ and $\sigma \in 2^{<\omega}$, then*

$$\mathcal{N} \models (A(X) \cap [\sigma]) \notin \mathcal{M}_{<\alpha}.$$

Proof

If $X$ is infinite, $A(X)$ is dense, so $A(X) \cap [\sigma] \notin \mathcal{M}_0$ the nowhere dense sets.

So suppose $\alpha > 0$ and in $V$ write $X$ as the disjoint union of sets $X_n$ for $n < \omega$ of smaller cardinality. Suppose there exists $\beta < \alpha$ and $p_0$ such that

$$p_0 \Vdash A(X) \cap [\sigma] = \bigcup_n Y_n \text{ where } (Y_n : n < \omega) \in (\mathcal{M}_{<\beta})^\omega.$$

Suppose $H_q$ fixes the hereditarily symmetric names $(\mathring{Y}_n : n < \omega)$. By refining the $X_n$ and $q$ we may assume that $q = (Z_n : n < \omega)$ is a partition with $Z_{2n} = X_n$ for all $n$. Choose $Z_{2n_0}$ with $|Z_{2n_0}| \geq \aleph_\beta$ and disjoint from the domain of $p_0$. Choose an arbitrary $\delta \in Z_{2n_0}$ and find an extension $p_1 \leq p_0 \cup \{(\delta, \sigma)\}$ and $n_1$ such that

$$p_1 \Vdash x_\delta \in Y_{n_1}.$$

Let $\tau = p_1(\alpha)$ and assume $\tau$ is incomparable with the other elements of the range of $p_1$.

**Claim.** $p_1 \Vdash A(Z_{2n_0}) \cap [\tau] \subseteq Y_{n_1}$.

Suppose not and take $p_2 \leq p_1$ and $\beta \in Z_{2n_0}$ such that $p_2(\beta) \supseteq \tau$ and

$$p_2 \Vdash x_\beta \notin Y_{n_1}.$$

Then the automorphism $\pi$ which swaps $\delta$ and $\beta$ is in $H_q$ and fixes $\mathring{Y}_{n_1}$ but $p_1$ and $\pi(p_2)$ are compatible and $\pi(p_2) \Vdash x_\delta \notin Y_{n_1}$. This proves the Claim.

The claim yields the lemma.
QED

The two lemmas together imply that the Borel hierarchy in $\mathcal{N}$ has length at least $\lambda$. Lemma 6.1 implies that $A(\aleph_\alpha)$ is Borel. Lemma 6.2 implies (just as in the proof of Theorem 1.1) that $A(\aleph_\alpha) \notin \mathcal{B}_{<\alpha}$. The proof of Lemma 5.7 also holds in this $\mathcal{N}$, so $\mathbf{\Pi}^0_{<\alpha} \cup \mathbf{\Sigma}^0_{<\alpha} \subseteq \mathcal{B}_{<\alpha}$ and the length of the Borel hierarchy is at least $\lambda$.

The last thing to show is that $\mathcal{N}$ and $V$ have the same $\aleph_\alpha$'s [10]. For $B \subseteq \kappa$ in $V$, let $\mathbb{P}_B = \{p \in \mathbb{P} \ : \ \mathrm{dom}(p) \subseteq B\}$ and let $G_B = G \cap \mathbb{P}_B$.

**Lemma 6.3** *Suppose $f : \alpha \to \beta$ is in $\mathcal{N}$ where $\alpha$ and $\beta$ are ordinals. Then there exist in $V$ a countable $B \subseteq \kappa$ such that $f \in V[G_B]$.*

Proof
Let $H_q$ fix $\mathring{f}$ where $q = (X_n : n < \omega)$. Let $B = \bigcup\{X_n \ : \ |X_n| < \omega\}$. Then $B$ is a countable subset of $\kappa$. By the usual automorphism argument $f \in V[G_B]$.
QED

The partial order $\mathbb{P}_B$ is countable in $V$ and so $V$ and $V[G_B]$ have the same cardinals, i.e., if $f : \gamma \to \beta$ is a map in $V[G_B]$, then in $V$ there is map $g : \gamma \times \omega \to \beta$ such that for every $\delta \quad f(\delta) = g(\delta, m)$ for some $m < \omega$. Hence, $\mathcal{N}$ and $V$ have the same $\aleph_\alpha$'s.

This finishes the proof of Theorem 1.2. Note that to use this method to get the Borel hierarchy to have length at least $\omega_2 + 1$ requires us to assume the consistency of $\omega_2 + 1$ strongly compact cardinals. We do not know if we can simply start with any model in which $\omega_1$ and $\omega_2$ both have countable cofinality.

# 7 Proof of Theorem 1.3

We prefer to use the hierarchy $\mathcal{A}_\alpha$ (see 5.1) instead of $\mathcal{G}_\alpha$ and so we will show that

$$\mathcal{N}_\alpha \models 2^\omega \in \mathcal{A}_\alpha \backslash \mathcal{A}_{<\alpha}.$$

As in the proof of Theorem 1.1 let $V$ be the Feferman-Lévy model and $T \in L$ be the well-founded tree of rank $(\aleph_{\omega+1})^L$. For each $\alpha < \omega_2^V$ define

$$T_\alpha = \{s \ : \ \langle\alpha\rangle\hat{\ }s \in T\}.$$

Then the rank of $\langle\rangle$ in $T_\alpha$ is exactly the rank of $\langle\alpha\rangle$ in $T$ which was $\alpha$. Let $\mathcal{N}_\alpha$ be defined exactly as $\mathcal{N}$ but using the tree $T_\alpha$ in place of $T$.

[10] We do not know if $V$ and $V[G]$ have the same cardinals.

**Lemma 7.1** $\mathcal{N}_\alpha \models A_{\langle\rangle} \in \mathcal{A}_\alpha \backslash \mathcal{B}_{<\alpha}$.

Proof
This is exactly the same proof as in Theorem 1.1.
QED

Since $\mathcal{A}_\beta \subseteq \mathcal{B}_\beta$ this implies that $A_{\langle\rangle} \notin \mathcal{A}_{<\alpha}$ and since $A_{\langle\rangle} \subseteq 2^\omega$, and $X \subseteq Y \in \mathcal{A}_\beta$ implies $X \in \mathcal{A}_\beta$, thus we have that

$$\mathcal{N}_\alpha \models 2^\omega \notin \mathcal{A}_{<\alpha}.$$

Most of the remainder of the proof of Theorem 1.3 (Lemmas 7.2-7.7), is to show that

$$\mathcal{N}_\alpha \models 2^\omega \in \mathcal{A}_\alpha.$$

The intuitive reason this is true is because $A_{\langle\rangle} \in \mathcal{A}_\alpha$ and the reals in $\mathcal{N}_\alpha$ can somehow be easily obtained from $A_{\langle\rangle}$ and the reals in $V$.

Let $\langle\cdot,\cdot\rangle$ be a computable pairing function from $\omega \times \omega$ to $\omega$. For example,

$$\langle n, m\rangle = 2^n(2m+1) - 1.$$

Using this define a bijection from $2^\omega$ to $(2^\omega)^\omega$ by

$$x \mapsto (x_n \in 2^\omega : n < \omega) \text{ where } x_n(m) = x(\langle n, m\rangle).$$

Hopefully, we will not confuse the notation $x_n$ with the Cohen reals $x_s$ which are attached to the nodes $s \in \text{Leaf}(T_\alpha)$.

For sets $A, B \subseteq 2^\omega$ define: $A\#B =$

$$\{x \in 2^\omega \ : \ \exists N < \omega \ \exists y \in B \ \ (\forall n < N \ x_n \in A \text{ and } \forall n \geq N \ \ x_n = y_n)\}.$$

**Lemma 7.2** *For any $\alpha \geq 1$ if $A, B \in \mathcal{A}_\alpha$, then $A\#B \in \mathcal{A}_\alpha$.*

Proof
For $\alpha = 1$ note that for $A$ and $B$ countable, the set $A\#B$ is countable (without using choice). Recall that the $\mathcal{A}_\alpha$ families are closed under finite unions. Given increasing sequences $A_n$ and $B_n$ for $n < \omega$ note that:

$$(\bigcup_{n<\omega} A_n)\#(\bigcup_{n<\omega} B_n) = \bigcup_{n<\omega}(A_n\#B_n).$$

So now the result follows by induction.
QED

For $A \subseteq 2^\omega$ define

$$A^{<\omega} = \{x \in 2^\omega \ : \ \exists N < \omega \ \ \forall n < N \ x_n \in A \text{ and } \forall n \geq N \ \ x_n \equiv 0\}$$

where $x \equiv 0$ means $x$ is identically zero.

**Lemma 7.3** *For any $\alpha \geq 1$ if $A \in \mathcal{A}_\alpha$, then $A^{<\omega} \in \mathcal{A}_\alpha$.*

Proof
Note that $A^{<\omega} = A\#\{\underline{0}\}$ where $\underline{0}$ is the identically zero function.
QED

In the model $V[G_\alpha]$ for each $t \in T_\alpha \setminus \text{Leaf}(T_\alpha)$, define

$$B_t = \{x \in 2^\omega : \exists s \supseteq t \ \ \text{rank}(s) = 1 \quad \text{and } \forall n < \omega \quad x_n = x_{s\hat{}\langle n\rangle}\}.$$

Recall that $A_t = \{x_s : s \in \text{Leaf}(t)\}$. Define $C_t = A_t\#B_t$.

**Lemma 7.4** *$C_t \in \mathcal{N}_\alpha$, in fact, $C_t \in (\mathcal{A}_\beta)^{\mathcal{N}_\alpha}$ where $\beta = \text{rank}(t)$.*

Proof
Working in $V$ consider the set $P_t$ of sequences of names, $\langle \mathring{x}_n : n < \omega\rangle$ such that there exists $N < \omega$ and $s \supseteq t$ with $\text{rank}(s) = 1$ such that

1. for all $n < N$ there exists $r \in \text{Leaf}(t)$ such that $\mathring{x}_n = \mathring{x}_r$ and
2. for all $n \geq N$ $\ \mathring{x}_n = \mathring{x}_{s\hat{}\langle n\rangle}$.

Recall that all $\pi \in \mathcal{H}$ have finite support and the $\pi \in H_t$ permute the set of names for elements of $A_t$, i.e., $\{\mathring{x}_s : s \in \text{Leaf}(t)\}$, moving only finitely many of them. It follows that any $\pi \in H_t$ permutes around the elements of $P_t$. From $P_t$ it is an exercise to construct a name for $\mathring{C}_t$ which is fixed by $H_t$.

But $\pi \in H_t$ also map $\mathring{A}_{t\hat{}\langle\delta\rangle}$ to itself for each $\delta \in \text{Child}(t)$. Hence $H_t$ fixes the sequence $(\mathring{C}_{t\hat{}\langle\delta\rangle} : \delta \in \text{Child}(t))$. Recall that $\text{Child}(t)$ is countable in $V \subseteq \mathcal{N}_\alpha$ and since

$$C_t = \bigcup\{(\cup_{s\in F} A_s)\#C_{t\hat{}\langle\delta\rangle} \ : \ \delta \in \text{Child}(t) \text{ and } F \in [\text{Child}(t)]^{<\omega}\}$$

the lemma follows by induction.
QED

Since the lemmas yield $C_{\langle\rangle} \in \mathcal{A}_\alpha$ in $\mathcal{N}_\alpha$ we have:

**Corollary 7.5** $C_{\langle\rangle}^{<\omega} \in \mathcal{A}_\alpha$.

Working in $V$ define $\mathcal{Q}$ to be the set of all $f : \omega \times \omega \to 2^{<\omega} \cup \{*\}$. Since $\mathcal{Q}$ is essentially the same as $\omega^\omega$ we know that $\mathcal{Q}$ is the countable union of countable sets. Given any $f \in \mathcal{Q}$ and $x \in 2^\omega$ define $f(x) \in 2^\omega$ by

$$f(x)(n) = \begin{cases} 1 & \text{if } \exists m \;\; f(n,m) \subseteq x \\ 0 & \text{otherwise.} \end{cases}$$

We assume that $*$ is not a subsequence of any $x$. For example, if $M$ is a model of ZF and $x$ is $2^{<\omega}$-generic over $M$, then for any $y \in M[x] \cap 2^\omega$ there exists $f \in M$ such that $f(x) = y$. To see this, work in $M$, and construct $f$ so that for any $n < \omega$

$$\{f(n,m) \;:\; m < \omega\} = \{p \in 2^{<\omega} \;:\; p \Vdash \mathring{y}(n) = 1\}.$$

**Lemma 7.6** *In $V[G]$, for all $y \in 2^\omega$*

$$y \in \mathcal{N}_\alpha \textit{ iff } \exists f \in \mathcal{Q}^V \; \exists z \in C_{\langle\rangle}^{<\omega} \;\; f(z) = y.$$

Proof
The implication $\leftarrow$ is trivial because both $\mathcal{Q}^V$ and $C_{\langle\rangle}^{<\omega}$ are in $\mathcal{N}_\alpha$.

For the nontrivial direction, we will find $z \in B_{\langle\rangle}^{<\omega}$. Suppose that $y \in 2^\omega \cap \mathcal{N}_\alpha$ and suppose $H_Q$ fixes $\mathring{y}$ where $Q$ is a finite subset of $T_\alpha$.

At this point it would simplify our argument to assume that for any $s \in T$ if $\text{rank}(s) > 1$ , then the $\text{rank}(s\hat{\ }\langle\delta\rangle) > 0$ for all $\delta \in \text{Child}(s)$. Equivalently, the parent of any leaf node has rank one. Obviously we could have built $T$ with this property, so we assume we did.

Assume that $Q$ contains the rank one parent of every rank zero node in $Q$. Let $(s_i : i < N)$ list all rank one nodes in $Q$. Define

1. $\text{Leaf}(Q) = \bigcup\{\text{Leaf}(s_i) : i < N\}$ and
2. $\mathbb{P}_Q = \{p \in \mathbb{P} \;:\; \text{dom}(p) \subseteq \text{Leaf}(Q)\}$.

We claim that $y$ has a $\mathbb{P}_Q$-name. To see this note that for any pair of finite sets $F_0$ and $F_1$ of leaf nodes disjoint from $\text{Leaf}(Q)$ there is a $\pi \in H_Q$ for which $\hat{\pi}(F_0)$ is disjoint from $F_1$. From this it follows that for any $n, i$, and $p \in \mathbb{P}$

$$p \Vdash \mathring{y}(n) = i \text{ iff } p \restriction_{\text{Leaf}(Q)} \;\Vdash \mathring{y}(n) = i.$$

Hence $y$ has a $\mathbb{P}_Q$-name.

Define $z^i \in 2^\omega$ for each $i < N$ so that $z^i_n = x_{s_i\hat{\ }\langle n\rangle}$ for every $n$. So

1. each $z^i$ is in $B_{\langle\rangle}$,
2. $y \in V[\langle z_i : i < N\rangle]$ and
3. $\langle z_i : i < N\rangle$ is $(2^{<\omega})^N$-generic over $V$.

As in the argument of Lemma 4.2, let

$$A = \{(p, n, i) \in (2^{<\omega})^N \times \omega \times \{0, 1\} \ : \ p \Vdash \mathring{y}(n) = i\}.$$

Since there exists $n < \omega$ with $A \in L[G_n]$, we can construct $f \in L[G_n] \subseteq V$ such that $f(\langle z_i : i < N\rangle) = y$.
QED

**Lemma 7.7** *In $\mathcal{N}$, for any set $A \in \mathcal{A}_\alpha$ where $\alpha \geq 2$ the set*

$$\mathcal{Q} \circ A =^{\text{def}} \{f(x) : f \in \mathcal{Q} \text{ and } x \in A\}$$

*is in $\mathcal{A}_\alpha$.*

Proof
For $\alpha = 2$ $\mathcal{A}_\alpha$ is the family of sets which are the countable union of countable sets. Let $A = \bigcup_n A_n$ and let $\mathcal{Q} = \bigcup_n \mathcal{Q}_n$ where $A_n$ and $\mathcal{Q}_n$ are countable. Then for each $n, m < \omega$ the set

$$\{f(x) : x \in A_n \text{ and } f \in \mathcal{Q}_m\}$$

is countable, so $\mathcal{Q} \circ A$ is the countable union of countable sets.

For larger $\alpha$ note that

$$\mathcal{Q} \circ (\bigcup_{n<\omega} A_n) = \bigcup_{n<\omega} \mathcal{Q} \circ A_n$$

so the result follows by induction.
QED

By Corollary 7.5 and Lemmas 7.6 and 7.7, we have that in $\mathcal{N}_\alpha$

$$2^\omega = \mathcal{Q} \circ C_{\langle\rangle}^{<\omega} \in \mathcal{A}_\alpha$$

hence this concludes the proof that

$$\mathcal{N}_\alpha \models 2^\omega \in (\mathcal{A}_\alpha \setminus \mathcal{A}_{<\alpha}).$$

Finally we consider the Borel hierarchy in $\mathcal{N}_\alpha$ when $\alpha$ is a limit ordinal. Note that, $\mathcal{P}(2^\omega) \subseteq \mathcal{A}_\alpha$, because the $\mathcal{A}_\alpha$ families are closed under taking subsets. If $\alpha$ is a limit ordinal then:

$$\mathcal{A}_{<\alpha} \subseteq \mathbf{\Sigma}^0_{<\alpha} \cap \mathbf{\Pi}^0_{<\alpha} \subseteq \mathcal{B}_{<\alpha}.$$

Since the set $A_{\langle\rangle}$ is not in $\mathcal{B}_{<\alpha}$, the Borel hierarchy has to have length at least $\alpha$. But every subset of $2^\omega$ is in $\mathcal{A}_\alpha$, so every subset of $2^\omega$ is in $\mathbf{\Sigma}^0_\alpha$ and hence $\mathbf{\Pi}^0_\alpha$. Hence the Borel hierarchy has exactly $\alpha + 1$ levels.

# 8 A Question

In Theorem 1.3 for successor ordinals $\alpha$ we get a weaker result for the Borel hierarchy. Suppose $\alpha = \lambda + n$ for $\lambda$ limit ordinal and $0 < n < \omega$, then the Borel hierarchy in $\mathcal{N}_\alpha$ has length $\gamma$ where $\lambda + n \leq \gamma \leq \lambda + 2n$. We are not sure what it is exactly. The problem is that in the definition of $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$ we forced an alternation between union and intersection. Hence

$$\mathcal{A}_{\lambda+n} \subseteq \mathbf{\Pi}^0_{\lambda+2n} \cap \mathbf{\Sigma}^0_{\lambda+2n}.$$

If instead we allow taking unions and then more unions, e.g., redefined $\mathbf{\Sigma}^0_\alpha$ (and similarly $\mathbf{\Pi}^0_\alpha$) as:

$$\mathbf{\Sigma}^0_\alpha = \{\bigcup_{n<\omega} A_n \; : \; (A_n : n < \omega) \in (\mathbf{\Sigma}^0_{<\alpha} \cup \mathbf{\Pi}^0_{<\alpha})^\omega\},$$

then this problem disappears and the Borel hierarchy has length exactly $\alpha$ even for successor ordinal case.

On the other hand, we could instead define $\mathbf{\Sigma}^0_\alpha$ to be the smallest class of sets containing $\mathbf{\Pi}^0_{<\alpha}$ and closed under countable unions, and similarly, $\mathbf{\Pi}^0_\alpha$ to be the smallest class of sets containing $\mathbf{\Sigma}^0_{<\alpha}$ and closed under countable intersections. Then in our models for Theorem 1.3, $\mathbf{\Sigma}^0_2$ contains all subsets of $2^\omega$. Similarly the sets $A_s$ and $A(\aleph_\alpha)$ from Theorems 1.1 and 1.2 would be $\mathbf{\Sigma}^0_2$.

**Question 8.1** *Using this alternative definition of the length of the Borel hierarchy, can it be greater than $\omega_1$?*

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# Appendix

The appendix is not intended for final publication but for the on-line electronic version only.

## The Feferman-Lévy model

The ground model satisfies $V = L$, let us call it $L$. In $L$ let $\mathbb{C}ol$ be the following version of the Lévy collapse of $\aleph_\omega$:

$$\mathbb{C}ol = \{p : F \to \aleph_\omega \ : \ F \in [\omega \times \omega]^{<\omega} \text{ and } \forall (n,m) \in F \ \ p(n,m) \in \aleph_n\}.$$

The group $\mathcal{H}$ of automorphisms of $\mathbb{C}ol$ are those which are determined by finite support permutations of $\omega \times \omega$ which preserve the first coordinate, that is, $\pi \in \mathcal{H}$ iff there exists a finite support permutation $\hat{\pi} : \omega \times \omega \to \omega \times \omega$ such that $\hat{\pi}(n,m) = (n', m')$ implies $n = n'$ and $\pi(p)(s) = p(\hat{\pi}(s))$ for all $p \in \mathbb{C}ol$. The normal filter $\mathcal{F}$ of subgroups is generated by

$$H_n = \{\pi \in \mathcal{H} \ : \ \hat{\pi} \restriction n \times \omega \text{ is the identity } \}$$

for $n < \omega$.

The Feferman-Lévy model, $V$, is the symmetric model $L \subseteq V \subseteq L[G]$ determined by $\mathbb{C}ol$, $G$, and the group $\mathcal{H}$ and filter of subgroups $\mathcal{F}$.

For any $n < \omega$ let

$$\mathbb{C}ol_n = \{p \in \mathbb{C}ol \ : \ \text{dom}(p) \subseteq n \times \omega\}.$$

For $G$ $\mathbb{C}ol$-generic over $L$ let $G_n = G \cap \mathbb{C}ol_n$. Note that $H_n$ fixes the canonical name for $G_n$,

$$\mathring{G}_n = \{(p, \check{p}) : p \in \mathbb{C}ol_n\}$$

so $L[G_n] \subseteq V$. If we let

$$\mathring{X}_n = \{(1, \tau) \ : \ \tau \subseteq \mathbb{C}ol_n \times \{\check{k} : k < \omega\}\}$$

then $X_n = L[G_n] \cap \mathcal{P}(\omega)$ and every $\pi \in \mathcal{H}$ fixes $\mathring{X}_n$. It follows that the sequence $(L[G_n] \cap \mathcal{P}(\omega) \ : \ n < \omega)$ is in $V$. Note that each $L[G_n] \cap \mathcal{P}(\omega)$ is countable in $V$.

**Theorem 9.1**
$$\mathcal{P}(\omega) \cap V = \bigcup_{n<\omega} (L[G_n] \cap \mathcal{P}(\omega)).$$

*More generally, if $X \subseteq Y \in L$ and $X \in V$, then for some $n < \omega$ we have that $X \in L[G_n]$*

Proof
We prove the last statement. Suppose

$$p_0 \Vdash \mathring{X} \subseteq \check{Y} \in L \text{ and } \mathring{X} \in V.$$

Choose $n$ large enough so that $H_n$ fixes $\mathring{X}$ and $p_0 \in \mathbb{C}ol_n$.

Note that for each $k \geq n$ that $\pi \in H_n$ can arbitrarily permute $\{k\} \times \omega$. It follows that for any $y \in Y$ and $p \leq p_0$ that

$$p \Vdash \check{y} \in \mathring{X} \text{ iff } p \restriction_{(n \times \omega)} \ \Vdash \check{y} \in \mathring{X}$$

and similarly

$$p \Vdash \check{y} \notin \mathring{X} \text{ iff } p \restriction_{(n \times \omega)} \ \Vdash \check{y} \notin \mathring{X}.$$

Define

$$\mathring{W} = \{(p,\ \check{y}) \in \mathbb{C}ol_n \times \{\check{y} : y \in Y\} \ : \ p \leq p_0 \text{ and } p \Vdash \check{y} \in \mathring{X}\}.$$

It follows that $p_0 \Vdash \mathring{X} = \mathring{W}$. But clearly, $W^G \in L[G_n]$.
QED

## A variant of the Feferman-Lévy model

We show that the following variant of the Feferman-Lévy model has the property that $\mathcal{P}(\omega) \in \mathcal{G}_2 \setminus \mathcal{G}_1$ using an argument similar to Gitik's. Redefine the Lévy Collapse as follows:

$$\mathbb{C}ol = \{p : F \to \aleph_\omega \ : \ F \in [\aleph_\omega \times \omega]^{<\omega} \text{ and } \forall (\alpha, m) \in F \ \ p(\alpha, m) \in \alpha\}.$$

The group $\mathcal{H}$ is defined similarly, the normal filter of subgroups, $\mathcal{F}$, is defined to be the filter generated by subgroups of the form

$$H_F = \{\pi \in \mathcal{H} \ : \ \hat{\pi} \restriction F \times \omega \text{ is the identity}\}$$

where $F \in [\aleph_\omega]^{<\omega}$. Call this alternative Feferman-Lévy model $V'$.

**Theorem 9.2** *In $V'$ we have that $\mathcal{P}(\omega)$ is not the countable union of countable sets but is the countable union of countable unions of countable sets.*

Proof

For any finite $F \subseteq \aleph_\omega$ define

$$\mathbb{C}ol_F = \{p \in \mathbb{C}ol \ : \ \mathrm{dom}(p) \subseteq F \times \omega\}$$

and for $G$ which is $\mathbb{C}ol$-generic define

$$G_F = G \cap \mathbb{C}ol_F.$$

**Claim.** $\mathcal{P}(\omega) \cap V' = \bigcup\{L[G_F] \cap \mathcal{P}(\omega) \ : \ F \in [\omega_1^V]^{<\omega}\}$.

This claim follows from a similar argument to the ordinary Feferman-Lévy model.

Each $\mathbb{C}ol_F$-name is fixed by $H_F$. The set of all $\mathbb{C}ol_F$-names:

$$\mathring{X}_F = \{(1, \tau) : \tau \text{ is a } \mathbb{C}ol_F\text{-name}\}$$

is fixed by every $\pi \in \mathcal{H}$. Note that $L[G_F] \cap \mathcal{P}(\omega) = X_F^G$ is a countable set in $V'$ and the sequence $(X_F^G : F \in [\aleph_\omega^L]^{<\omega})$ is in $V'$. Note that

$$\bigcup_{n<\omega} \cup\{L[G_F] \cap \mathcal{P}(\omega) \ : \ F \in [\aleph_n^L]^{<\omega}\}$$

is a countable union of countable unions of countable sets.

Now we prove that in $V'$ the power set of $\omega$ is not the countable union of countable sets. This follows from the

**Claim.** If $Y \subseteq X \in L$ and $Y \in V'$, then there exists $F$ finite such that $Y \in L[G_F]$.

This claim is proved similarly to Theorem 9.1.

In $V'$, suppose for contradiction that $\mathcal{P}(\omega) = \bigcup_{n<\omega} Y_n$ where each $Y_n$ is countable. Working in $L$ let $(\mathring{Y}_n : n < \omega)$ and $(\mathring{f}_n : n < \omega)$ be sequences of hereditarily symmetric names and $p \in \mathbb{C}ol$ such that for each $n$

$$p \Vdash \mathring{f}_n : \omega \to \mathring{Y}_n \text{ is onto.}$$

By the Claim we can find in $L$ a sequence $(F_n : n < \omega)$ of finite sets such that

$$p \Vdash \mathring{f}_n \in L[G_{F_n}].$$

Choose any $\alpha \notin \bigcup_n F_n$ and let $x \subseteq \omega$ code the generic map $g_\alpha : \omega \to \alpha$. Then $x \notin \bigcup_n Y_n$.
QED

## A remark on descriptive set theory

Lévy [10] shows that in any model of ZF in which $\omega_1 = \aleph_\omega^L$ there is a $\Pi^1_2$ predicate $Q(n, x)$ on $\omega \times 2^\omega$ such that

$$\forall n \exists x \, Q(n, x) \quad \wedge \quad \neg\exists(x_n : n < \omega)\forall n \, Q(n, x_n).$$

The predicate $Q$ says that $x$ is a code for a countable model of the form $(L_\alpha, \in)$ with $n$ infinite cardinals and there is no real $y$ coding a model of the form $(L_\beta, \in)$ with $\beta > \alpha$ in which these cardinals are collapsed. He notes that such an example cannot be done for a $\mathbf{\Sigma}^1_2$ predicate because the Kondo-Addison Theorem can be proved without the axiom of choice.

## Proof of Specker's Proposition

At the suggestion of one of the referees the proof was omitted from the published paper. We include it here for the convenience of the online reader.

**Proposition**. (Specker)

1. $\omega_2$ is not the countable union of countable sets.

   In fact, more generally

2. $\aleph_\alpha \notin \mathcal{G}_{<\alpha}$ for any ordinal $\alpha$.

   Similarly

3. $\mathcal{P}(\aleph_\alpha) \notin \mathcal{G}_\alpha$ for any ordinal $\alpha$.

   If every $\aleph$ has cofinality $\omega$, then

4. $\aleph_\alpha \in \mathcal{G}_\alpha$ for every ordinal $\alpha$.

Proof

(1) Suppose for contradiction that $\omega_2 = \bigcup_{n<\omega} X_n$ where each $X_n$ is countable. For each $n < \omega$ there exists a unique countable ordinal $\alpha_n < \omega_1$ and unique order preserving bijection $f_n : \alpha_n \to X_n$. Therefore there is no choice required to define the onto map $f : \omega \times \omega_1 \to \omega_2$ by:

$$f(n,\alpha) = \begin{cases} f_n(\alpha) & \text{if } \alpha < \alpha_n \\ 0 & \text{otherwise.} \end{cases}$$

But there is a definable bijection between $\omega \times \omega_1$ and $\omega_1$ so this would be a contradiction.

(2) Left to the reader.

(3) In ZF there is a bijection between $\kappa$ and $\kappa \times \kappa$ for any infinite ordinal $\kappa$. Also there is a map from $\mathcal{P}(\kappa \times \kappa)$ onto $\kappa^+$ (map each well-ordering onto its order type). Since $\mathcal{G}_\alpha$ is closed under taking images and $\aleph_{\alpha+1} \notin \mathcal{G}_\alpha$ the claim follows.

(4) $\aleph_0 \in \mathcal{G}_0$. Given $\aleph_\alpha$ we have by induction that for every ordinal $\beta < \aleph_\alpha$ that $\beta \in \mathcal{G}_{<\alpha}$ and since the cofinality of $\aleph_\alpha$ is $\omega$ the proposition follows.
QED

## The width of the Borel hierarchy

At the suggestion of one of the referees, this section was omitted from the published paper.

Rather than using the terminology, $F_{\sigma\sigma\delta\sigma\sigma}$, for example, let us consider the following. For $f \in 2^{<\omega_1}$ define the class $\Gamma_f$ as follows:

1. $\Gamma = \Gamma_{\langle\rangle}$ be the family of clopen subsets of $2^\omega$.
2. For $f : \delta \to 2$ where $\delta$ is a limit ordinal, define:

$$\Gamma_f = \bigcup\{\Gamma_{f\restriction\alpha} \;:\; \alpha < \delta\}.$$

3. For $f : \alpha + 1 \to 2$ define:

$$\text{if } f(\alpha) = 0 \text{ then } \Gamma_f = \{\bigcup_{n<\omega} A_n \;:\; (A_n : n < \omega) \in (\Gamma_{f\restriction\alpha})^\omega\},$$

$$\text{if } f(\alpha) = 1 \text{ then } \Gamma_f = \{\bigcap_{n<\omega} A_n \;:\; (A_n : n < \omega) \in (\Gamma_{f\restriction\alpha})^\omega\}.$$

Hence $F_{\sigma\sigma\delta\sigma\sigma} = \Gamma_{\langle 1,0,0,1,0,0\rangle}$.

Note that $\Gamma_{\langle 0,0\rangle} = \Gamma_{\langle 0\rangle}$ = open sets and $\Gamma_{\langle 1,1\rangle} = \Gamma_{\langle 1\rangle}$ = closed sets. To rule out these trivial collapses, we define nontrivial $f : \delta \to 2$ to be admissible if $f(0) \neq f(1)$.

For $f$ and $g$ admissible define $f \trianglelefteq g$ iff there exists a strictly increasing

$$\pi : \mathrm{dom}(f) \to \mathrm{dom}(g) \text{ such that } \forall \alpha \in \mathrm{dom}(f) \;\; f(\alpha) = g(\pi(\alpha)).$$

Note that if $f \trianglelefteq g$, then $\Gamma_f \subseteq \Gamma_g$. Instead of looking for very long Borel hierarchies we can ask instead for very wide Borel hierarchies:

**Conjecture 9.3** *It is relatively consistent with ZF that for every $f$ and $g$ admissible*

$$f \trianglelefteq g \textit{ iff } \Gamma_f \subseteq \Gamma_g.$$

However, it is impossible that it be infinitely wide, by which we mean:

**Theorem 9.4** *For any infinite set $X$ of admissables there exists distinct $f, g \in X$ with $f \trianglelefteq g$, hence $\Gamma_f \subseteq \Gamma_g$.*

Proof

The ordering $\trianglelefteq$ is a well-quasiordering. This is due to Nash-Williams [12]. We show how to avoid using the axiom of choice.

A well-quasi ordering $(Q, \trianglelefteq)$ is a reflexive transitive relation such that for every sequence $(f_n : n < \omega) \in Q^\omega$ there exists $n < m$ with $f_n \trianglelefteq f_m$. Besides the fact that Nash-Williams proof may use the axiom of choice, the set $X$ might be infinite but not contain an infinite sequence, i.e., $X$ is Dedekind finite.

This particular quasi-ordering is absolute; take $\pi$ witnessing $f \trianglelefteq g$ by choosing the least possible value:

$$\pi(\alpha) = \min \beta \geq \sup\{\pi(\gamma) + 1 : \gamma < \alpha\} \text{ such that } f(\alpha) = g(\beta).$$

If any $\pi$ works, the least possible value $\pi$ works. It follows that for any two models $M \subseteq N$ of set theory and $f, g \in M$,

$$M \models f \trianglelefteq g \text{ iff } N \models f \trianglelefteq g.$$

This is true even if $M$ and $N$ are nonwell-founded models. To see that ZF proves our proposition, suppose not. Then there is a countable model $(M, E)$

of ZF which models $M \models X$ is an infinite pairwise $\trianglelefteq$-incomparable family. Using forcing we can generically add a sequence $(f_n \in X \;:\; n < \omega^M)$ and get a model $N \supseteq M$ which thinks there is an infinite sequence $(\omega^N = \omega^M)$ which is an $\trianglelefteq$-antichain. But the inner model of $N$, $((L[f_n \in X \;:\; n < \omega^N])^N, E^N)$, satisfies the axiom of choice and hence the Nash-Williams Theorem is true, which is a contradiction.
QED

## An erroneous attribution

Kunen's Set Theory contains an erroneous attribution to "A.Miller" for Exercises E3 and E4 of Chapter VII page 245. These are due to Paul E. Cohen [1]. Cohen gives an easy proof that $L[R]$ of the Cohen real model fails to satisfy the axiom of choice.

## Other interesting references.

Gregory H. Moore [11] has an interesting book on the history of the axiom of choice. The book by Herrlich [7] contains many results on the theme "Disasters without choice".

Hájek [4] shows the independence of Church's axioms (although I have not been able to see a copy of this paper). Hardy 1904 [5, 6] shows without AC that $\omega_1$ can be embedded into $\omega^\omega$ if there is a ladder sequence on $\omega_1$, i.e., $\langle C_\alpha \subseteq \alpha : \alpha \in \lim(\omega_1)\rangle$ where $C_\alpha$ is a cofinal $\omega$-sequence in $\alpha$.

Gitik and Lowe [3] investigate models of ZF in which there are linear orders in which there are no cofinal well-ordered subsets.

Gitik [2] shows that it is consistent to have a model of ZF in $\aleph_0$ and $\aleph_1$ are the only two regular cardinals.

Howard [8] proves that the axiom of choice of countable families of countable sets does not imply that the countable union of countable sets is countable.

Jech [9] shows that in ZF every hereditarily countable set has rank less than $\omega_2$ and if $\omega_1$ is singular, then there are hereditarily countable sets of all ranks less than $\omega_2$.

Truss [13] also has a model of ZF in which every set is Borel. We do not know what the length of the Borel hierarchy is in his model.

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