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Set theoretic properties of Loeb measure [2]

Abstract

In this paper we ask the question: to what extent do basic set theoretic properties of Loeb measure depend on the nonstandard universe and on properties of the model of set theory in which it lies? We show that assuming Martin's axiom and κ-saturation the smallest cover by Loeb measure zero sets must have cardinality less than κ. In contrast to this we show that the additivity of Loeb measure cannot be greater than ω_1. Define $cof(H)$ as the smallest cardinality of a family of Loeb measure zero sets which cover every other Loeb measure zero set. We show that $card(\lfloor \log_2(H) \rfloor) \leq cof(H) \leq card(2^H)$ where $card$ is the external cardinality. We answer a question of Paris and Mills concerning cuts in nonstandard models of number theory. We also present a pair of nonstandard universes M and N and hyperfinite integer $H \in M$ such that H is not enlarged by N, 2^H contains new elements, but every new subset of H has Loeb measure zero. We show that it is consistent that there exists a Sierpiński set in the reals but no Loeb-Sierpiński set in any nonstandard universe. We also show that is consistent with the failure of the continuum hypothesis that Loeb-Sierpiński sets can exist in some nonstandard universes and even in an ultrapower of a standard universe.

Let H be a hyperfinite set in an ω_1−saturated nonstandard universe. Let μ be the counting measure on H, i.e. for any internal subset A of H let $\mu(A)$ be the nonstandard rational: $\frac{|A|}{|H|}$ where $|A|$ is the internal cardinality of A, a hyperinteger. Loeb (1975)[12] showed that the standard part of μ has a natural extension to a countably additive measure on the σ-algebra generated

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by the internal subsets of H. For more background on nonstandard measure theory and its applications, see the survey article Cutland (1983)[3]. References on general properties of Loeb measure and the σ-algebra generated by the internal sets include Henson (1979)[5][6] and Keisler et. al. (1989)[8].

Some properties of Loeb measure are the following. For every Loeb measurable set X there exists an internal set $A \subseteq H$ such that $(X - A) \cup (A - X)$ has Loeb measure zero. Also every Loeb measure zero set can be covered by one of the form $\cap_{n\in\omega} A_n$ where each A_n is an internal set of measure less than $\frac{1}{n+1}$. It is related to Lebesgue measure on the unit interval $[0,1]$. Identify H with the time line $T = \{n\triangle t : n < |H|\}$ where $\triangle t = \frac{1}{|H|}$. Let $st : T \to [0,1]$ be the standard part map. Then for any Lebesgue measurable $Z \subseteq [0,1]$ the set $st^{-1}(Z)$ is Loeb measurable with the same measure as Z.

Theorem 0.1 *Keisler-Leth (19..)[9] If F is a family of internal Loeb measure zero subsets of an infinite hyperfinite set H, the nonstandard universe is $\kappa-$saturated, and F has external cardinality less than κ, then there exists an internal set $A \subseteq H$ of Loeb measure zero which covers every element of F.*

proof:Consider the sentences $\Sigma(A)$:

$$\{B \subseteq A : B \in F\} \cup \{\frac{|K|}{|H|} < \frac{1}{n+1} : n \in \omega\}$$

where A is a variable. Clearly this set of sentences is finitely satisfiable and has cardinality the same as F. It follows by $\kappa-$saturation that some internal A satisfies them all simultaneously.

□

Note that this implies that in a κ-saturated universe any external $X \subseteq H$ of cardinality less that κ has Loeb measure zero.

Theorem 0.2 *Suppose the universe is ω_1-saturated and every set of reals of cardinality $< \kappa$ has Lebesgue measure zero, then for any infinite hyperfinite set H and $X \subseteq H$ of an external cardinality $< \kappa$, X has Loeb measure zero.*

proof:The standard part map shows this.

□

In this case it may be impossible to cover X with an internal set of Loeb measure zero.

Theorem 0.3 *For any M an ω_1-saturated universe and H an infinite hyperfinite set in M, there exists any elementary ω_1-saturated extension N of M which has the property that there exists a family $\{A_\alpha \subseteq H : \alpha < \omega_1\}$ of internal Loeb measure zero subsets of N such that for every internal Loeb measure zero $B \subseteq H$ in N there exists $\alpha < \omega_1$ with $B \subseteq A_\alpha$.*

proof:Build an elementary chain of ω_1-saturated universes M_α for $\alpha < \omega_1$. Use the proof of Theorem 0.1 to get $A_\alpha \in M_{\alpha+1}$ such that every Loeb measure zero set $B \in M_\alpha$ is covered by A_α.
$\square$

By using a simple diagonal argument in a universe given by the last theorem there will be an $X \subseteq H$ of cardinality ω_1 which is not covered by any internal Loeb measure zero set. In fact, the set X will have the property that for every internal Loeb measure zero $A \subseteq H$, $X \cap A$ is countable. We say that $X \subseteq H$ is a Loeb-Sierpiński set iff it is uncountable and meets every Loeb measure zero set in a countable set. Thus what we have here is a weak kind of Loeb-Sierpiński set. If MA+$\neg$CH is true, then every set of reals of cardinality ω_1 has measure zero. So by Theorem 0.2 this weak Loeb-Sierpiński set would not be a Loeb-Sierpiński set.

Now we ask for the smallest cardinality of a cover of H by Loeb measure zero sets. Note that since monads ($st^{-1}\{p\}$ for some $p \in [0,1]$) have measure zero, H can always be covered by continuum many Loeb measure zero sets. MA$_\kappa$ stands for the version of Martin's Axiom which says that for any partially ordered set $\mathbb{P}$ which has the countable chain condition and any family $\mathcal{D}$ of dense subsets of $\mathbb{P}$ of cardinality less than κ there exists a $\mathbb{P}$-filter G which meets all the dense sets in $\mathcal{D}$.

Theorem 0.4 *Suppose MA_κ and the nonstandard universe is κ-saturated, then H cannot be covered by fewer than κ sets of Loeb measure zero.*

proof:Let $\mathbb{P}$ be the partial order of all internal subsets of H of positive Loeb measure ordered by inclusion (where stronger conditions are smaller). Forcing with $\mathbb{P}$ is the same as using the measure algebra formed by taking the $\sigma-$algebra generated by the internal subsets of H and dividing out by the Loeb measure zero sets. It has the countable chain condition because it is impossible to find $n+1$ sets of measure greater than $\frac{1}{n}$ which have pairwise intersections of measure zero.

For $\lambda < \kappa$ let $\{\bigcap_{n\in\omega} K_n^\alpha : \alpha < \lambda\}$ be a family of λ many Loeb measure zero sets where each K_n^α is an internal subset of H of Loeb measure less than $\frac{1}{n+1}$. For each α define a dense $D_\alpha \subseteq \mathbb{P}$ by

$$D_\alpha = \{b \in \mathbb{P} : \exists n\ b \cap K_n^\alpha = \emptyset\}$$

Using MA_κ let G be a $\mathbb{P}$-filter meeting every D_α for $\alpha < \lambda$. For each α let n_α be such that $(H - K_\alpha^{n_\alpha}) \in G$. Consider the family of sentences $\Sigma(x)$:

$$\{x \in (H - K_\alpha^{n_\alpha}) : \alpha < \lambda\}$$

Since G is a $\mathbb{P}$-filter this family of sentences is finitely satisfiable. Hence by κ-saturation some internal $x \in H$ satisfies them all. But this shows that the family of Loeb measure zero sets did not cover H.

□

This partial order was also used in Kaufmann and Schmerl (1987)[10]. Next we consider the additivity of Loeb measure. Here we show that it is always as small as possible.

Theorem 0.5 *For any infinite hyperfinite H in an ω_1-saturated universe there exists a family of ω_1 Loeb measure zero sets whose union does not have measure zero.*

proof:Without loss of generality we may assume $H = 2^K$ for some internal hyperinteger K, since H may be replaced by its internal cardinality and for some hyperinteger K we have $2^K \leq H \leq 2^{K+1}$ and 2^K has Loeb measure at least 1/2 in H. Suppose $\Sigma \in [K]^\omega$ where $[K]^\omega$ is the infinite countable subsets of K and let $\Sigma = \{\alpha_n : n \in \omega\}$. For each $n \in \omega$ define

$$H_n = \{h \in 2^K : h(\alpha_0) = h(\alpha_1) = \ldots = h(\alpha_{n-1}) = 0\}$$

Then each H_n is an internal set of measure $\frac{1}{2^n}$.

Lemma 0.6 *If A is an internal set and $\bigcap_{n\in\omega} H_n \subseteq A$, then there exists $n \in \omega$ such that $H_n \subseteq A$.*

This follows from the fact that the H_n are a descending sequence and the universe is ω_1-saturated.

□

Let $\Sigma_\beta \in [K]^\omega$ for $\beta < \omega_1$ be a family of disjoint countable subsets of K. Let H_n^β be defined as H_n was but using Σ_β instead of Σ. Define $A_\beta = \cap_{n\in\omega} H_n^\beta$.

So each A_β has Loeb measure zero. We show that the union $\cup_{\beta<\omega_1} A_\beta$ cannot have Loeb measure zero. Suppose A is an internal set and

$$\cup_{\beta<\omega_1} A_\beta \subseteq A$$

By the Lemma there exists $n \in \omega$ and $\{\beta_m : m \in \omega\} \in [\omega_1]^\omega$ such that for every $m \in \omega$ $H_n^{\beta_m} \subseteq A$. Let $K_m = H_n^{\beta_m}$. Note that each K_m for $m \in \omega$ has Loeb measure $\epsilon = \frac{1}{2^n}$ and they are independent, i.e. for any $m_1 < m_2 < \ldots < m_k < \omega$ the Loeb measure of

$$K_{m_1} \cap K_{m_2} \cap \ldots \cap K_{m_k}$$

is ϵ^k. Consequently $\cup_{m\in\omega} K_m$ has Loeb measure one (since the measure of $\cap_{m<N}(H - K_m)$ is $(1-\epsilon)^N$ and $(1-\epsilon)^N \to 0$ as $N \to \infty$). Consequently the Loeb measure of A is one and the theorem is proved.
□

The last cardinal associated with Loeb measure we will consider is the cofinality, $cof(H)$. This is the smallest cardinality of a family F of Loeb measure zero sets such every Loeb measure zero set is covered by a member of the family F.

Theorem 0.7 *Let card be the external cardinality function and H an infinite hyperinteger in some ω_1-saturated universe, then*

$$card(\lfloor \log_2(H) \rfloor) \leq cof(H) \leq card(2^H)$$

proof:The first inequality is proved by a similar argument to the proof of the last theorem. Let the hyperfinite integer $K = \lfloor \log_2(H) \rfloor$ so $2^K \leq H \leq 2^{K+1}$. Let $\{\Sigma_\alpha : \alpha < card(K)\}$ be disjoint countable subsets of K and define $A_\alpha \subseteq H$ as above. The argument above shows that no Loeb measure zero set covers uncountably many of the A_α, hence $card(K) \leq cof(H)$.

The second inequality follows from the fact that every Loeb measure zero set is covered by a Loeb measure zero set in the σ-algebra generated by the internal subsets of H and the following result of Shelah (1970)[17]: if L is an infinite hyperfinite set in an ω_1-saturated universe and $\kappa = card(L)$, then $\kappa^\omega = \kappa$ (take $L = 2^H$ the number of internal subsets of H).
□

In Keisler (1967)[7] it is shown under GCH that for any set of infinite successor cardinals C there exists a nonstandard universe M in which

$$C = \{card(H) : H \text{ is a hyperfinite set in } M\}$$

For finite C the nonstandard universe M can be an ultrapower of a standard universe.

In Shelah (1975)[16] it is shown that for any countable theory T with distinguished unary predicates Q and P, if for every $n < \omega$ T has a model M such that $n^n \leq |Q^M|^n \leq |P^M| < \omega$, then T has a model M where $|Q^M| = \omega$ and $|P^M| = 2^\omega$. Taking T to be any theory containing arithmetic and Q to be n^2 and P to be 2^{n^2} we see that there is a nonstandard universe with a hyperinteger H where $card(H) = \omega$ and $card(2^H) = 2^\omega$. If follows from Chang's two cardinal theorem and its proof that assuming the continuum hypothesis there is an ω_1-saturated nonstandard universe with a hyperinteger H where $card(H) = \omega_1$ and $card(2^H) = \omega_2$.

This result was also proved by Paris and Mills (1979)[15] using a method similar to the MacDowell-Specker theorem. For any infinite cardinal κ and nonstandard universe M let

$$I_\kappa^M = \{H : card(H) \leq \kappa \text{ and } H \text{ an integer of M}\}$$

Since H^2 has the same internal cardinality as the cartesian product $H \times H$ which has the same external cardinality as H for infinite H it is clear that I_κ^M must be closed under multiplication. Paris and Mills show this is sufficient by showing: If M is any countable nonstandard universe and I is a proper initial segment of the integers of M closed under multiplication, then there exists an elementary extension N of M in which $I_\omega^N = I$ and every integer H of N not in I has cardinality 2^ω. The following theorem answers a question raised by Paris and Mills concerning cuts in nonstandard models of number theory.

Theorem 0.8 *Assume the continuum hypothesis. Then there exists an ω_1-saturated universe N with a hyperinteger H which has external cardinality ω_1 but every integer greater than H^n for all $n < \omega$ has cardinality ω_2.*

proof:Let M be any ω_1-saturated universe of cardinality ω_1 and let H be any infinite hyperinteger of M. Let

$$I = \{K \text{ a hyperinteger of M } : \exists n \in \omega \; K < H^n\}$$

and let J be the complement of I (with respect to the hyperintegers of M). We construct a sequence $< X_\alpha : \alpha < \omega_1 >$ of elements of M such each X_α is a hyperfinite set of hyperintegers of M such that

- the internal cardinality of X_α is an element of J
- if $\alpha < \beta$ then $X_\beta \subseteq X_\alpha$
- for any set X in M there exists an α such that either $X_\alpha \subseteq X$ or X_α is disjoint from X
- for any α and function f in M whose domain includes X_α there exists a $\beta \geq \alpha$ such that $f \restriction X_\beta$ is either one-to-one or constant
- for any $d \in J$ some X_α has internal cardinality less than d

Note that since M is ω_1-saturated and the cofinality of I is ω the coinitiality of J is ω_1, i.e. for any countable set $B \subseteq J$ there exists $c \in J$ such that for every $b \in B$ we have $c < b$. To obtain X_λ for $\lambda < \omega_1$ a limit ordinal, first find $b \in J$ such that for every $\alpha < \lambda$ the internal cardinality of X_α is greater than b. Now use ω_1-saturation to find X_λ of internal cardinality greater than b and for all $\alpha < \lambda$ $X_\lambda \subseteq X_\beta$. To make functions one-to-one or constant use the following lemma.

Lemma 0.9 *Suppose $f : X \to Y$ is an onto function in M such that the internal cardinality of X is in J, then there exists an internal set $X' \subseteq X$ with internal cardinality in J and $f \restriction X'$ is either one-to-one or constant.*

proof:Note that $X = \bigcup_{y \in Y} f^{-1}(y)$. If some $f^{-1}(y)$ has internal cardinality $b \in J$, then let $X' = f^{-1}(y)$ and hence $f \restriction X'$ is constant. Otherwise (since internally f is a finite function and the maximum of a hyperfinite set of integers is always achieved) there exists $c \in I$ such that for every $y \in Y$ the internal cardinality of $f^{-1}(y)$ is less than c. Since I is closed under multiplication, it follows that the internal cardinality of Y is in J. In M choose a set $X' \subseteq X$ such that f maps X' one-to-one onto Y and hence the internal cardinality of X' is in J.
$\Box$

This ends the construction of the sequence $< X_\alpha : \alpha < \omega_1 >$. Let c be a new constant symbol and let T be the theory which consists of the elementary diagram of M plus all statements of the form "$c \in X_\alpha$" for $\alpha < \omega_1$. Let M_1 be any model of T which is an elementary superstructure of M and let M_0 be the set of all $f^{M_1}(c)$ such that $f \in M$ is a function whose domain contains some X_α.

Claim: For any formula $\psi(x_1, x_2, \ldots, x_n)$ and $\vec{f} = \langle f_1, f_2, \ldots, f_n \rangle$ a sequence of functions from M

$$M_1 \models \psi(\vec{f}(c)) \text{ iff } \exists \alpha < \omega_1 \, \forall b \in X_\alpha \, M \models \psi(\vec{f}(b))$$

This is proved just like Łos's theorem.

Claim: M is an elementary substructure of M_0.

We have that $M \subseteq M_0$ because of the constant functions in M. So it is enough to note that M_0 is an elementary substructure of M_1. This follows from the Tarski-Vaught criterion. Suppose

$$M_1 \models \exists x \, \theta(x, f_1(c), \ldots, f_n(c))$$

then there must be some X_α contained in the domain of each f_i such that

$$M \models \forall b \in X_\alpha \exists x \, \theta(x, f_1(b), \ldots, f_n(b))$$

In M find g with domain X_α such that

$$M \models \forall b \in X_\alpha \, \theta(g(b), f_1(b), \ldots, f_n(b))$$

and hence

$$M_1 \models \theta(g(c), f_1(c), \ldots, f_n(c))$$

It follows that M is an elementary substructure of M_0.

Claim: If $a \in I$ and $b \in M_0$ with $b < a$ then $b \in M$, i.e. the initial segment of the hyperintegers of M_0 determined by I is not enlarged.

To see this let $f(c) = b$ for some $f \in M$ and let X_α be contained in the domain of f have the property that for every $x \in X_\alpha$ we have $f(x) < a$. By our construction we may assume that $f \restriction X_\alpha$ is one-to-one or constant. However it cannot be one-to-one since the internal cardinality of X_α is in J and a is in I, so it must be constant. Hence this constant must be in M and therefore in I.

Claim: for every $d \in J$ there exists $b \in M_0 - M$ with $b < d$, i.e. there are arbitrarily small new elements of J.

Some X_α has internal cardinality less than d. Let $f \in M$ be a one-to-one function from X_α into d. Clearly $f^{M_0}(c) < d$. $f^{M_0}(c)$ must be new because for any $b \in M$ it must be true that some $X_\beta \subseteq (X_\alpha - f^{-1}(b))$. It is also true that no new element of M_0 is beneath every element of J.

Claim: M_0 is an ω_1-saturated model of cardinality ω_1.

Clearly M_0 has cardinality ω_1, so it is enough to check that it is ω_1-saturated. So let $\Sigma(x) = \{\theta_n(x, f_n(c)) : n \in \omega\}$ be a finitely realizable type in M_0. We may assume parameters are singletons since n-tuples are elements of our universe. Let α be sufficiently large so that X_α is a subset of the domain of f_n for every $n < \omega$. Since $\Sigma(x)$ is finitely realizable we can also choose α large enough so that for every $n \in \omega$

$$M \models \forall b \in X_\alpha \exists x \bigwedge_{m<n} \theta_m(x, f_m(b))$$

Consider the following type $\Gamma(g)$ with variable g over M. $\Gamma(g)$ contains the sentence "g is a function with domain X_α"and for each $n \in \omega$ the sentence $\forall b \in X_\alpha \ \theta_m(g(b), f_n(b))$. Since the type $\Gamma(g)$ is finitely realizable in M and M is ω_1-saturated some function g in M realizes it and so $g(c)$ realizes $\Sigma(x)$ in M_0. (Kotlarski (1983)[11] shows that every simple cofinal extension of an ω_1-saturated model is ω_1-saturated.)

Now we prove Theorem 0.8. This is proved similarly to the standard proof of Chang's two cardinal Theorem (see Chang and Keisler (1973)[2] Theorem 7.2.7 p.438). If M is an elementary substructure of N let

$$I^N = \{b \text{ a hyperinteger of } N : \exists a \in I \ b < a\}$$

Construct an elementary chain of models N_α for $\alpha < \omega_2$ such that $N_0 = M$, each N_α is isomorphic M, and $I^{N_\alpha} = I$.

For successor steps to obtain $N_{\alpha+1}$ just use the pair M, M_0. So the only thing to do is the step for λ a limit ordinal. Letting N be the union of N_α for $\alpha < \lambda$ and I^N be the initial segment of N determined by I we need to see that N can be embedded into M in such a way that I^N is mapped onto I. This is done by showing that N (while not necessarily ω_1-saturated) is ω_1-saturated for types realizable in I^N:

Claim: Any countable finitely realizable type $\Sigma(x)$ over N, which contains for some $b \in I^N$ the formula $x < b$, is realized in N.

Let $\Sigma(x) = \{\theta_n(x, a_n) : n \in \omega\}$. Choose a sequence $\langle b_n : n \in \omega\rangle$ from N such that for each $n \in \omega$ $b_n < b$ and for $m < n$ $N \models \theta_m(b_n, a_m)$. Since $I^N = I^{N_0}$ and N_0 is ω_1-saturated, there exists an internal sequence $\langle b_n : n \in K\rangle$ for an infinite hyperinteger K in N_0 which is an extension of $\langle b_n : n \in \omega\rangle$. Since N_0 is substructure of N we have that $\langle b_n : n \in K\rangle \in N$. Working in N for each $n \in \omega$ let K_n be the least $m < K$ (if any) such that $N \models \neg\theta_n(b_m, a_n)$. By construction each K_n is an infinite integer of N. Since the $I^N = I^{N_0}$ the coinitiality of the nonstandard integers of N must be ω_1 and so there exists some hyperfinite L in N less than all K_n for $n \in \omega$. It follows that b_L realizes $\Sigma(x)$.

N_λ is now obtained by a back-and-forth argument starting with taking H to itself where $I = \{b : \exists n \in \omega \;\; b < H^n\}$. This concludes the proof of Theorem 0.8.

□

I do not know what either $cof(H)$ or $cof(2^H)$ are in this model. This argument needs only that the coinitiality of J is ω_1. I do not know how to do it if the coinitiality of J is ω, as for example in the proof of Theorem 0.10. The result easily generalizes to (κ, κ^+) in place of (ω_1, ω_2) if $\kappa^{<\kappa} = \kappa$.

This technique can also be used to prove the following theorem.

Theorem 0.10 *Let M be a countable nonstandard universe and H an infinite hyperinteger in M. Then there exists an elementary extension N of M such that there exists new subsets of H, i.e. $X \subseteq H$ with $X \in (N - M)$ but every new $X \subseteq H$ has nonzero Loeb measure, i.e. if $X \in N$ and $\frac{|X|}{H}$ is infinitesimal, then $X \in M$, i.e. the internal Loeb measure zero sets are the same in M and N.*

proof:Let $J = \{K \text{ a hyperinteger of M } : \exists n \in \omega \;\; K > 2^{\frac{H}{n}}\}$ and let I be the complement of J in the hyperintegers of M. Similar to the last proof construct an ω-sequence $\langle X_n : n \in \omega \rangle$ of hyperfinite sets of integers in M such that

- the internal cardinality of each X_n is an element of J
- if $n < m$, then $X_m \subseteq X_n$
- for any set X in M there exists an n such that either $X_n \subseteq X$ or X_n is disjoint from X
- for any n and function f in M whose domain includes X_n there exists an $m > n$ such that $f \restriction X_m$ is either one-to-one or constant
- for any $d \in J$ some X_n has internal cardinality less than d (of course this will automatically be true if we take $X_0 = 2^H$)

Let c be a new constant symbol and let T be the theory which consists of the elementary diagram of M plus all statements of the form "$c \in X_n$" for $n < \omega$. Let M_1 be any model of T which is an elementary superstructure of M and let N be the set of all $f^{M_1}(c)$ such that $f \in M$ is a function whose domain contains some X_n for $n < \omega$. The following set of claims also go thru:

Claim: For any formula $\psi(x_1, x_2, \ldots, x_n)$ and $\vec{f} = \langle f_1, f_2, \ldots, f_n \rangle$ a sequence of functions from M

$$M_1 \models \psi(\vec{f}(c)) \text{ iff } \exists n < \omega \, \forall b \in X_n \; M \models \psi(\vec{f}(b))$$

Claim: M is an elementary substructure of N.

Claim: If $a \in I$ and $b \in N$ with $b < a$ then $b \in M$, i.e. the initial segment of N determined by I is not enlarged.

Claim: for every $d \in J$ there exists $b \in N - M$ with $b < d$, i.e. there are arbitrarily small new elements of J.

It remains only to show that if $X \subseteq H$ in N has the property that $\frac{|X|}{H}$ is infinitesimal, then $X \in M$. To do that we need the following claim from nonstandard calculus:

Claim: Suppose $K < H$ are infinite hyperintegers and $\frac{K}{H} \approx 0$, then

$$\binom{H}{K}^{\frac{1}{H}} \approx 1$$

i.e. if $\frac{K}{H}$ is infinitesimal, then the H^{th} root of the number of subsets of H of size K is infinitesimally close to 1.

We will use Stirling's approximation for $n!$

$$n! = (\frac{n}{e})^n \sqrt{2\pi n}(1 + \epsilon_n)$$

where $\epsilon_n \approx 0$ if n is infinite. Since H, K, and $H - K$ are all infinite we get that

$$\binom{H}{K} = \frac{H!}{(H-K)!K!} = \frac{H^H}{K^K(H-K)^{H-K}} \left(\frac{H}{K(H-K)} \right)^{\frac{1}{2}} x$$

where $x \approx \frac{1}{\sqrt{2\pi}}$. For any positive finite but not infinitesimal real number x we have that $x^{\frac{1}{H}} \approx 1$. Consequently we need only show:

$$\left(\frac{H^H}{K^K (H-K)^{H-K}} \left(\frac{H}{K(H-K)} \right)^{\frac{1}{2}} \right)^{\frac{1}{H}} \approx 1$$

Calculating this let $\epsilon = \frac{K}{H}$ so that $K = \epsilon H$ and $\frac{1}{H} < \epsilon \approx 0$:

$$\begin{aligned} &\left(\frac{H^H}{K^K (H-K)^{H-K}} \left(\frac{H}{K(H-K)} \right)^{\frac{1}{2}} \right)^{\frac{1}{H}} \\ &\quad = \frac{H}{K^{K/H}(H-K)^{1-K/H}} \left(\frac{H}{K(H-K)} \right)^{\frac{1}{(2H)}} \\ &\quad = \frac{H}{(\epsilon H)^{\epsilon}(H-\epsilon H)^{(1-\epsilon)}} \left(\frac{H}{\epsilon H(H-\epsilon H)} \right)^{\frac{1}{(2H)}} \\ &\quad = \frac{1}{\epsilon^{\epsilon}(1-\epsilon)^{(1-\epsilon)}} \left(\frac{1}{\epsilon(1-\epsilon)H} \right)^{1/(2H)} \\ &\quad = \frac{1}{\epsilon^{\epsilon}(1-\epsilon)^{(1-\epsilon)}} \left(\frac{1}{\epsilon^{1/H}(1-\epsilon)^{1/H}} (\frac{1}{H})^{1/H} \right)^{1/2} \end{aligned}$$

Since $\epsilon \approx 0$ we have $(1-\epsilon)^{(1-\epsilon)} \approx 1$, and $(1-\epsilon)^{1/H} \approx 1$. Using L'Hopitol's rule it easy to check that for any positive infinitesimal δ that $\delta^{\delta} \approx 1$, hence $\epsilon^{\epsilon} \approx 1$, $(\frac{1}{H})^{1/H} \approx 1$, and since $\frac{1}{H} < \epsilon < 1$ we have that

$$1 \approx (1/H)^{1/H} < \epsilon^{1/H} < 1$$

and so $\epsilon^{1/H} \approx 1$. This proves the Claim.

Now we prove the theorem. Suppose $X \subseteq H$ is in N but not M, and suppose $X = f^{M_1}(c)$ where f is a function from M whose domain includes X_n. Clearly f cannot be constant so we may assume it is a one-to-one map from X_n to the (internal) set of all subsets of H. Working in M define the function $g : X_n \to H$ by letting $g(x)$ be the internal cardinality of $f(x)$. As H is in I the function g cannot be made one-to-one and hence for some $m \geq n$, $g \restriction X_m$ is constantly equal to some hyperinteger K. Since the internal cardinality of X is K and X is new, clearly K must be infinite. If $\frac{K}{H}$ is not infinitesimal, then we are done. So assume that $\frac{K}{H} \approx 0$. By our last claim we have that for any $n < \omega$

$$\left(\begin{array}{c} H \\ K \end{array} \right)^{\frac{1}{H}} < 2^{\frac{1}{n}}$$

so that for any $n < \omega$

$$\left(\begin{array}{c} H \\ K \end{array} \right) < 2^{\frac{H}{n}}$$

It follows that $\left(\begin{array}{c} H \\ K \end{array} \right)$ is in I. This contradicts the fact that the range of f on X_m must have internal cardinality the same as X_m, an element of J.
□

A set of real numbers X is a Sierpiński set iff it is uncountable but meets every measure zero set in a countable set. Keisler-Leth (19..)[9] have introduced the analogous notion of a Loeb-Sierpiński set. A set $X \subseteq H$ is a Loeb-Sierpiński set iff it is uncountable but meets every Loeb measure zero set in a countable set.

Keisler and Leth have proved that in an ω_1-saturated universe in which the external cardinality of 2^H is ω_1 there exists a Loeb-Sierpiński set $X \subseteq H$. Such a universe can exist iff the continuum hypothesis holds. They also note that the standard part map takes a Loeb-Sierpiński set to a Sierpiński set in the reals. Note that by Theorem 0.1 if there is a Loeb-Sierpiński set in a nonstandard universe that universe cannot be ω_2-saturated.

For any cardinal κ let μ_κ be the product measure on 2^κ. This measure is determined by: for any $F \in [\kappa]^{<\omega}$ and $s : F \to 2$

$$\mu_\kappa(\{x \in 2^\kappa : s \subseteq x\}) = \frac{1}{2^{|F|}}$$

We define $X \subseteq 2^\kappa$ is a μ_κ-Sierpiński set iff X is uncountable but meets every μ_κ measure zero set in a countable set. We begin by establishing a relation between Loeb-Sierpiński sets and Sierpiński sets in 2^κ.

Theorem 0.11 *If there exists a Loeb-Sierpiński set $X \subseteq H$ in some ω_1-saturated universe and infinite cardinal $\kappa \leq card(\lceil \log_2(H) \rceil)$ (where card is the external cardinality), then there exists an μ_κ-Sierpiński set in 2^κ.*

proof:Let $K = \lceil\log_2(H)\rceil)$, so that K is a hyperinteger satisfying $2^{K-1} \leq H \leq 2^K$. Since H has measure at least half in 2^K a Loeb-Sierpiński set in H is also Loeb-Sierpiński set in 2^K. Without loss of generality we may assume $H = 2^K$. Let $\{x_\alpha : \alpha < \kappa\} \subseteq K$ be distinct and define $\rho : H \to 2^\kappa$ by $\rho(h)(\alpha) = h(x_\alpha)$, where we identify 2^K with internal maps from K into $2 = \{0, 1\}$. Note that for any $\alpha < \kappa$ and $i = 0$ or 1

$$\rho^{-1}(\{x \in 2^\kappa : x(\alpha) = i\}) = \{h \in 2^K : h(x_\alpha) = i\}$$

which is an internal set of Loeb measure 1/2.

Claim: If $Y \subseteq 2^\kappa$ has μ_κ measure zero, then $\rho^{-1}(Y)$ has Loeb measure zero.

Suppose $Y \subseteq \bigcap_{n<\omega} Y_n$ where $Y_n = \bigcup_{t\in X_n}[t_n]$, each $t \in X_n$ is a finite partial function from κ to 2, $[t] = \{x \in 2^\kappa : t \subseteq x\}$ with $\mu_\kappa([t]) = 1/2^{|t|}$, each X_n countable, and $\Sigma_{t\in X_n}\mu_\kappa([t]) < 1/n$. But then $\rho^{-1}(Y) \subseteq \rho^{-1}(Y_n)$ and $\rho^{-1}(Y_n) = \bigcup_{t\in X_n} \rho^{-1}([t])$. Since each $\rho^{-1}([t])$ is an internal set with Loeb measure $\mu_\kappa([t])$, it follows that the Loeb measure of $\rho^{-1}(Y_n) < 1/n$. This proves the claim.

Hence if $X \subseteq H$ is a Loeb-Sierpiński set, then $\rho''X \subseteq 2^\kappa$ is a μ_κ-Sierpiński set.

□

Theorem 0.12 *Suppose card*$(\lceil\log_2(H)\rceil) > 2^{\omega_1}$*, then there cannot be a Loeb-Sierpiński set in* H*.*

proof:It suffices to show there cannot be a μ_κ-Sierpiński set for $\kappa = (2^{\omega_1})^+$. Suppose for contradiction that $X = \{x_\alpha \in 2^\kappa : \alpha < \omega_1\}$ is a μ_κ-Sierpiński set and let $\{y_\beta \in 2^{\omega_1} : \beta < \kappa\}$ be defined by $y_\beta(\alpha) = x_\alpha(\beta)$. Since $\kappa = (2^{\omega_1})^+$ there exists $A \in [\kappa]^\omega$ and $y \in 2^{\omega_1}$ such that for all $\alpha \in A$ we have $y_\alpha = y$. Choose $B \in [\omega_1]^{\omega_1}$ and $i \in \{0, 1\}$ such that $y \upharpoonright B$ is constantly i. It follows that for all $\alpha \in A$ that $x_\alpha \upharpoonright B$ is constantly equal to i, however the set

$$\{x \in 2^\kappa : x \upharpoonright B = i\}$$

has measure zero.

□

Theorem 0.13 *It is consistent with ZFC that there exists a Sierpiński set in* 2^ω*, but no Loeb-Sierpiński set in any* ω_1*-saturated nonstandard universe.*

It suffices to find a model of set theory which contains a Sierpiński set, but does not contain a μ_c-Sierpiński set where c is the cardinality of the continuum. The model is obtained as follows. Let M be a countable standard model of ZFC+GCH and let $\kappa = \aleph_\omega^M$ and let $\mathbb{B}_\kappa$ be the measure algebra on 2^κ, i.e. the complete boolean algebra of Borel subsets of 2^κ modulo the μ_κ-measure zero sets. Let G be $\mathbb{B}_\kappa$ generic over M. Then

$$M[G] \models \text{There is a Sierpiński set in } 2^\omega \text{ but none in } 2^c$$

We need only show there is no Sierpiński set in 2^c. The argument will be similar to one found in Miller (1982)[14]. We will use the following Lemma of Kunen from that paper:

Lemma 0.14 *(Kunen) Suppose $B_i \subseteq X$ for $i < n$ are μ-measurable sets with $\mu(B_i) \geq 3/4$, then*

$$\mu\left(\left\{r \in X : |\{i < n : r \in B_i\}| \geq \frac{5}{8}n\right\}\right) \geq 1/3$$

For any sentence θ in the forcing language let $[|\theta|]$ be the boolean value of θ, i.e. $[|\theta|] = \Sigma\{b \in \mathbb{B}_\kappa : b \Vdash \theta\}$

Lemma 0.15 *Suppose $f \in (2^{\omega_1})^{M[G]}$, then $\exists n < \omega\ \exists g \in M[G_n]$ such that domain(g) $\in ([\omega_1]^{\omega_1})^M$ and $\forall \alpha \in$ domain(g) $\mu_\kappa([|f(\alpha) = g(\alpha)|]) \geq 3/4$.*

proof:Working in M for each $\alpha < \omega_1$ let $C_\alpha \subseteq 2^\kappa$ be a clopen set such that $\mu_\kappa([|f(\alpha) = 1|] \triangle C_\alpha) \leq 1/4$, where $\triangle$ denotes symmetric difference. Since C_α is clopen there exists $F_\alpha \in [\kappa]^{<\omega}$ and $T_\alpha \subseteq 2^{F_\alpha}$ such that $C_\alpha = \bigcup_{t \in T_\alpha}[t]$, i.e. a finite union of cylinders over finitely many coordinates. Since $\kappa = \bigcup_{n\in\omega} \omega_n$ there exists $n \in \omega$ and $Y \in [\omega_1]^{\omega_1}$ such that for every $\alpha \in Y$ we have $F_\alpha \subseteq \omega_n$. Define $g : Y \to 2$ by $g(\alpha) = 1$ iff $C_\alpha \in G$. Then $g \in M[G_n]$ since $C_\alpha \in G$ iff $C_\alpha \in G_n$. Also $[|f(\alpha) \neq g(\alpha)|] \leq ([|f(\alpha) = 1|] \triangle C_\alpha)$ which has measure less than 1/4. This proves the Lemma.
□

In $M[G]$ we have that $\kappa^+ = c = \aleph_{\omega+1}$. Now suppose $\{x_\alpha \in 2^{\kappa^+} : \alpha < \omega_1\}$ is a Sierpiński set in 2^c in $M[G]$. Let $\{y_\alpha \in 2^{\omega_1} : \alpha < \kappa^+\}$ be defined by $y_\alpha(\beta) = x_\beta(\alpha)$. By the Lemma and the fact that

$$M[G_n] \models 2^{\omega_1} = \omega_n$$

there exists $Q \in [\kappa^+]^\omega \cap M$ (in fact one of size κ^+) and $g : Y \to 2$ with $g \in M[G_n]$ and $Y \in ([\omega_1]^{\omega_1})^M$ such that $\forall \alpha \in Q\ \forall \beta \in Y$

$$\mu_\kappa([|y_\alpha(\beta) = g(\beta)|]) \geq \frac{3}{4}$$

equivalently

$$\mu_\kappa([|x_\beta(\alpha) = g(\beta)|]) \geq \frac{3}{4}$$

Now $\{x_\beta \restriction Q : \beta \in Y\}$ must be a Sierpiński set in 2^Q. Consequently letting $Q = \{\alpha_n : n \in \omega\}$, by the strong law of large numbers for all but countably many $\beta \in Y$

$$\lim_{n\to\infty} \frac{|\{i < n : x_\beta(\alpha_i) = 0\}|}{n} = \frac{1}{2}$$

Choose $\beta \in Y$ and $n \in \omega$ so that if

$$b = [|\frac{7}{16} < \frac{|\{i < n : x_\beta(\alpha_i) = 0\}|}{n} < \frac{9}{16}|]$$

then $\mu_\kappa(b) > \frac{2}{3}$. Let $B_i = [|x_\beta(\alpha_i) = g(\beta)|]$ and note that $\mu_\kappa(B_i) \geq 3/4$, so by Kunen's Lemma:

$$\mu_\kappa\left(\left\{r \in 2^\kappa : |\{i < n : r \in B_i\}| \geq \frac{5}{8}n\right\}\right) \geq 1/3$$

Let

$$c = \left\{r \in 2^\kappa : |\{i < n : r \in B_i\}| \geq \frac{5}{8}n\right\} = [|\{i < n : x_\beta(\alpha_i) = g(\beta)\}| \geq \frac{5}{8}n|]$$

I claim that $b \wedge c = 0$, which contradicts the fact that $\mu_\kappa(b) > 2/3$ and $\mu_\kappa(c) \geq 1/3$. This is because it is impossible that

$$|\{i < n : x_\beta(\alpha_i) = 0\}| \geq \frac{5}{8}n \ \text{ or } \ |\{i < n : x_\beta(\alpha_i) = 1\}| \geq \frac{5}{8}n$$

and

$$\frac{7}{16} < \frac{|\{i < n : x_\beta(\alpha_i) = 0\}|}{n} < \frac{9}{16}$$

This proves the theorem.

□

In a model due to Bartoszynski and Ihoda (1989)[1] there exists a subset of 2^ω of cardinality ω_1 which is a Sierpiński set, but also every Sierpiński set in 2^ω has cardinality ω_1. This does not mean that there is no Sierpiński set in $2^{\mathfrak{c}}$. To construct their model start with a model M of ZFC+MA+¬CH, then add ω_1 random reals, i.e. force with the measure algebra of 2^{ω_1}. In this model there is a Sierpiński set in 2^{ω_2}. To see this let $D_\alpha \in [\omega_1]^{\omega_1}$ for $\alpha < \omega_2$ be almost disjoint sets (uncountable sets with pairwise intersection countable) and define $x_\alpha \in 2^{\omega_2}$ for $\alpha < \omega_1$ by $x_\alpha(\beta) = G(D_\beta(\alpha))$ where $G : \omega_1 \to 2$ is the generic map and $D_\beta(\alpha)$ is the α^{th} element of D_β. It is not hard to show that $\{x_\alpha : \alpha < \omega_1\} \subseteq 2^{\omega_2}$ is a Sierpiński set. We do not know of model for Theorem 0.13 where the continuum is ω_2.

Theorem 0.16 *It is relatively consistent with ZFC that the continuum hypothesis is false but in some ω_1-saturated universe there is a Loeb-Sierpiński set.*

proof:Suppose M is a countable standard model of ZFC+¬CH. And let $W \in M$ be an ω_1-saturated nonstandard universe, and $H \in W$ some hyperfinite set. Let $\mathbb{B}$ be the boolean algebra obtained by taking the σ-algebra generated by the internal subsets of H and dividing out by the Loeb measure zero sets. Then $\mathbb{B}$ has the countable chain condition. Let G be $\mathbb{B}$-generic over M. Working in $M[G]$ find an elementary extension of W say W' such that the type $\Sigma(x) = \{x \in A : [A] \in G\}$ is realized in W'. It is easy to check that for any $\langle A_n : n \in \omega \rangle \in M$ such that each $A_n \subseteq H$ is internal and the Loeb measure of A_n less than $1/n$, then there exists $n \in \omega$ with $[H - A_n] \in G$. It follows that $x \notin \bigcap_{n\in\omega} A_n$. Iterate this forcing ω_1 times using finite support at limits and obtain $W_\alpha \in M_\alpha$ and $x_\alpha \in H \in W_\alpha$, then using the countable chain condition, in the final model M_{ω_1} the universe $W_{\omega_1} = \bigcup_{\alpha<\omega_1} W_\alpha$ is ω_1-saturated and $\{x_\alpha : \alpha < \omega_1\}$ is a Loeb-Sierpiński set.
□

We say that a nonstandard universe W is an ω-power iff there exists an infinite ordinal α (a typical example is $\alpha = \omega + \omega$) and a nonprincipal ultrafilter U on ω such that W is the ultrapower V_α^ω/U, where V_α is the set of all sets of rank less than α. All ω-powers are ω_1-saturated.

Theorem 0.17 *It is relatively consistent with ZFC that continuum hypothesis is false and we have an ω-power in which there exists a Loeb-Sierpiński set.*

proof:We will use the following lemmas.

Lemma 0.18 *If M is a countable standard model of ZFC and $U \in M$ a nonprincipal ultrafilter on ω, then there exists a generic extension N of M which satisfies the countable chain condition and*

$$N \models \exists Z \in [\omega]^\omega \ \forall X \in U \ Z \subseteq^* X$$

where $A \subseteq^* B$ means inclusion modulo finite.

proof:This is a well-known forcing $\{(s, X) : s \in [\omega]^{<\omega}$ and $X \in U\}$, see Mathias (1977)[13].
□

Suppose $F \subseteq P(\omega)$ is a field of sets and $\mu : F \to [0, 1]$ is a finitely additive measure with $\mu(\omega) = 1$ and $\mu(n) = 0$ for each $n \in \omega$. Let $\mathbb{P} = \{b \in F : \mu(b) > 0\}$ ordered by inclusion.

Lemma 0.19 *$\mathbb{P}$ has the countable chain condition. If G is $\mathbb{P}$- generic over a model M of ZFC , then for every $\langle a_n : n \in \omega\rangle \in M \cap F^\omega$ with $\lim_{n\to\omega} \mu(a_n) = 0$ there exists $n \in \omega$ with $(\omega - a_n) \in G$.*

proof:The countable chain condition holds because there cannot be n sets of measure greater than $1/n$ and pairwise intersection measure zero. The second sentence is an easy density argument.
□

Given $Z \in [\omega]^\omega$ and $h \in \omega^\omega$ with $h(n) \geq 2$ all $n \in Z$, then define $X = \Pi_{n\in Z} h(n)$ and give X the product measure μ determined as follows. Let $A_{i,n} = \{x \in X : x(n) = i\}$ for $i < h(n)$ then $\mu(A_{i,n}) = 1/h(n)$ and for $n_1, n_2, \ldots, n_k$ distinct

$$\mu(A_{i_1,n_1} \cap \ldots \cap A_{i_k,n_k}) = \frac{1}{h(n_1)} \cdot \frac{1}{h(n_2)} \cdots \frac{1}{h(n_k)}$$

Let $\mathbb{B}$ be the measure algebra determined by X and μ, i.e. the Borel subsets of X modulo the μ measure zero sets. Then $\mathbb{B}$ is isomorphic to the usual random real forcing and a generic filter G determines and is determined by a "random real" $r \in X$.

Lemma 0.20 *Suppose h and Z are elements of M a countable standard model of ZFC, $r \in X$ is a random real over M, $g : Z \to [\omega]^{<\omega}$ with $g \in M$,*

$g(n) \subseteq h(n)$ *all* n, *and* $\lim_{n\to\omega\ n\in Z} \frac{|g(n)|}{h(n)} = p$. *Then the following limit exists and equals* p*:*

$$\lim_{n\to\omega} \frac{|\{m < n : m \in Z \text{ and } r(m) \in g(m)\}|}{|\{m < n : m \in Z\}|} = p$$

proof:This follows from the strong law of large numbers for variable distributions (see Feller (1968)[4] X.7 page 258). For each $n \in \omega$ let $\mathbf{X_n}$ be a random variable which is 1 with probability p_n and 0 with probability $1 - p_n$ where $p_n = \frac{|g(z_n)|}{h(z_n)}$ and $\{z_n : n < \omega\}$ is an enumeration of Z. The sequence is mutually independent and satisfies Kolmogorov's Criterion (variances are less than 1) so that the strong law of large numbers holds. This implies that with probability 1 the sequence $\frac{S_n - m_n}{n}$ tends to zero where $S_n = \mathbf{X_0} + \mathbf{X_1} + \ldots + \mathbf{X_{n-1}}$ and $m_n = p_0 + \ldots + p_{n-1}$. Since any random real r must be in any measure one set coded in the ground model we must have that for

$$a_n = |\{m < n : r(k_m) \in g(k_m)\}|$$

that $\lim_{n\to\infty} \frac{a_n - m_n}{n} = 0$. To finish the proof it is enough to see that

$$\lim_{n\to\infty} \frac{a_n}{n} = p$$

This is true because $\lim_{n\to\infty} \frac{m_n}{n} = p$ since

$$\begin{aligned} \frac{m_n}{n} - p &= \frac{p_0 + p_1 + \ldots + p_{n-1} - np}{n} \\ &= \frac{(p_0 - p) + \ldots + (p_m - p)}{n} + \frac{(p_{m+1} - p) + \ldots + (p_{n-1} - p)}{n} \end{aligned}$$

and choosing m large makes the second quotient small and n large drives the first down.

□

Now suppose H is an infinite hyperinteger in an ω-power via a nonprincipal ultrafilter U from M a countable standard model of ZFC, and $h \in \omega^\omega$ represents H in this ultrapower, i.e. $[h]_U = H$. We combine the last three lemmas:

Lemma 0.21 *There exists a countable chain condition generic extension N of M and an ultrafilter $U^* \in N$ extending U and $f \in \omega^\omega$ such that $[f]_{U^*} \in H = [h]_{U^*}$ and $[f]_{U^*}$ is not in any Loeb measure zero set coded in M, i.e. $\langle g_n : n \in \omega \rangle \in M$ such that $[g_n]_U \subseteq H$ and such that Loeb measure of $[g_n]$ is less than $1/n$ for each n, there exists $n \in \omega$ with $[f]_{U^*} \notin [g_n]_{U^*}$.*

proof:Apply Lemma 0.18 to obtain $Z \in [\omega]^\omega$ so that $\forall X \in U$ we have $Z \subseteq^* X$. Let $r \in X = \Pi_{n\in Z} h(n)$ be a random real over $M[Z]$. Working in $M[Z][r]$ define the field of sets

$$F = \{X \subseteq \omega : \exists g \in M \; g : \omega \to [\omega]^{<\omega} \text{ and } X \cap Z = \{m \in \omega : r(m) \in g(m)\}\}$$

Define $\mu(X)$ for any $X \in F$ by

$$\mu(X) = \lim_{n\to\omega} \frac{|X \cap Z \cap n|}{|Z \cap n|}$$

By Lemma 0.20 this limit exists and in fact equals the Loeb measure of the $[g]_U$ that put X into F. Use Lemma 0.19 to obtain G $\mathbb{P}$-generic over $M[Z][r]$ (where $\mathbb{P} = \{b \in F : \mu(b) > 0\}$) and let U^* be any ultrafilter extending G. Let $f \in \omega^\omega$ be any map extending r. Since $\mu(Z) = 1$ we have $U^* \supset U$. If $\langle g_n : n \in \omega \rangle \in M$ is the code for a Loeb measure zero subset of H, then letting $X_n = \{m \in \omega : f(m) \in g_n(m)\}$ ($f = r$ on Z) we have that the Loeb measure of $[g_n]_U$ is $\mu(X_n)$ which goes to zero as $n \to \infty$. So by Lemma 0.19 there exists $n \in \omega$ such that $(\omega - X_n) \in G \subseteq U^*$. This means that

$$\{m : f(m) \notin g_n(m)\} \in U^*$$

so $[f]_{U^*} \notin [g_n]_{U^*}$.
□

Now we prove Theorem 0.17. Start with any M_0 countable standard model of ZFC+¬CH, nonprincipal ultrafilter $U \in M_0$ on ω, and $H = [h]_U$ an infinite hyperinteger with $h \in \omega^\omega \cap M$. Iterate Lemma 0.21 with finite support ω_1 times to obtain $\langle f_\alpha : \alpha < \omega_1 \rangle$ and $\langle U_\alpha : \alpha < \omega_1 \rangle$ with $[f_\alpha]_{U_\alpha} \in [h]_{U_\alpha} = H$ and $[f_{\alpha+1}]_{U_{\alpha+1}}$ not in any Loeb measure zero set coded in M_α. Since the entire iteration has the countable chain condition, at the end $U = \bigcup_{\alpha<\omega_1} U_\alpha$ is an ultrafilter in M_{ω_1} and $\{f_\alpha : \alpha < \omega_1\}$ will be a Loeb-Sierpiński set.
□

I do not know if a Loeb-Sierpiński set can have cardinality ω_2.

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