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LINDELÖF MODELS OF THE REALS: SOLUTION TO A PROBLEM OF SIKORSKI

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ABSTRACT

We show that it is consistent with ZFC that there is a model M of ZF + DC such that the integers of M are ω_1-like, the reals of M have cardinality ω_2, and the unit interval $[0,1]^M$ is Lindelöf (i.e. every open cover has a countable subcover). This answers an old question of Sikorski.

Introduction and history

Sikorski in a series of papers ([16], [17], [18]) attempted to generalize algebraic (i.e. ordered archimedean fields) and topological (i.e. Bolzano–Weierstrauss, metric) properties of the real number line. The basic theme of his program seemed to be the idea that the cardinals (ω, ω_1) should be replaceable by (ω_1, ω_2).

Two definitions of his are pertinent here:

(1) An ordered field has *character* ω_1 if it has an unbounded subset of order type ω_1.

(2) An ordered field has the BW_1 *property* (after Bolzano–Weierstrauss) if it has character ω_1 and every bounded ω_1-sequence of elements of F contains a convergent subsequence.

Previous results:

(1) (Sikorski [16]) There are BW_1-fields.

(2) (Sikorski [17]) The real closure and the algebraic closure of a BW_1-field is a BW_1-field.

The generalization of a metric to that of ω_1-metrizable space was done by Sikorski [18]. He also defined a space to be ω_1-compact (= Lindelöf) if every open cover has a countable subcover.

Sikorski asked [18, p. 132] if there was an example of an ω_1-metrizable ω_1-compact space of cardinality $> \omega_1$.

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(3) (Juhasz–Weiss [8]) There is an ω_1-metrizable Lindelöf space of cardinality $> \omega_1$ if and only if there is a Kurepa tree with no Aronszajn subtrees. (Thus by results of Silver [19] and Jensen (see Devlin [7]) the existence of such a space is independent of the usual axioms for set theory.)

Sikorski [16, p. 88] also raised the question as to the existence of a BW_1-field of cardinality $> \omega_1$.

(4) (Cowles–LeGrange [10]) Showed that any closed bounded interval of a BW_1-field is an ω_1-metrizable Lindelöf space. (Thus by (3) a positive answer to Sikorski's question would imply the existence of a Kurepa tree with no Aronszajn subtrees.)

Our main result here, as stated in the abstract, thus answers Sikorski's question by showing that it is independent of the usual notions of set theory.

Essentially the idea of our proof is that *if* we can build for every model of ZF + DC an elementary extension such that

(a) the natural numbers are an end-extension of the old integers and

(b) every new real in $[0,1]$ is infinitesimally close to an old real (this is Lemma 1)

then under iteration property (b) will give us a model with the Lindelöf property while property (a) will give us natural numbers which are ω_1-like. (This is Theorem 2.)

Our full result though requires that the reals have cardinality $\geqq \omega_2$. We obtain this via a forcing argument. The idea is that our conditions are essentially models of set theory and we can use Lemma 1 and a non-standard version of an embedding result of Friedman and Woodin to show that a model with the desired reals and integers is added.

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The results

1. LEMMA. *Suppose M is a countable model of ZF + DC. Then there exists N an elementary extension of M such that ω^N is a proper end extension of ω^M and every $x \in [0,1]^N$ is infinitesimally close to some $y \in [0,1]^M$ (i.e. there exists $n \in \omega^N - \omega^M$ with $|x - y| < 1/n$).*

PROOF. Build $Z_n \in M$ for $n < \omega$ such that $Z_{n+1} \subseteq Z_n$, $M \models$ "$Z_n \in [\omega]^\omega$", and for every $f \in M$ such that $M \models$ "$f : \omega \to [0,1]$", there exists $n < \omega$ such that

$$M \models \text{“}f(Z_n) \text{ is a convergent sequence”}.$$

Also, construct the Z_n so that for every $m \in \omega^M$ there exists $n < \omega$ such that $Z_n \cap m = \emptyset$. Now let $T = \mathrm{Th}(M) + \{c \in Z_n\}_{n<\omega}$ where $\mathrm{Th}(M)$ is the full theory M and c is a new constant symbol. Clearly T is consistent and c is a new integer. Let N^* be any model of T. Now let $N = \{x \in N^* : \exists f \in M,\ M \models \text{“}f : \omega \to M\text{”}$ and $N^* \models \text{“}f(c) = x\text{”}\}$. Because of the constant functions we have that $M \subseteq N \subseteq N^*$.

1.1. Claim. $N \precsim N^*$.

Proof. This is true because of countable choice holding in M and the Tarski–Vaught criteria. Suppose $N^* \models \exists x \theta(x, f(c))$. Then in M find $g : \omega^M \to M$ such that $M \models \text{“}\forall n \in \omega$ if $\exists x\, \theta(x, f(n))$, then $\theta(g(n), f(n))\text{”}$. Since $M \preccurlyeq N^*$ we have that $N^* \models \text{“}\theta(g(c), f(c))\text{”}$.

1.2. Claim. *ω^N is an end extension of ω^M.*

Proof. If $n < \omega^M$ and $f(c) < n$, then for some m, $f(Z_m)$ is eventually constant, so $f(c) \in M$. □

1.3. Claim. *Every $y \in [0,1]^N$ is infinitesimally close to some $x \in [0,1]^M$.*

Proof. If $y = f(c)$, then for some $n \in \omega$, $f(Z_n)$ converges to $x \in M$ and therefore y is infinitesimally close to x. □

These claims prove the Lemma. □

Kunen pointed out to us that if M contains a non-principal ultrafilter U on ω^M, then an easier proof can be given. Just let N be the M-ultrapower of M with respect to U. (I.e. $N = (M \cap M^\omega)/U$.) Note that our proof only needed that M model countable choice. Is any choice needed? (U could be added generically since M in a model of DC.)

As an application of Lemma 1 (before using it for our main result) we prove the following Theorem.

2. Theorem. *Let M be any countable model of* ZF + DC. *Then there exists an elementary extension N of M such that ω^N is ω_1-like and $[0,1]^N$ is Lindelöf.*

Proof. Build a chain of countable elementary extensions of M, M_α for $\alpha < \omega_1$ such that $M_0 = M$, $M_\lambda = \bigcup_{\alpha<\lambda} M_\alpha$ for each limit ordinal λ, for $\alpha < \beta$, ω^{M_β} is an end extension of ω^{M_α}, and for each $\alpha < \omega_1$ every $y \in [0,1]^{M_{\alpha+1}}$ is infinitesimally close to some $x \in [0,1]^{M_\alpha}$. Now let $N = \bigcup_{\alpha<\omega_1} M_\alpha$. To see that $[0,1]^N$ is Lindelöf, let $\mathcal{U}$ be an open cover of $[0,1]^N$. Without loss of generality we may assume that the elements of $\mathcal{U}$ are open intervals with rational endpoints and so $\mathcal{U} \subseteq N$. By a Lowenheim–Skolem argument find $\alpha < \omega_1$ such that $\forall x \in [0,1]^{N_\alpha}$ there exists $I \in \mathcal{U} \cap N_\alpha$ such that $x \in I$.

2.1. Claim. *$\mathcal{U} \cap N_\alpha$ covers $[0,1]^N$.*

Proof. Since every $y \in [0,1]^{N_{\alpha+1}}$ is infinitesimally close to some $x \in [0,1]^{N_\alpha}$, it follows that $\mathcal{U} \cap N_\alpha$ still covers $[0,1]^{N_{\alpha+1}}$. Also since $\omega^{N_{\alpha+1}}$ is a proper end extension of ω^{N_α} there exists $k \in \omega^{N_{\alpha+1}} - \omega^{N_\alpha}$. It is easy to see that $\{[l/k, (l+1)/k] : 0 \leqq l \leqq k\}$ refines $\mathcal{U} \cap N_\alpha$ and since ω^N is an end extension of $\omega^{N_{\alpha+1}}$, for every $x \in [0,1]^N$ there exists $l \in \omega^N$ such that $l/k \leqq x \leqq (l+1)/k$. □

Now since $\mathcal{U}$ was an arbitrary open cover the Theorem is proved. □

This result is similar to that of Keisler [9] section 3 where it is shown that $\diamondsuit_{\omega_1}$ implies that there is an ω_1-like model of ZFC such that every class over the model is definable. (Shelah [13] eliminates $\diamondsuit_{\omega_1}$ from the proof.) It is also like Schmerl [11] theorem 1.5 which proves that there are ω_1-like models of arithmetic in which every class is definable.

We are now going to try to construct models of ZF, M such that ω^M is ω_1-like and $\mathbf{R}^M$ has cardinality $\geqq \omega_2$. Note however that the set $(2^{<\omega})^M$ contains a Kurepa tree (i.e. a tree of countable width, height ω_1, and having $\geqq \omega_2$ branches). Hence we can only give consistency results. It is easy to modify the results of Keisler [9] corollary 4.4 to obtain such a model from the existence of a Kurepa tree. Also the existence of such a model follows the transfer theorem:

$$(\omega\text{-like}, \omega_1\text{-like}) \to (\omega_1\text{-like}, \omega_2\text{-like})$$

which Burgess and Silver have shown is true if V = L (see Burgess [1]).

We also desire that $[0,1]^M$ be Lindelöf (thus transferring ω-compactness up to ω_1-compactness). The method of proof is similar to that of forcing a generic Kurepa tree. Such a tree contains no Aronszajn subtrees (see Todorčević [21]).

The next Lemma is needed for our proof. It is the nonstandard version of an unpublished result due to H. Friedman and H. Woodin. BP stands for the proposition that every set of reals has the Baire property. BUNIF stands for the property that for every relation $R \subseteq \mathbf{R} \times X$ ($\mathbf{R}$ is the reals and X any set), if for every $r \in \mathbf{R}$ there exists $x \in X$ such that $(r, x) \in R$, then there exists a comeager set G and a function $f : G \to X$ such that for all $r \in \mathbf{R}$, $(r, f(r)) \in R$. That is, BUNIF says that every relation is uniformizable on a comeager set. BUNIF is true in Solovay's model (Solovay [20]) where an inaccessible is collapsed to ω_1. It is also true in Shelah's model in which every set of reals has the Baire property (Shelah [14]). This was pointed out to us by M. Magidor.

3. LEMMA (Friedman, Woodin). *Suppose* $M \models$ "ZF + DC + V = L[**R**] + BP + BUNIF" *and G is Cohen generic over M (i.e. via the usual countable partial order). Then there is an elementary embedding of M into* $L[\mathbf{R}]^{M[G]}$.

PROOF. The partial order is the set of open intervals in **R** with rational end points. Working in $M[G]$, define $U = \{A \in P(\mathbf{R}) \cap M\colon \exists p \in G,\ p \cap A$ is comeager in $p\}$. Form the ultrapower

$$N = (M \cap M^{\mathbf{R}^M})/U.$$

This structure is gotten by starting with all functions $f : \mathbf{R}^M \to M$ which are in M and then defining an equivalence relation

$$f \equiv g \text{ iff } \{x \in \mathbf{R}^M : f(x) = g(x)\} \in U$$

and a binary relation

$$[f] \in [g] \text{ iff } \{x \in \mathbf{R}^M : f(x) \in g(x)\} \in U.$$

As usual define $j : M \to N$ by taking $x \in M$ into the equivalence class of the constant function everywhere equal to x. Working in $M[G]$ we prove the following three claims.

3.1. CLAIM. *j is an elementary embedding.*

PROOF. As usual, it is proved by induction on logical complexity that for any formula $\theta(v_1, v_2, \cdots, v_n)$ and $f_1, f_2, \cdots, f_n$:

$$N \models \theta([f_1], [f_2], \cdots, [f_n])$$

iff

$$\{x \in \mathbf{R}^M : M \models \theta(f_1(x), f_2(x), \cdots, f_n(x))\} \in U.$$

For θ an atomic formula this follows from the definition of N. The $\neg$ and $\wedge$ case are handled by noticing that BP and the genericity of G imply U is an ultrafilter. The existential case is easily proved using BUNIF. □

3.2. CLAIM. *N is well-founded.*

PROOF. It is enough to note that j maps the ordinals of M onto the ordinals of N. We use the well known fact that BP implies that the well-ordered union of meager sets is meager. Suppose α is an ordinal of M and $f : \mathbf{R}^M \cap p \to \alpha$, $f \in M$, and $p \in \mathbf{P}$.

Then there exists $q \leqq p$ and $\beta < \alpha$ such that $f^{-1}(\beta) \cap q$ is comeager in q. Hence by genericity of G evey ordinal of N less than $j(\alpha)$ is a $j(\beta)$ for some $\beta < \alpha$. □

3.3. CLAIM. *N is isomorphic to* $L[\mathbf{R}]^{M[G]}$ *via the transitive collapse of E.*

PROOF. It suffices to show $G \in N$ since every real in $M[G]$ is in $L[x, G]$ for some real x in M. But [Id] is mapped to G where $\mathrm{Id}: \mathbf{R} \to \mathbf{R}$ is the identity mapping. □

Although we have used "$\in$" for the membership relation in M there is nothing in the proof that requires that M be a standard model. Most popular expositions of forcing (e.g. Shoenfield [15] and Burgess [2]) construct the model $M[G]$ by induction on rank, which can't be done if M is nonstandard. However forcing can be defined syntactically in M and for a generic G, $M[G]$ can be taken to be what it is forced to be. That is, if we look at all terms in forcing the language and define $\tau^G = \sigma^G$ iff there exists $p \in G$ such that $p \Vdash$ "$\tau = \sigma$" and $\tau^G \in \sigma^G$ iff there exists $p \in G$ such that $p \Vdash$ "$\tau \in \sigma$", then we get the model $M[G]$. See Cohen [5] for a result using forcing over nonstandard models. This finishes the proof of Lemma 3. □

Let DAD stand for the proposition that every set of reals definable from an ω-sequence of ordinals is determined. Friedman and Woodin used this lemma to prove that Con(ZFC + DAD) implies Con(ZFC + $\neg$ CH + DAD).

After receiving an earlier version of this paper Hugh Woodin pointed out to us that we could have gotten by with a simpler version of Lemma 3. All we need is some condition on our models of set theory, M, so that if G is Cohen over M, then there is an elementary embedding of M into $L[\mathbf{R}]^{M[G]}$. This will be true if $M = L[\mathbf{R}]^{L[H]}$ where H is generic over L for adding ω_1 Cohen reals. Also, it suffices that M models ZF + DC + V = L[**R**] + for every nonconstructible real x there exists a real c, Cohen over L, such that $L[x] = L[c]$. Woodin's remarks also reminded us of a paper of P. E. Cohen [6] where some related results are proved.

Now we state the main result of this paper.

4. THEOREM. *Suppose* CH *is true and* M_0 *is a countable model of* "ZF + DC + BP + BUNIF + V = L[**R**]". *Then there exist a partial order* **P** *such that* **P** *has the* ω_2*-c.c., forcing with* **P** *adds no subset of* ω, *and forcing with* **P** *adds a model N which is an elementary extension of M such that* ω^N *is* ω_1*-like,* $[0,1]^N$ *has cardinality* ω_2, *and* $[0,1]^N$ *is Lindelöf.*

PROOF. A condition $p \in \mathbf{P}$ consists of a countable model M_p with universe a countable ordinal which elementarily extends M_0 and a function $f_p : \Sigma_p \to M_p$ where Σ_p is a countable subset of ω_2 and for each $\alpha \in \Sigma_p$, $f_p(\alpha)$ is an open interval of $[0,1]^{M_p}$ with rational end points. We define $\bar{p} \leqq p$ iff M_p is an elementary substructure of $M_{\bar{p}}$, $\omega^{M_{\bar{p}}}$ is an end extension of ω^{M_p}, $\Sigma_{\bar{p}} \supseteq \Sigma_p$, and for each $\alpha \in \Sigma_p$, $f_{\bar{p}}(\alpha) \subseteq f_p(\alpha)$.

4.1. LEMMA. **P** *has* ω_2-c.c.

PROOF. Given ω_2 conditions p_α for $\alpha < \omega_2$, then if we assume the continuum hypothesis, then there are only ω_1 possible M_p. Consequently we can assume $M_{p_\alpha} = M_{p_\beta}$ for all α and β. By the Δ-systems lemma we can find Σ and $\Gamma \in [\omega_2]^{\omega_2}$ such that for all $\alpha, \beta \in \Gamma$, $\Sigma_\alpha \cap \Sigma_\beta = \Sigma$ and $f_{p_\alpha} \upharpoonright \Sigma = f_{p_\beta} \upharpoonright \Sigma$. (See, for example, Burgess [2], 3.6.) But now for any $\alpha, \beta \in \Gamma$, p_α and p_β are compatible. □

For the next two lemmas it may be useful to refer to the following diagram of models

$$(\omega^{M_i^*} = \omega^{M_{i+1}^*})$$

$M \subseteq M_1^* \subseteq M_2^* \cdots \subseteq M^* \subseteq \hat{M}_p$
$\vdots$ (Every real in $\hat{M}$ infinitesimally close to a real of M^*)
M_{p_2}
$\cup|$
M_{p_1} ($\omega^{M_{i+1}}$ end extends ω^{M_i})
$\cup|$
M_{p_0}

4.2. LEMMA. **P** *is* ω_1-*Baire* (*i.e. the countable intersection of open dense sets is dense*).

PROOF. Suppose $D_n \subseteq \mathbf{P}$ for $n < \omega$ are open dense. For any $p_0 \in \mathbf{P}$ construct a sequence $p_{n+1} \leqq p_n$ with $p_{n+1} \in D_n$. Let $M = \bigcup_{n<\omega} M_{p_n}$ and $\Sigma = \bigcup_{n<\omega} \Sigma_{p_n}$. Let Q be the usual Cohen order, i.e. the partial order of open intervals with rational end points. For each $\alpha \in \Sigma$ let G_α be the filter generated in Q^M by $\{f_n(\alpha) : \alpha \in \Sigma_{p_n}\}$. By a dovetailing argument make sure that for each $n < \omega$

$$(\alpha_1, \alpha_2, \cdots, \alpha_n) \in \Sigma^n, \quad \text{and} \quad D \in M$$

such that $M \models$ "D is dense open $\subseteq Q^n$", there exists some $m < \omega$ such that

$$M_{p_m} \models \text{"}(f_{p_m}(\alpha_1), f_{p_m}(\alpha_2), \cdots, f_{p_m}(\alpha_n)) \in D\text{"}.$$

Thus any finite sequence of the G_α's is Cohen generic over M. Let $\Sigma = \{\alpha_n : n < \omega\}$. Let $M_0^* = M$ and for each $n < \omega$ let

$$M_{n+1}^* = L[\mathbf{R}]^{M_n^*[G_{\alpha_n}]}.$$

By Lemma 3, M_n^* is elementarily embedded into M_{n+1}^* (by a map fixing ordinals and therefore reals). Let $M^* = \bigcup_{n<\omega} M_n^*$ (or more precisely the direct limit). Then

$$M \lesssim M^*$$

and $\omega^M = \omega^{M^*}$. Now let $\hat{M}$ be an elementary extension of M^* such that $\omega^{\hat{M}}$ is a proper end extension of ω^{M^*} and $\varepsilon = 1/k$ where $k \in \omega^{\hat{M}} - \omega^{M^*}$. Define $f: \Sigma \to \hat{M}$ by $f(\alpha) = (r_\alpha, s_\alpha)$ where (r_α, s_α) is any rational interval about the Cohen real determined by G_α of diameter less than ε. Then $(\hat{M}, f)$ is condition extending each p_n. □

Suppose G is **P**-generic over the universe V and let $N^* = \bigcup\{M_p : p \in G\}$. Clearly ω^{N^*} is ω_1-like and by an easy genericity argument for all finite sequences $(\alpha_1, \alpha_2, \cdots, \alpha_n) \in \omega_2$, $\langle G_{\alpha_1}, G_{\alpha_2}, \cdots, G_{\alpha_n}\rangle$ is Cohen-generic over N^*. Hence by Lemma 3 and an extension of the elementary chain lemma to directed families (see Chang and Keisler [4], 3.1.9) we can construct a model N which is an elementary extension N^*, each $G_\alpha \in N$ and every real of N is in $N^*[G_{\alpha_1}, \cdots, G_{\alpha_n}]$ for some finite sequence $\langle \alpha_1, \alpha_2, \cdots, \alpha_n\rangle$ in ω_2. Our final lemma finishes the proof of Theorem 4.

4.3. Lemma. *$[0,1]^N$ is Lindelöf.*

Proof. This proof is just a slightly more complicated version of the proof of Lemma 4.2. Suppose

$$p_0 \Vdash \text{“}U \text{ is a cover of } [0,1]^N \text{ by open intervals with rational end points”}.$$

Build the sequence p_n as in Lemma 4.2. Also make sure (by dovetailing) that for all $n < \omega$, finite sequences $\langle \alpha_1, \alpha_2, \cdots, \alpha_n\rangle$ from Σ, and for all terms τ for an element of $[0,1]^{M[G_{\alpha_1}, G_{\alpha_2}, \cdots, G_{\alpha_n}]}$ there exists $m < \omega$ and a rational interval I in M_{p_m} such that $p_m \Vdash$ "$I \in U$" and

$$(f_{p_m}(\alpha_1), f_{p_m}(\alpha_2), \cdots, f_{p_m}(\alpha_n)) \Vdash \text{“}\tau \in I\text{”}.$$

As in the proof of Lemma 4.2 let $M_0^* = M$ and $M_{n+1}^* = L[\mathbf{R}]^{M^*[G_{\alpha_n}]}$ where $\Sigma = \{\alpha_n : n < \omega\}$, and let $M^* = \bigcup_{n<\omega} M_n^*$. Now let $\hat{M}$ be an elementary end extension of M^* such that $\omega^{\hat{M}}$ is a proper end extension of ω^{M^*} and for all $x \in [0,1]^{\hat{M}}$ there exists $y \in [0,1]^{M^*}$ and $k \in \omega^{\hat{M}} - \omega^{M^*}$ such that $|x - y| < 1/k$ (such an extension is given by Lemma 1). As before let $\varepsilon = 1/k$ for some $k \in \omega^{M^*} - \omega^{\hat{M}}$ and let $f: \Sigma \to \hat{M}$ be defined by choosing any rational interval $f(\alpha)$ about G_α of diameter less than ε. Then $p = (\hat{M}, f)$ is a condition extending each p_n and by construction there exists a countable set V of open intervals with rational end points such that $V \subseteq M^*$, V covers $[0,1]^{M^*}$, and $p \Vdash$ "$V \subseteq U$". Since the diameters of things in V are not infinitesimal, it is clear that V covers $[0,1]^{M^*}$ also. By the same argument as Claim 2.1 we have that V covers $[0,1]^N$. □

Concluding remarks

Reading Sikorski's papers leaves one with the idea that he had some general cardinality transfer principle in mind. Thus compact becomes Lindelöf, metrizable becomes ω_1-metrizable. As we pointed out a model M with ω^M ω_1-like and the reals of cardinality $\geqq \omega_2$ does indeed follow from the two-cardinal transfer theorem of Burgess and Silver (Burgess [1]) (ω-like, ω_1-like)$\rightarrow$(ω_1-like, ω_2-like). It would be quite interesting to find a transfer result (possibly a three cardinal one) that holds in L and which would imply the existence of a model which has the above properties and is also Lindelöf.

In an earlier version of this paper we conjectured that our results were true in L and probably provable from "morass with built-in ◇". At our urging Dan Velleman has in fact shown this to be the case. He uses a stationary simplified morass with linear limits (see Velleman [22]). Also the paper of Shelah and Stanley on morasses with built-in ◇ may shed some light on this.

It would also be nice to remove the assumption that the model thinks that every set of reals has the Baire property or some other hypothesis as suggested by H. Woodin (see remark just above Theorem 4). We know of no result which suggests such an hypothesis is necessary.

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