

On the Length of Borel Hierarchies

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This is a survey paper on the lengths of Borel hierarchies and related hierarchies. It consists of lecture notes of a lecture given in July 2016 at the Kurt Godel Research Center, Vienna, Austria and at the TOPOSYM, Prague, Czech Republic.

Notation for the Borel hierarchy is as follows:

$$\text{open} = \underline{\Sigma}_1^0 = G$$

$$\text{closed} = \underline{\Pi}_1^0 = F$$

$$\underline{\Sigma}_2^0 = F_\sigma = \text{countable unions of closed sets}$$

$$\underline{\Pi}_2^0 = G_\delta = \text{countable intersections of open sets} = \text{complements of } \underline{\Sigma}_2^0\text{-sets}$$

$$\underline{\Sigma}_\alpha^0 = \{ \bigcup_{n < \omega} A_n : A_n \in \underline{\Pi}_{<\alpha}^0 = \bigcup_{\beta < \alpha} \underline{\Pi}_\beta^0 \}$$

$$\underline{\Pi}_\alpha^0 = \text{complements of } \underline{\Sigma}_\alpha^0 \text{ sets}$$

$$\text{Borel} = \underline{\Pi}_{\omega_1}^0 = \underline{\Sigma}_{\omega_1}^0$$

In a metric space closed sets are G_δ , i.e., $\underline{\Pi}_1^0 \subseteq \underline{\Pi}_2^0$. Similarly for $1 \leq \alpha < \beta$

$$\underline{\Sigma}_\alpha^0 \cup \underline{\Pi}_\alpha^0 \subseteq \underline{\Sigma}_\beta^0 \cap \underline{\Pi}_\beta^0 =^{def} \underline{\Delta}_\beta^0$$

Theorem 1 (Lebesgue 1905) *For every countable $\alpha > 0$*

$$\underline{\Sigma}_\alpha^0(2^\omega) \neq \underline{\Pi}_\alpha^0(2^\omega).$$

Define $\text{ord}(X)$ to be the least α such that $\underline{\Sigma}_\alpha^0(X) = \underline{\Pi}_\alpha^0(X)$.

Hence $\text{ord}(2^\omega) = \omega_1$. If X is any topological space which contains a homeomorphic copy of 2^ω , then $\text{ord}(X) = \omega_1$. More generally, if $Y \subseteq X$, then $\text{ord}(Y) \leq \text{ord}(X)$. This is because

$$\underline{\Sigma}_\alpha^0(Y) = \{A \cap Y : A \in \underline{\Sigma}_\alpha^0(X)\} \text{ and } \underline{\Pi}_\alpha^0(Y) = \{A \cap Y : A \in \underline{\Pi}_\alpha^0(X)\}.$$

If X countable, then $\text{ord}(X) \leq 2$.

Theorem 2 (Bing, Bledsoe, Mauldin 1974) *Suppose $(2^\omega, \tau)$ refines the usual topology and is second countable. Then $\text{ord}(2^\omega, \tau) = \omega_1$.*

Theorem 3 (Reclaw 1993) *If X is a second countable space and X can be mapped continuously onto any space containing 2^ω , then $\text{ord}(X) = \omega_1$.*

Q 1. It is consistent that for any $2 \leq \beta \leq \omega_1$ there are $X, Y \subseteq 2^\omega$ and $f : X \rightarrow Y$ continuous, one-to-one, and onto such that $\text{ord}(X) = 2$ and $\text{ord}(Y) = \beta$, see Theorem 23. What other pairs of orders (α, β) are possible?

Corollary 4 *If X is separable metric space which is not zero-dimensional, then $\text{ord}(X) = \omega_1$.*

To see why, let $x_0 \in X$ be any point and consider the map $x \mapsto d(x_0, x)$. If the image of this map contains an interval then $\text{ord}(X) = \omega_1$. Otherwise there are arbitrarily small clopen balls centered at x_0 .

If X is separable, metric, and zero-dimensional, then it is homeomorphic to a subspace of 2^ω . So from now on we consider only X which are subsets of 2^ω .

Define $X \subseteq 2^\omega$ is a Luzin set iff it is uncountable and $X \cap M$ is countable for every meager set $M \subseteq 2^\omega$.

Theorem 5 (Poprougenko and Sierpiński 1930) *If $X \subseteq 2^\omega$ is a Luzin set, then $\text{ord}(X) = 3$.*

If $X \subseteq 2^\omega$ is a Luzin set, then for every Borel set B there are $\underline{\Pi}_2^0 C$ and $\underline{\Sigma}_2^0 D$ such that $B \cap X = (C \cup D) \cap X$. This is because every Borel sets is open mod meager and hence relative to X open mod countable. It cannot have order 2 because a dense G_δ would meet it in an uncountable set. We might think of the order of X as being $2 + \epsilon$, since these sets are at level one of the Hausdorff difference hierarchy.

Q 2. Can we have $X \subseteq 2^\omega$ with $\text{ord}(X) = 4$ and for every Borel set B there are $\underline{\Pi}_3^0 C$ and $\underline{\Sigma}_3^0 D$ such that $B \cap X = (C \cup D) \cap X$?

Q 3. Same question for the β^{th} level of the Hausdorff difference hierarchy inside the $\underline{\Delta}_{\alpha+1}^0$ sets?

Theorem 6 (Szpilrajn 1930) *If $X \subseteq 2^\omega$ is a Sierpiński set, then the order of X is 2.*

This is because every Borel set contains an F_σ of the same measure.

Theorem 7 (Miller 1979) *The following are each consistent with ZFC:*

- for all $\alpha < \omega_1$ there is $X \subseteq 2^\omega$ with $\text{ord}(X) = \alpha$.
- $\text{ord}(X) = \omega_1$ for all uncountable $X \subseteq 2^\omega$.
- $\{\alpha : \alpha_0 < \alpha \leq \omega_1\} = \{\text{ord}(X) : \text{unctbl } X \subseteq 2^\omega\}$.

Q 4. What other sets can $\{\text{ord}(X) : \text{unctbl } X \subseteq 2^\omega\}$ be?
 $\{\alpha : \omega \leq \alpha \leq \omega_1\}$? Even ordinals?

Theorem 8 (Miller 1979) For any $\alpha \leq \omega_1$ there is a complete ccc Boolean algebra \mathbb{B} which can be countably generated in exactly α steps.

Theorem 9 (Kunen 1979) (CH) For any $\alpha < \omega_1$ there is an $X \subseteq 2^\omega$ with $\text{ord}(X) = \alpha$.

Theorem 10 (Fremlin 1982) (MA) For any $\alpha < \omega_1$ there is an $X \subseteq 2^\omega$ with $\text{ord}(X) = \alpha$.

Theorem 11 (Miller 1979a) For any α with $1 \leq \alpha < \omega_1$ there is a countable set G_α of generators of the category algebra, $\text{Borel}(2^\omega)$ mod meager, which take exactly α steps.

Suppose $\text{Borel}(2^\omega)/I_\alpha$ has order α . If $\alpha < \beta$ and there is an I_α -Luzin set, then does there exist a I_β -Luzin set? If X is I_ω -Luzin is it the clopen sum of sets of order $< \omega$?

Cohen real model and Random real model

Theorem 12 (Miller 1995) If there is a Luzin set of size κ , then for any α with $3 \leq \alpha < \omega_1$ there is an $X \subseteq 2^\omega$ of size κ and hereditarily of order α .

In the Cohen real model there is $X, Y \in [2^\omega]^{\omega_1}$ with hereditary order 2 and ω_1 respectively. Also, every $X \in [2^\omega]^{\omega_2}$ has $\text{ord}(X) \geq 3$ and contains $Y \in [X]^{\omega_2}$ with $\text{ord}(Y) < \omega_1$.

Theorem 13 (Miller 1995) In the random real model, for any α with $2 \leq \alpha \leq \omega_1$ there is an $X_\alpha \subseteq 2^\omega$ of size ω_1 with $\alpha \leq \text{ord}(X_\alpha) \leq \alpha + 1$.

Q 5. Presumably, $\text{ord}(X_\alpha) = \alpha$ but I haven't been able to prove this.

Sacks real model

Theorem 14 (Miller 1995) *In the iterated Sacks real model for any α with $2 \leq \alpha \leq \omega_1$ there is an $X \subseteq 2^\omega$ of size ω_1 with $\text{ord}(X) = \alpha$. Every $X \subseteq 2^\omega$ of size ω_2 has order ω_1 .*

In this model there is a Luzin set of size ω_1 . Also for every $X \subseteq 2^\omega$ of size ω_2 there is a continuous onto map $f : X \rightarrow 2^\omega$ (Miller 1983) and hence by (Reclaw 1993) $\text{ord}(X) = \omega_1$.

What if the Axiom of Choice fails?

Theorem 15 (Miller 2008) *It is consistent with ZF that $\text{ord}(2^\omega) = \omega_2$.*

This implies that ω_1 has countable cofinality, so the axiom of choice fails very badly in our model. We also show that using Gitik's model (1980) where every cardinal has countable cofinality, there are models of ZF in which the Borel hierarchy is arbitrarily long. It cannot be "class" long since then there would be a map from the family of Borel sets onto the ordinals.

Q 6. If we change the definition of Σ_α^0 so that it is closed under countable unions, then I don't know if the Borel hierarchy can have length greater than ω_1 .

Q 7. Over a model of ZF can forcing with $Fin(\kappa, 2)$ collapse cardinals?

No, see Klausner and Goldstern 2016 [6].

The ω_1 -Borel hierarchy of subsets of 2^ω

- $\Sigma_0^* = \Pi_0^* = \text{clopen subsets of } 2^\omega$
- $\Sigma_\alpha^* = \{\cup_{\beta < \omega_1} A_\beta : (A_\beta : \beta < \omega_1) \in (\cup_{\beta < \alpha} \Pi_\beta^*)^{\omega_1}\}$
- $\Pi_\alpha^* = \{2^\omega \setminus A : A \in \Sigma_\alpha^*\}$

CH $\rightarrow \Pi_2^* = \Sigma_2^* = \mathcal{P}(2^\omega)$

Theorem 16 (Miller 2011) *(MA+notCH) $\Pi_\alpha^* \neq \Sigma_\alpha^*$ for every positive $\alpha < \omega_2$.*

Q 8. This result generalizes: (MA) if $\kappa < \mathfrak{c}$ then the κ -Borel hierarchy has length κ^+ . What about the $< \mathfrak{c}$ -Borel hierarchy for \mathfrak{c} weakly inaccessible?

Theorem 17 (Miller 2011) *In the Cohen real model $\Sigma_{\omega_1+1}^* = \Pi_{\omega_1+1}^*$ and $\Sigma_\alpha^* \neq \Pi_\alpha^*$ for every $\alpha < \omega_1$.*

Q 9. I don't know if $\Sigma_{\omega_1}^* = \Pi_{\omega_1}^*$ holds in the Cohen real model.

Q 10. (Brendle, Larson, Todorćević 2008) Is it consistent with notCH to have $\Pi_2^* = \Sigma_2^*$?

If notCH, then $\mathcal{P}(2^\omega) \neq \Sigma_2^*$; consider a Bernstein set.

Theorem 18 (Steprans 1982) *It is consistent that $\Pi_3^* = \Sigma_3^* = \mathcal{P}(2^\omega)$ and $\Pi_2^* \neq \Sigma_2^*$.*

Theorem 19 (Carlson 1982) *If every subset of 2^ω is ω_1 -Borel, then the cofinality of the continuum must be ω_1 .*

Theorem 20 (Miller 2012) (1) *If $\mathcal{P}(2^\omega) = \omega_1$ -Borel, then $\mathcal{P}(2^\omega) = \Sigma_\alpha^*$ for some $\alpha < \omega_2$.*

(2) *For each $\alpha \leq \omega_1$ it is consistent that $\Sigma_{\alpha+1}^* = \mathcal{P}(2^\omega)$ and $\Sigma_{<\alpha}^* \neq \mathcal{P}(2^\omega)$, i.e. length α or $\alpha + 1$.*

Q 11. Can the ω_1 -Borel hierarchy have length α for some α with $\omega_1 < \alpha < \omega_2$?

Define $X \subseteq 2^\omega$ is a Q_α -set iff $\text{ord}(X) = \alpha$ and $\text{Borel}(X) = \mathcal{P}(X)$. Q -set is the same as Q_2 -set.

Theorem 21 (Fleissner, Miller 1980) *It is consistent to have $X \subseteq 2^\omega$, an uncountable Q -set, which is concentrated on $E = \{x \in 2^\omega : \forall^\infty n \ x(n) = 0\}$.*

Hence $X \cup E$ is a Q_3 -set.

Concentrated means that $X \setminus U$ is countable for any open $U \supseteq E$. Hence the order of $X \cup E$ is $2 + \epsilon$. It is also consistent to have a concentrated set with $\text{ord}(X) = \omega_1$.

Theorem 22 (Miller 2014) (CH) *For any α with $3 \leq \alpha < \omega_1$ there are $X_0, X_1 \subseteq 2^\omega$ with $\text{ord}(X_0) = \alpha = \text{ord}(X_1)$ and $\text{ord}(X_0 \cup X_1) = \alpha + 1$.*

- Q 12.** Is it consistent that the X_i be Q_α -sets?
Q 13. What about getting $\text{ord}(X_0 \cup X_1) \geq \alpha + 2$?

Theorem 23 (Judah and Shelah 1991) *It is consistent to have a Q -set and $\mathfrak{d} = \omega_1$ using an iteration of proper forcings with the Sacks property.*

- Q 14.** What about a Q_α -set for $\alpha > 2$?

In this model for any α with $2 < \alpha \leq \omega_1$ there are $X, Z \subseteq 2^\omega$ with $\text{ord}(X) = \alpha$, $\text{ord}(Z) = 2$, and a continuous one-to-one map $f : Z \rightarrow X$. The Sacks property implies that the cofinality of the meager ideal is ω_1 , i.e., there is a family of meager sets \mathcal{M} with $|\mathcal{M}| = \omega_1$ such that for every meager set X there is $Y \in \mathcal{M}$ with $X \subseteq Y$. It follows that there is a Luzin set X of size ω_1 . Hence by Theorem 12 for any α with $2 < \alpha < \omega_1$ there is a set X_α of size ω_1 with $\text{ord}(X_\alpha) = \omega_1$. If we put $X_{\omega_1} = \bigcup_{\alpha < \omega_1} X_\alpha$, then $\text{ord}(X_{\omega_1}) = \omega_1$.

Now let $Y = \{y_\beta : \beta < \omega_1\}$ be the Q -set and let $X_\alpha = \{x_\beta : \beta < \omega_1\}$. Put $Z = \{(x_\beta, y_\beta) : \beta < \omega_1\}$. Then $\text{ord}(Z) = 2$ and the projection map from Z to X_α is one-to-one and continuous.

Theorem 24 (Miller 2003) *It is consistent to have a Q -set $X \subseteq [\omega]^\omega$ which is a maximal almost disjoint family.*

- Q 15.** It is consistent to have Q -set $\{x_\alpha : \alpha < \omega_1\}$ and a non Q -set $\{y_\alpha : \alpha < \omega_1\}$ such that $x_\alpha =^* y_\alpha$ all α . Can such an $\{x_\alpha : \alpha < \omega_1\}$ be MAD?

Products of Q -sets

Theorem 25 (Brendle 2016) *It is consistent to have a Q -set X such that X^2 is not a Q -set.*

In this theorem $\text{ord}(X^2) = 3$, in fact, there is a subset $\Delta \subseteq X^2$ such that $\Delta \neq A \cap X^2$ for any $\underline{\Delta}_3^0$ set $A \subseteq (2^\omega)^2$, i.e., the order of X^2 is not $2 + \epsilon$. To see this let $\Delta = \{(x_\alpha, x_\beta) : \alpha < \beta < \omega_1\}$.

Define $\Delta \subseteq^* A$ iff $\exists \gamma_0 < \omega_1 \forall \alpha, \beta (\gamma_0 < \alpha < \beta < \omega_1) \rightarrow (\alpha, \beta) \in A$ and similarly define $\overline{\Delta} \subseteq^* A$ iff $\exists \gamma_0 < \omega_1 \forall \alpha, \beta (\gamma_0 < \beta < \alpha < \omega_1) \rightarrow (\alpha, \beta) \in A$.

Brendle shows that for his Q -set that for any G_δ set $A \subseteq (2^\omega)^2$ that $\Delta \subseteq^* A$ iff $\overline{\Delta} \subseteq^* A$. Now consider any $A \subseteq (2^\omega)^2$ which is $\underline{\Delta}_3^0$. The Hausdorff difference hierarchy Theorem implies that there are G_δ sets $(G_\alpha : \alpha < \lambda)$ for some limit $\lambda < \omega_1$ such that

1. $\alpha < \beta < \lambda$ implies $G_\beta \subseteq G_\alpha$,
2. $G_\gamma = \bigcap_{\alpha < \gamma} G_\alpha$ for limit $\gamma < \lambda$, and
3. $A = \bigcup \{G_\alpha \setminus G_{\alpha+1} : \alpha \text{ even} < \lambda\}$.

Suppose $\Delta = A \cap X^2$. Let $\Sigma = \{\alpha < \lambda : \Delta \subseteq^* G_\alpha \text{ and } \alpha \text{ even}\}$. Since $A \subseteq G_0$ it follows that $0 \in \Sigma$. Let $\beta = \sup(\Sigma)$. Then $\beta \in \Sigma$ or $\beta = \lambda$ by condition (2). We claim that $\beta = \lambda$. Assume not. Since $\Delta \subseteq^* G_\beta$ we have that $\overline{\Delta} \subseteq^* G_\beta$. But since β is even, $\overline{\Delta}$ is disjoint from $G_\beta \setminus G_{\beta+1}$ which is a subset of A . It follows that $\overline{\Delta} \subseteq^* G_{\beta+1}$ and so $\Delta \subseteq^* G_{\beta+1}$. But $G_{\beta+1} \setminus G_{\beta+2}$ is disjoint from Δ , hence $\beta + 2 \in \Sigma$ which contradicts $\beta = \sup(\Sigma)$.

So $\beta = \lambda$. But this implies $\Delta \subseteq^* \bigcap_{\alpha < \lambda} G_\alpha$ and hence is disjoint from A , contradiction.

Theorem 26 (Miller) (1) If X^2 Q_α -set and $|X| = \omega_1$, then X^n is a Q_α -set for all n .

(2) If X^3 Q_α -set and $|X| = \omega_2$, then X^n is a Q_α -set for all n .

(3) If $|X_i| < \omega_n$, $\prod_{i \in K} X_i$ a Q_α -set for every $K \in [N]^n$, then $\prod_{i \in N} X_i$ is a Q_α -set.

(1) Suppose $X = \{x_\alpha \in 2^\omega : \alpha < \omega_1\}$. Let

$$\Delta_3 = \{(x_\alpha, x_\beta, x_\gamma) : \alpha, \beta \leq \gamma\}.$$

Choose $F_n : \omega_1 \rightarrow \omega_1^2$ so that for each $\gamma < \omega_1$ $(\gamma + 1)^2 = \{F_n(\gamma) : n < \omega\}$. Define F_n^i by $F_n(\gamma) = (F_n^0(\gamma), F_n^1(\gamma))$. Then since $\text{ord}(X^2) = \alpha$ the set $H_n^i = \{(x_\delta, x_\gamma) : F_n^i(\gamma) = \delta\}$ is a relative Σ_α^0 in X^2 . And so

$$G_n = \stackrel{\text{def}}{=} \{(x, y, z) : (x, z) \in H_n^0\} \cap \{(x, y, z) : (y, z) \in H_n^1\}.$$

is Σ_α^0 in X^3 . Consider any $A \subseteq \Delta_3$. Define

$$D_n = \{x_\gamma : F_n(\gamma) = (\alpha, \beta) \text{ and } (x_\alpha, x_\beta, x_\gamma) \in A\}.$$

Since $\text{ord}(X) \leq \alpha$ the set D_n is a relative Σ_α^0 in X and

$$A = \bigcup_{n < \omega} (X^2 \times D_n) \cap G_n$$

is \sum_{α}^0 in X^3 . The same argument works for the sets

$$\Delta_1 = \{(x_{\alpha}, x_{\beta}, x_{\gamma}) : \gamma, \beta \leq \alpha\} \text{ and } \Delta_2 = \{(x_{\alpha}, x_{\beta}, x_{\gamma}) : \gamma, \alpha \leq \beta\}.$$

For any $A \subseteq X^3$ there are $A_i \subseteq \Delta_i$ so that $A = A_1 \cup A_2 \cup A_3$, therefore $\text{ord}(X^3) = \alpha$.

(2) Suppose $X = \{x_{\alpha} : \alpha < \omega_2\}$. For each $\alpha < \omega_2$ let $g_{\alpha} : \omega_1 \rightarrow \alpha + \omega_1$ be a bijection. Choose $F_n : \omega_2^2 \rightarrow \omega_2^2$ for $n < \omega$ so that for every $\alpha_3 \leq \alpha_4 < \omega_2$

$$(\gamma + 1)^2 = \{F_n(\alpha_3, \alpha_4) : n < \omega\} \text{ where } g_{\alpha_4}(\gamma) = \alpha_3.$$

Define F_n^i by $F_n^i(\alpha_3, \alpha_4) = (F_n^0(\alpha_3, \alpha_4), F_n^1(\alpha_3, \alpha_4))$. Since $\text{ord}(X^3) = \alpha$ we have that the graphs of the F_n^i are \sum_{α}^0 in X^3 and since the graph F_n is essentially the intersection of these it is \sum_{α}^0 in X^4 . It follows as above that if

$$A \subseteq \{(x_{\alpha_1}, x_{\alpha_2}, x_{\alpha_3}, x_{\alpha_4}) : \alpha_1, \alpha_2 \leq \alpha_3 \leq \alpha_4 < \omega_2\}$$

then A is \sum_{α}^0 in X^4 . Similarly for any permutation i, j, k, l of $1, 2, 3, 4$ if

$$A \subseteq \{(x_{\alpha_1}, x_{\alpha_2}, x_{\alpha_3}, x_{\alpha_4}) : \alpha_i, \alpha_j \leq \alpha_k \leq \alpha_l < \omega_2\}$$

then A is \sum_{α}^0 in X^4 . Hence any $A \subseteq X^4$ is a finite union of \sum_{α}^0 sets and so $\text{ord}(X^4) = \alpha$.

(3) is left to the reader.

Q 16. Can we have an $X \subseteq 2^{\omega}$ such that X^2 a Q -set but X^3 is not Q -set?

In Theorem 25 Brendle could have used a generic relation $R \subseteq \omega_1 \times \omega_1$ instead of a well-ordering to witness that X^2 is not a Q -set. Perhaps here a generic relation $R \subseteq \omega_2 \times \omega_2 \times \omega_2$ would work.

Q 17. For $\alpha > 2$ can we have X a Q_{α} -set but X^2 not a Q_{α} -set?

Theorem 27 (Miller 1995) (CH) For any α with $3 \leq \alpha < \omega_1$ there is a $Y \subseteq 2^{\omega}$ such that $\text{ord}(Y) = \alpha$ and $\text{ord}(Y^2) = \omega_1$.

Q 18. Can we have $\alpha < \text{ord}(Y^2) < \omega_1$ in this Theorem?

Theorem 28 (Miller 1979) If $\text{Borel}(X) = \mathcal{P}(X)$, then $\text{ord}(X) < \omega_1$. In other words, there can be no Q_{ω_1} -set.

Theorem 29 (Miller 1979) It is consistent to have: for every $\alpha < \omega_1$ there is a Q_{α} -set.

In this model the continuum is \aleph_{ω_1+1} .

Q 19. For $\alpha \geq 3$ can we have a Q_α of cardinality greater than or equal to some $Q_{\alpha+1}$ -set?

Q 20. If we have a Q_ω -set must there be Q_n -sets for infinitely many $n < \omega$?

Q 21. Can there be a Q_ω -set of cardinality ω_1 ?

Theorem 30 (Miller 1979, 2014) *For any successor α with $3 \leq \alpha < \omega_1$ it is consistent to have a Q_α -set but no Q_β -set for $\beta < \alpha$.*

In this model the continuum has cardinality ω_2 . The Q_α -set X has size ω_1 and has “strong order” α . Namely, even if you add countably many more sets to the topology of X its order is still α . Another way to say this is that in this model $\mathcal{P}(\omega_1)$ is a countably generated σ -algebra in α -steps but it cannot be countably generated in fewer steps. (In fact, not even with ω_1 -generators.) I don’t know about Q_β sets for $\beta > \alpha$ in this model. However, if Brendle’s argument can be generalized it would show that X^2 is a $Q_{\alpha+1}$ -set.

Abstract Rectangles

Theorem 31 (Rao 1969, Kunen 1968) *Assume the continuum hypothesis, then every subset of the plane is in the σ -algebra generated by the abstract rectangles at level 2: $\mathcal{P}(2^\omega \times 2^\omega) = \sigma_2(\{A \times B : A, B \subseteq 2^\omega\})$.*

Theorem 32 (Kunen 1968) *Assume Martin’s axiom, then*

$$\mathcal{P}(2^\omega \times 2^\omega) = \sigma_2(\{A \times B : A, B \subseteq 2^\omega\}).$$

In the Cohen real model or the random real model any well-ordering of 2^ω is not in the σ -algebra generated by the abstract rectangles.

Theorem 33 (Rothberger 1952 and Bing, Bledsoe, Mauldin 1974) *Suppose that $2^\omega = \omega_2$ and $2^{\omega_1} = \aleph_{\omega_2}$ then the σ -algebra generated by the abstract rectangles in the plane is not the power set of the plane.*

Theorem 34 (Bing, Bledsoe, Mauldin 1974) *Suppose every subset of the plane is in the σ -algebra generated by the abstract rectangles. Then for some countable α every subset of the plane is in the σ -algebra generated by the abstract rectangles by level α . $\mathcal{P}(2^\omega \times 2^\omega) = \sigma_\alpha(\{A \times B : A, B \subseteq 2^\omega\})$*

Theorem 35 (Miller 1979) For any countable $\alpha \geq 2$ it is consistent that every subset of the plane is in the σ -algebra generated by the abstract rectangles at level α but for every $\beta < \alpha$ not every subset is at level β . $\text{ord}(\sigma(\{A \times B : A, B \subseteq 2^\omega\})) = \alpha$

Theorem 36 (Miller 1979) Suppose $2^{<\mathfrak{c}} = \mathfrak{c}$ and $\alpha < \omega_1$. Then the following are equivalent:

(1) Every subset of $2^\omega \times 2^\omega$ is in the σ -algebra generated by the abstract rectangles at level α .

(2) There exists $X \subseteq 2^\omega$ with $|X| = \mathfrak{c}$ and every subset of X of cardinality less than \mathfrak{c} is Σ_α^0 in X .

Q 22. Can we have $2^{<\mathfrak{c}} \neq \mathfrak{c}$ and $\mathcal{P}(2^\omega \times 2^\omega) = \sigma(\{A \times B : A, B \subseteq 2^\omega\})$?

Q 23. Can we have $\text{ord}(\sigma(\{A \times B : A, B \subseteq 2^\omega\})) < \omega_1$ and $\mathcal{P}(2^\omega \times 2^\omega) \neq \sigma(\{A \times B : A, B \subseteq 2^\omega\})$?

Q 24. Can we have $\text{ord}(\sigma(\{A \times B : A, B \subseteq 2^\omega\}))$ be strictly smaller than $\text{ord}(\sigma(\{A \times B \times C : A, B, C \subseteq 2^\omega\}))$?

Borel Universal Functions

Theorem 37 (Larson, Miller, Steprans, Weiss 2014) Suppose $2^{<\mathfrak{c}} = \mathfrak{c}$ then the following are equivalent:

(1) There is a Borel universal function, i.e., a Borel function $F : 2^\omega \times 2^\omega \rightarrow 2^\omega$ such that for every abstract $G : 2^\omega \times 2^\omega \rightarrow 2^\omega$ there are $h : 2^\omega \rightarrow 2^\omega$ and $k : 2^\omega \rightarrow 2^\omega$ such that for every $x, y \in 2^\omega$ $G(x, y) = F(h(x), k(y))$.

(2) Every subset of the plane is in the σ -algebra generated by the abstract rectangles.

Furthermore the universal function has level α iff every subset of the plane is in the σ -algebra generated by the abstract rectangles at level α .

Abstract Universal Functions

Theorem 38 (Larson, Miller, Steprans, Weiss 2014) If $2^{<\kappa} = \kappa$, then there is an abstract universal function $F : \kappa \times \kappa \rightarrow \kappa$, i.e.,

$$\forall G \exists h, k \forall \alpha, \beta \ G(\alpha, \beta) = F(h(\alpha), k(\beta)).$$

Theorem 39 (Larson, Miller, Steprans, Weiss 2014) It is relatively consistent that there is no abstract universal function $F : \mathfrak{c} \times \mathfrak{c} \rightarrow \mathfrak{c}$.

Q 25. Is it consistent with $2^{<\mathfrak{c}} \neq \mathfrak{c}$ to have a universal $F : \mathfrak{c} \times \mathfrak{c} \rightarrow \mathfrak{c}$?

Higher Dimensional Abstract Universal Functions

Theorem 40 (Larson, Miller, Steprans, Weiss 2014) *Abstract universal functions $F : \kappa^n \rightarrow \kappa$ of higher dimensions reduce to countably many cases where the only thing that matters is the arity of the parameter functions, e.g.*

- (1) $\exists F \forall G \exists h, k \forall x, y \quad G(x, y) = F(h(x), k(y))$
- (2) $\dots G(x, y, z) = F(h(x, y), k(y, z), l(x, z))$
- (n) $\dots G(x_0, \dots, x_n) = F(h_S(x_S) : S \in [n+1]^n)$

Theorem 41 (Larson, Miller, Steprans, Weiss 2014) *In the Cohen real model for every $n \geq 1$ there is a universal function on ω_n where the parameter functions have arity $n+1$ but no universal function where the parameter functions have arity n .*

A set is Souslin in X iff it has the form $\bigcup_{f \in \omega^\omega} \bigcap_{n < \omega} A_{f \upharpoonright n}$ where each A_s for $s \in \omega^{<\omega}$ is Borel in X .

Theorem 42 (Miller 1981) *It is consistent to have $X \subseteq 2^\omega$ such that every subset of X is Souslin in X and $\text{ord}(X) = \omega_1$. A Q_S -set.*

Theorem 43 (Miller 1995) *(CH) For any α with $2 \leq \alpha \leq \omega_1$ there is exists an uncountable $X \subseteq 2^\omega$ such that $\text{ord}(X) = \alpha$ and every Souslin set in X is Borel in X .*

Theorem 44 (Miller 2005) *It is consistent that there exists $X \subseteq 2^\omega$ such $\text{ord}(X) \leq 3$ and there is a Souslin set in X which is not Borel in X .*

Q 26. Can we have $\text{ord}(X) = 2$ here?

Theorem 45 (Miller 1981) *It is consistent that for every subset $A \subseteq 2^\omega \times 2^\omega$ there are abstract rectangles $B_s \times C_s$ with $A = \bigcup_{f \in \omega^\omega} \bigcap_{n < \omega} (B_{f \upharpoonright n} \times C_{f \upharpoonright n})$ but $\mathcal{P}(2^\omega \times 2^\omega) \neq \sigma(\{A \times B : A, B \subseteq 2^\omega\})$.*

Theorem 46 (Miller 1979) *It is consistent that the universal Σ_1^1 set $U \subseteq 2^\omega \times 2^\omega$ is not in the σ -algebra generated by the abstract rectangles.*

Theorem 47 (Miller 1981) *It is consistent that there is no countably generated σ -algebra which contains all Σ_1^1 subsets of 2^ω .*

Q 27. Thm 3 is stronger than Thm 2. Is the converse true?

For X a separable metric space define:

- $\Sigma_0^X = \Pi_0^X =$ Borel subsets of X^m some m .
- Σ_{n+1}^X the projections of Π_n^X sets.
- Π_{n+1}^X the complements of Σ_{n+1}^X sets.
- $\text{Proj}(X) = \bigcup_{n < \omega} \Sigma_n^X$.

Theorem 48 (Miller 1990) *It is consistent there are $X, Y, Z \subseteq 2^\omega$ of projective orders 0, 1, 2:*

- $\text{ord}(X) = \omega_1$ and $\Sigma_0^X = \text{Proj}(X)$
- $\Sigma_0^Y \neq \Sigma_1^Y = \text{Proj}(Y)$
- $\Sigma_0^Z \neq \Sigma_1^Z \neq \Sigma_2^Z = \text{Proj}(Z)$

Q 28. (Ulam) What about projective order 3 or higher?

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