On λ'-sets

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Abstract

A set $X \subseteq 2^\omega$ is a λ'-set iff for every countable set $Y \subseteq 2^\omega$ there exists a G_δ set G such that $(X \cup Y) \cap G = Y$. In this paper we prove two forcing results about λ'-sets. First we show that it is consistent that every λ'-set is a γ-set. Secondly we show that it is independent whether or not every $(\dagger)$-λ'-set is a λ'-set.

1 λ'-sets and γ-sets

A set $X \subseteq 2^\omega$ is a λ'-set iff for all countable $A \subseteq 2^\omega$ there exists a G_δ set G such that

$$(X \cup A) \cap G = A$$

An ω-cover of X is a countable set of open sets such that every finite subset of X is contained in an element of the cover. A γ-cover of X is a countable sequence of open subsets of X such that every element of X is in all but finitely many elements of the sequence.

Define X to be a γ-**set** iff any ω-cover of X contains a γ-cover of X.

In this section we answer a question of Gary Gruenhage who asked if there is always a λ'-set which is not a γ-set. We answer this in the negative.

It is well known (see Gerlits and Nagy [4]) that MA(σ-centered) implies that every set of reals of cardinality less than the continuum is a γ-set. The standard model for MA(σ-centered) (see Kunen and Tall [8]) is obtained as follows:

Suppose that M is a countable standard model of ZFC+CH and we iterate σ-centered forcings of size ω_1 in M with a finite support iteration of length ω_2. In the final model M_{ω_2}, we have that MA(σ-centered) is true and the continuum is ω_2.

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Theorem 1.1 *In the standard model for MA(σ-centered) every λ' set has cardinality $\leq \omega_1$, and (it follows from MA(σ-centered)) every set of size ω_1 is a γ-set. Hence, in this model, every λ'-set is a γ-set.*

Proof

We will use the following Lemma in our proof.

Lemma 1.2 *Suppose that $\mathbb{P}$ is a σ-centered forcing such that*

$$|\vdash \tau \in 2^\omega$$

Then there exists a countable set $A \subseteq 2^\omega$ in the ground model such that for every $p \in \mathbb{P}$ and open set $U \supseteq A$ coded in the ground model there exists $q \leq p$ such that $q|\vdash \tau \in U$.

Proof

To prove the Lemma we will use the following Claim.

Claim. Suppose $\Sigma \subseteq \mathbb{P}$ is a centered subset. Then there exists $x \in 2^\omega$ such that for every $p \in \Sigma$ and for every $n < \omega$ there exists $q \leq p$ such that

$$q|\vdash \check{x} \upharpoonright n = \tau \upharpoonright n.$$

pf: Otherwise by the compactness of 2^ω there exists a finite set

$$\{p_m : m < N\} \subseteq \Sigma \text{ and } \{s_m : m < N\} \subseteq 2^{<\omega}$$

such that $\{[s_m] : m < N\}$ covers 2^ω and for each $m < N$ we have that

$$p_m|\vdash \tau \notin [s_m].$$

But this is a contradiction since there exists some $p \in \mathbb{P}$ below all of the p_m. This proves the Claim.

Let $\mathbb{P} = \bigcup_{n<\omega} \Sigma_n$ be a sequence of centered sets. Then for each n there exists $x_n \in 2^\omega$ such that for every $p \in \Sigma_n$ and for every $m \in \omega$ there exists $q \leq p$ such that

$$q|\vdash \check{x}_n \upharpoonright m = \tau \upharpoonright m.$$

Now let $A = \{x_n : n < \omega\}$. This proves the Lemma.

QED

Suppose $X \subseteq 2^\omega$ is a λ'-set in M_{ω_2}. For each $\alpha \leq \omega_2$ define

$$\mathbf{X}_\alpha = X \cap M_\alpha$$

By a standard Lowenheim-Skolem argument we can find $\alpha < \omega_2$ such that

1. $X_\alpha \in M_\alpha$ and
2. for every countable $A \subseteq 2^\omega$ which is in M_α there exists a G_δ-set G coded in M_α such that

$$(X_{\omega_2} \cup A) \cap G = A$$

We claim that $X = X_{\omega_2} = X_\alpha$ and hence has cardinality $\leq \omega_1$. Suppose that τ is any term for an element of 2^ω in M_{ω_2}. Since τ is added at some latter stage β with $\alpha \leq \beta < \omega_2$ and the iteration of σ-centered forcings of length $< \omega_2$ is σ-centered, it follows that τ is added by a σ-centered forcing over M_α. Let $A \subseteq 2^\omega$ be the countable set given by the Lemma. By the Lemma it follows that τ must be an element of any G_δ set coded in M_α which contains A. Using item (2) above we see that τ must be in A if it is in X_{ω_2}. Therefore $X_{\omega_2} \setminus X_\alpha = \emptyset$.
QED

Remark. This argument is similar to the proof that there are no λ'-sets of size ω_2 in Laver's model, see Miller [12].

Remark. A set of reals X is a λ-set iff every countable subset of X is a relative G_δ. In ZFC we must always have a λ-set which is not a γ-set. To see this let

$$X = \{f_\alpha \in \omega^\omega : \alpha < \mathfrak{b}\}$$

be well-ordered by eventual dominance and unbounded. Then Rothberger [15] (or see Miller [11]) showed that X is a λ-set. However X is not a γ-set as is witnessed by the sequences of ω-covers

$$\mathcal{U}_m = \{U_n^m : n \in \omega\} \text{ where } U_n^m = \{f \in \omega^\omega : f(m) < n\}.$$

In fact the set X is a λ'-set with respect to ω^ω.

Remark. A Hausdorff gap is an example of a λ' set of cardinality ω_1. γ-sets have strong measure zero and Laver [9] proved that it consistent that every strong measure zero set is countable.

Suppose there exists $X, Y \subseteq 2^\omega$ such that $|X| = |Y|$ and X is a λ'-set and Y is not a γ-set. Then there exists Z which is a λ'-set and not a γ-set. To see this let $X = \{x_\alpha : \alpha < \kappa\}$ and $Y = \{y_\alpha : \alpha < \kappa\}$. Put $Z = \{(x_\alpha, y_\alpha) : \alpha < \kappa\}$. The first κ for which MA(σ-centered) fails is $\mathfrak{p}$ (Bell

[1]) and $\mathfrak{p}$ is also the size of the smallest non γ-set. Hence any model where every λ'-set is γ-set and $\mathfrak{c} \leq \omega_2$ must satisfy MA(σ-centered) and $\mathfrak{c} = \omega_2$.

Remark. Gruenhage and Szeptycki [6] were interested in obtaining a set of reals $X \subseteq 2^\omega$ which is γ-set and not a λ'-set because of the following two topological games.

Let X be a topological space and $x \in X$ and define the following games:

$G_{\mathcal{O},\mathcal{P}}(X,x)$: On round n player $\mathcal{O}$ chooses an open neighborhood U_n of x and player $\mathcal{P}$ chooses a point $p_n \in U_n$. Player $\mathcal{O}$ wins iff the sequence p_n converges to x.

$G^{fin}_{\mathcal{O},\mathcal{P}}(X,x)$: The same except we allow player $\mathcal{P}$ to choose a finite set of points $P_n \subseteq U_n$ on his move and $\mathcal{O}$ wins iff $\cup_{n<\omega} P_n$ converges to x.

It is not hard to check that player $\mathcal{O}$ has a winning strategy in $G_{\mathcal{O},\mathcal{P}}(X,x)$ iff player $\mathcal{O}$ has a winning strategy in $G^{fin}_{\mathcal{O},\mathcal{P}}(X,x)$. Also if player $\mathcal{P}$ has a winning strategy in $G_{\mathcal{O},\mathcal{P}}(X,x)$, then it is a winning strategy in $G^{fin}_{\mathcal{O},\mathcal{P}}(X,x)$.

Given $X \subseteq 2^\omega$ consider the topology on $2^{<\omega} \cup \infty$ generated by

1. $\{\sigma\}$ for each $\sigma \in 2^{<\omega}$ and
2. $\{\infty\} \cup (2^{<\omega} \setminus \{x \restriction n : n < \omega\})$ for each $x \in X$.

Let X_F denote this countable topological space.

Gruenhage [5], Nyikos [14], Sharma [16], and Gruenhage and Szeptycki [6] can be combined to show that:

X is not a γ-set iff player $\mathcal{P}$ has a winning strategy in $G^{f}_{\mathcal{O},\mathcal{P}}(X_F,\infty)$.

If X is a λ'-set, then $\mathcal{P}$ has no winning strategy in $G_{\mathcal{O},\mathcal{P}}(X_F,\infty)$.

Hence, if there is a set X which is a λ'-set and not a γ-set, then $\mathcal{P}$ has a winning strategy in $G^{f}_{\mathcal{O},\mathcal{P}}(X_F,\infty)$ but not in $G_{\mathcal{O},\mathcal{P}}(X_F,\infty)$.

Daniel Ma [10] has a clearer proof of the connection between γ-sets and such games.

Dow [2] results imply that in Laver's model [9]:

X is a λ'-set iff $\mathcal{P}$ has no winning strategy in $G_{\mathcal{O},\mathcal{P}}(X_F,\infty)$.

But, it also consistent that they are not the same. In Galvin and Miller [3] it is shown that assuming MA(σ-centered) there is a γ-set X which is concentrated on a countable subset of itself. Hence $\mathcal{P}$ has no winning strategy in $G^f_{\mathcal{O},\mathcal{P}}(X_F,\infty)$ hence none in $G_{\mathcal{O},\mathcal{P}}(X_F,\infty)$, but X is not a λ'-set.

Question 1.3 *Is it consistent with ZFC that for every $X \subseteq 2^\omega$ that*
$\mathcal{P}$ has no winning strategy in $G_{\mathcal{O},\mathcal{P}}(X_F,\infty)$
iff
$\mathcal{P}$ has no winning strategy in $G^f_{\mathcal{O},\mathcal{P}}(X_F,\infty)$?

After the first version of this paper was written, Gruenhage [7] constructed (in ZFC) using a gap construction an example of a countable space which distinguishes the two games.

2 (†)-λ'-set

In this section we answer Problem 2.12 from Nowik and Weiss [13] which asks basically whether it is true that every (†)-λ'-set is a λ'-set.

Definition. For any $a \in [\omega]^\omega$ let $a = \{a_0, a_1, \ldots\}$ be its increasing enumeration, then for any $f \in \omega^\omega$ let

$$G_f = \{a \in [\omega]^\omega \subseteq 2^\omega \ : \ \forall n \ \exists m > n \ \ a_n < f(n)\}$$

Definition. A set $X \subseteq 2^\omega$ is a **(†)-λ'-set** iff for every $f \in \omega^\omega$ we have $X \cap G_f$ is a λ'-set.

Theorem 2.1 *Suppose that the continuum hypothesis is true or even just $\mathfrak{b} = \mathfrak{d}$. Then there exists a (†)-λ'-set which is not a λ'-set.*

Theorem 2.2 *In the Cohen real model (Cohen's original model for not CH) every (†)-λ'-set is a λ'-set.*

Proof of Theorem 2.1

Assume CH. Let $\{f_\alpha \in \omega^\omega : \alpha < \omega_1\}$ be a scale. That is, for $\alpha < \beta$ we have that $f_\alpha <^* f_\beta$ and for all $g \in \omega^\omega$ there exists $\alpha < \omega_1$ such that $g <^* f_\alpha$. We may also assume that the f_α are strictly increasing. Let $X \subseteq [\omega]^\omega$ be the set of ranges of the elements of the scale. Then for any $g \in \omega^\omega$ we have that $G_g \cap X$ is countable and hence a λ'-set. On the other hand X is not a λ'-set because of the countable set $[\omega]^{<\omega}$. If $U \subseteq P(\omega)$ is an open set containing $[\omega]^{<\omega}$, then $K = P(\omega) \setminus U$ is a compact subset of $[\omega]^{\omega}$. If we identify $[\omega]^{\omega}$ with the strictly increasing elements of ω^{ω} (via the homeomorphism $a \mapsto \{a_0, a_1, \ldots\})$), then there exists $f \in \omega^{\omega}$ such that for all $g \in K$ we have $\forall n \;\; g(n) < f(n)$. It follows that for all but countably many α we have that the range$(f_\alpha) \in U$.

The proof using $\mathfrak{b} = \mathfrak{d}$ is similar. Start with a scale indexed by $\mathfrak{b}$ and note that any set $Y \subseteq P(\omega)$ of size less than $\mathfrak{b}$ is a λ'-set (this is due to Rothberger, see the proof of Lemma 2.4).

QED

Proof of Theorem 2.2

Assume that M is a countable transitive standard model of ZFC+CH.

For any $\alpha \leq \omega_2^M$ let $\mathbb{P}_\alpha$ be the finite partial functions from α into 2. We claim that for any G a $\mathbb{P}_{\omega_2}$-generic filter over M that in the model $M[G]$ every $(\dagger)$-λ'-set is a λ'-set.

Lemma 2.3 *Suppose N is a countable standard model of ZFC+CH, $\mathbb{P}$ is a countable poset in N, and*

$$N \models X \subseteq \omega^{\omega} \textit{ is unbounded in } \leq^*$$

Then for any G which is $\mathbb{P}$-generic over N we have that

$$N[G] \models X \textit{ is unbounded in } \leq^*$$

Proof

Let $\{g_\alpha : \alpha < \omega_1^N\}$ be a scale in N. Working in N choose $f_\alpha \in X$ so that

$$\exists^\infty n \;\; f_\alpha(n) > g_\alpha(n)$$

Note that for every $g \in \omega^\omega \cap N$ there exists $\alpha < \omega_1$ such that

$$\forall \beta > \alpha \;\; \exists^\infty n \;\; f_\beta(n) > g(n).$$

Suppose by way of contradiction that for some $g \in N[G] \cap \omega^\omega$ and all $\alpha < \omega_1$ we have that $f_\alpha \leq^* g$. Then for some $\Sigma \in [\omega_1]^{\omega_1}$ and $n < \omega$ we have that

$$\forall m > n \;\; \forall \alpha \in \Sigma \;\; f_\alpha(m) \leq g(m)$$

Let $q \in G$ force this fact. Now since $\mathbb{P}$ is a countable poset, there exists some $p \in G$ with $p \leq q$ such that

$$\Gamma = \{\alpha < \omega_1 : p \Vdash \alpha \in \dot{\Sigma}\}$$

is uncountable (and by definability of forcing it is in N). But note that $\{f_\alpha : \alpha \in \Gamma\}$ is unbounded and so for some $m > n$ the set $\{f_\alpha(m) : \alpha \in \Gamma\}$ is unbounded in ω.

Let $r \leq p$ decide $g(m)$, i.e., for some $k < \omega$ suppose

$$r \Vdash \dot{g}(m) = k.$$

Choose $\alpha \in \Gamma$ such that $f_\alpha(m) > k$, then r forces a contradiction and the Lemma is proved.
QED

Lemma 2.4 *Suppose N is a countable standard model of ZFC+CH, $\mathbb{P}$ is a countable poset in N, and*

$$N \models Y \subseteq 2^\omega \textit{ is not a } \lambda' \textit{ - set}$$

Then for G $\mathbb{P}$-generic over N we have that

$$N[G] \models Y \textit{ is not a } \lambda' \textit{ - set}$$

Proof
Let $D \subseteq 2^\omega$ be countable in N and witness that Y is not a λ'-set, i.e.,. there is no G_δ set $\bigcap_n U_n$ coded in N with

$$\bigcap_n U_n \cap (Y \cup D) = D$$

Working in N let $D = \{x_n : n < \omega\}$ and let $Z = Y \setminus D$ and for each $z \in Z$ define $f_z \in \omega^\omega$ such that $f_z(n)$ is the least m such that $x_n \restriction m \neq z \restriction m$. Now the family $X = \{f_z : z \in Z\}$ must be unbounded in $\leq^*$ in N. Suppose not, then there exists $g \in \omega^\omega \cap N$ which eventually dominates each element of X. It follows that if we let

$$U_n = \bigcup_{m<\omega} [x_m \restriction \max\{n, g(m)\}]$$

then

$$(\bigcap_{n<\omega} U_n) \cap (Y \cup D) = D$$

which is a contradiction.

It follows from Lemma 2.3 that X is unbounded in $N[G]$. I claim that D cannot be G_δ in $Y \cup D$ in the model $N[G]$. Suppose it is, and let $\bigcap_{n<\omega} U_n$ be a G_δ in $N[G]$ such that

$$\bigcap_{n<\omega} U_n \cap (Y \cup D) = D$$

For each n let $g_n \in \omega^\omega$ be such that for every m we have that

$$[x_m \restriction g_n(m)] \subseteq U_n.$$

Now for any $z \in Z$ there exist a n such that $z \notin U_n$. But this means that $f_z(m) \leq g_n(m)$ for every m since otherwise

$$x_m \restriction g_n(m) = z \restriction g_n(m)$$

and then $z \in U_n$. This proves the Lemma.
QED

Now we prove Theorem 2.2. Suppose that $X \subseteq 2^\omega$ is in $M[G]$ where G is $\mathbb{P}_{\omega_2}$-generic over M and

$$M[G] \models X \text{ is not a } \lambda'\text{-set}$$

By Lowenheim-Skolem arguments there exists $\alpha < \omega_2$ such that

$$X_\alpha =^{def} X \cap M[G_\alpha], \;\; X_\alpha \in M[G_\alpha], \text{ and } M[G_\alpha] \models X_\alpha \text{ is not a } \lambda'\text{-set}$$

Since being a λ'-set only depends on codes for G_δ-sets and reals are added by countable suborders of $\mathbb{P}_{[\alpha,\omega_2)}$ it follows from Lemma 2.4 that

$$M[G] \models X_\alpha \text{ is not a } \lambda'\text{-set}$$

But if $f \in \omega^\omega \in M[G]$ is $\omega^{<\omega}$-generic over $M[G_\alpha]$ then $X_\alpha \subseteq G_f$. It follows that

$$M[G] \models X \text{ is not } (\dagger)\text{-}\lambda'\text{-set}$$

as was to be proved.
QED

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The appendices are not intended for final publication but for the electronic version only.

Appendix A

Here is a proof of

Theorem *(Nyikos, Gruenhage, Sharma) If $X \subseteq 2^\omega$ is a λ'-set which is dense in 2^ω, then $\mathcal{P}$ has no winning strategy in $G_{\mathcal{O},\mathcal{P}}(X_F,\infty)$.*

Proof

For contradiction let τ be a winning strategy for $\mathcal{P}$.

Claim. For any play $(F_1,\ldots,F_n)$ by $\mathcal{O}$ there exists $y \in 2^\omega$ so that for all finite $F \subseteq X$ and $k < \omega$ there exists $F_{n+1} \supseteq F$ a finite subset of X such that

$$y \restriction k \subseteq \tau(F_1,F_2,\ldots,F_n,F_{n+1})$$

proof: Let U be a regular ultrafilter on the $[X]^{<\omega}$, ie.,

$$\{F \in [X]^{<\omega} : x \in F\} \in U \text{ for all } x \in X.$$

Define $y \in 2^\omega$ by

$$y(k) = 1 \text{ iff } \{F \in [X]^{<\omega} : \tau(F_1,F_2,\ldots,F_n,F)(k) = 1\} \in U$$

Note that the length of $\tau(F_1,F_2,\ldots,F_n,F)$ is greater than k for U almost all F because X is dense. Also, for any $k<\omega$ and finite $F \subseteq X$ we have that

$$\{F_{n+1} : F \subseteq F_{n+1},\ y \restriction k \subseteq \tau(F_1,F_2,\ldots,F_n,F_{n+1})\} \in U$$

Hence the Claim is proved.

Let $\omega^{<\omega} = \{s_i : i < \omega\}$. Using the Claim construct $\langle F_s, y_s : s \in \omega^{<\omega}\rangle$ so that

1. F_s contains all y_{s_i} which are in X for $i < |s|$ and
2. $y_s \restriction k \subseteq \tau(F_{s\restriction 1}, F_{s\restriction 2},\ldots,F_s,F_{sk})$

Let $Y = \{y_s : s \in \omega^{<\omega}\}$ and suppose U_n is a descending sequence of open sets such that

$$\bigcap_{n<\omega} U_n \cap (X \cup Y) = Y$$

Construct $f \in \omega^\omega$ as follows. Since $y_{f\restriction n} \in U_n$ there exists $k < \omega$ such that $[y_{f\restriction n} \restriction k] \subseteq U_n$. Let $f(n) = k$ and thus we have that for each n

$$\tau(F_{f\restriction 1}, F_{f\restriction 1}, \ldots, F_{f\restriction (n+1)}) =^{def} s_n \text{ and } [s_n] \subseteq U_n$$

But this is a losing play for $\mathcal{P}$. Suppose $x \in X$ and $\exists^\infty n \quad s_n \subseteq x$. Then $x \in \bigcap_{n<\omega} U_n$ and so $x \in Y$. But by our construction each element of Y is in $F_{f\restriction n}$ for all but finitely many n and since $s_n \not\subseteq x$ for $x \in F_{f\restriction n}$ we have a contradiction.
QED

Remark. For a kind of weak converse suppose that there exists a countable

$$A \subseteq 2^\omega \setminus X$$

which is not G_δ in $X \cup A$, then $\mathcal{P}$ has a winning strategy. Let

$$A = \{a_n : n < \omega\}$$

be listed with infinitely many repetitions. Let $\mathcal{P}$ play some $\sigma_n \subseteq a_n$ such that $[\sigma_n] \cap F_n = \emptyset$. To see that it is winning let

$$G = \{z \in 2^\omega : \exists^\infty n \;\; \sigma_n \subseteq z\}$$

Since $A \subseteq G$ and is not relatively G_δ there exists $x \in X$ such that $\sigma_n \subseteq x$ for infinitely many n. Hence σ_n does not converge to ∞.

Player $\mathcal{P}$ also has a winning strategy if X contains a perfect subset Q. Just let player $\mathcal{P}$ play a sequence σ_n such that

$$G = \{x \in Q : \exists^\infty n \; \sigma_n \subseteq x\}$$

is comeager in Q.

Appendix B

If $X = \{f_\alpha \in \omega^\omega : \alpha < \mathfrak{b}\}$ is well-ordered by eventual dominance and unbounded. Then Rothberger showed the set X is a λ'-set with respect to ω^ω.

This follows from the following lemma.

Lemma 2.5 *(Rothberger) Suppose $Z_\beta = \{f_\alpha : \alpha < \beta\} \subseteq \omega^\omega$ is well-ordered by eventual dominance, and $A \subseteq \omega^\omega$ is countable and for every $g \in A$ there exists $\alpha < \beta$ such that $\exists^\infty n\; g(n) < f_\alpha(n)$. Then there exists a G_δ set G with*

$$G \cap (Z_\beta \cup A) = A$$

Proof

This is proved by induction on β. and assume the lemma is true for all $\delta < \beta$. If β is a successor ordinal, then the induction is trivial.

Case 1. β is a limit ordinal of uncountable cofinality.

Find $\delta_0 < \beta$ so that for each $g \in A$ $\exists^\infty n \quad g(n) < f_{\delta_0}(n)$. Then by induction there exists a G_δ set G with

$$G \cap (Z_{\delta_0} \cup A) = A$$

Let $H = \{g \in \omega : \exists^\infty n \;\; g(n) < f_{\delta_0}(n)\}$ Then H is a G_δ set containing A and missing $Z_\beta \setminus Z_\delta$ and so

$$(G \cap H) \cap (Z_\beta \cup A) = A$$

Case 2. β is a limit ordinal of countable cofinality.

Let β_n be an increasing ω-sequence with limit β and let

$$A_n = \{g \in A : \exists^\infty m \;\; g(m) < f_{\beta_n}(m)\}$$

By inductive assumption there exists G_δ sets G_n so that

$$G_n \cap (Z_{\beta_n} \cup A_n) = A_n$$

Define

$$G_n^* = G_n \cup \{g \in \omega^\omega : \exists^\infty m \;\; f_{\beta_n}(m) \leq g(m)\}$$

Note that G_n^* is a G_δ set which contains A but still

$$G_n^* \cap (Z_{\beta_n} \cup A_n) = A_n$$

Define $G = \cap_{n<\omega} G_n^*$. Then G is a G_δ-set with

$$G \cap (Z_\beta \cup A) = A$$

QED

Appendix C

To better see the connection with γ-sets consider the following game:

Game: $G^{\gamma}_{\mathcal{F},\mathcal{C}}(X)$: Two players $\mathcal{F}$ finite and $\mathcal{C}$ clopen alternate plays as follows. On round n player $\mathcal{F}$ plays a finite set $F_n \subseteq X$ and player $\mathcal{C}$ responds with a clopen set C_n in 2^ω with $F_n \subseteq C_n$. Player $\mathcal{F}$ wins iff $\langle C_n : n < \omega \rangle$ is a γ-cover of X, ie. for all $x \in X$ for all but finitely many n we have $x \in C_n$.

This game is exactly the same as $G^{f}_{\mathcal{O},\mathcal{P}}(X_F, \infty)$. A neighborhood basis for ∞ in X_F consists of sets of the form $2^{<\omega} \setminus \{x \upharpoonright n : x \in F, n < \omega\}$ for $F \subseteq X$ finite. So we can regard $\mathcal{O}$ as player $\mathcal{F}$ playing a finite subset of X. Instead of $\mathcal{P}$ playing a finite set $P_n \subseteq 2^{<\omega}$ just regard him as $\mathcal{C}$ playing the clopen set

$$C_n = 2^\omega \setminus \bigcup \{[s] : s \in P_n\}.$$

Theorem 2.6 *(Daniel Ma) For $X \subseteq 2^\omega$ the following are equivalent:*

1. *X is not a γ-set*
2. *$\mathcal{C}$ has a winning strategy in $G^{\gamma}_{\mathcal{F},\mathcal{C}}(X)$.*

Proof
Suppose X is not a γ-set and let $\mathcal{U}$ be an ω-cover with no γ-subcover. Without loss of generality we may assume the elements of $\mathcal{U}$ are clopen. Given any F_n let $\mathcal{C}$ choose $C_n \in \mathcal{U}$ with $F_n \subseteq C_n$. Then since $\langle C_n : n < \omega \rangle$ is not a γ-cover, $\mathcal{C}$ wins.

For the other direction suppose Player $\mathcal{C}$ has a winning strategy τ in $G^{\gamma}_{\mathcal{F},\mathcal{C}}(X)$. Construct $\langle F_s, C_s : s \in \omega^{<\omega} \rangle$ so that

1. for each $s \in \omega^{<\omega}$ the set $\mathcal{U}_s = \{C_{sn} : n < \omega\}$ is an ω-cover of X and
2. for each $s \in \omega^{<\omega}$ and the set C_s is the response of player $\mathcal{C}$ using the strategy τ against the play $F_{s\upharpoonright 1}, F_{s\upharpoonright 2}, \ldots, F_s$.

To do this just let

$$\mathcal{U}_s = \{C : \exists F \;\; C = \tau(F_{s\upharpoonright 1}, F_{s\upharpoonright 2}, \ldots, F_s, F)\}$$

This is countable since there are only countably many clopen sets and by the rules of the game it must be an ω-cover. For each element of $\mathcal{U}_s$ choose a witness F.

Suppose for contradiction that X is a γ-set. It is well known (Gerlits and Nagy [4]) that for a γ set X that given a sequence of ω-covers, we may choose one element of each to get a γ-cover. This is denoted $X \in S_1(\omega, \Gamma)$. Hence we may choose C_{sn_s} for each $s \in \omega^{<\omega}$ such that every $x \in X$ is in all but finitely many C_{sn_s}. But now just look at the branch

$$m_0, m_1, m_2, \ldots \text{ where } m_0 = n_{\langle\rangle}, \ldots, m_{k+1} = n_{\langle m_0, m_1, m_2, \ldots, m_k \rangle}$$

But

$$F_{\langle m_0 \rangle}, C_{\langle m_0 \rangle}, \ldots, F_{\langle m_0, m_1, \ldots, m_k \rangle}, C_{\langle m_0, m_1, \ldots, m_k \rangle}, \ldots$$

is a play using the strategy τ with yields a γ cover. This is a contradiction.