# Infinite Combinatorics and Definability [1]

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**Abstract**

The topic of this paper is Borel versions of infinite combinatorial theorems. For example it is shown that there cannot be a Borel subset of $[\omega]^\omega$ which is a maximal independent family. A Borel version of the delta systems lemma is proved. We prove a parameterized version of the Galvin-Prikry Theorem. We show that it is consistent that any $\omega_2$ cover of reals by Borel sets has an $\omega_1$ subcover. We show that if V=L then there are $\Pi^1_1$ Hamel bases, maximal almost disjoint families, and maximal independent families.

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# 1 Introduction

Notation and background for reading this paper is in section 11.

Section 2 is concerned with the Galvin-Prikry Theorem. This theorem says that for any Borel set $B \subset [\omega]^\omega$ there exists a set $H \in [\omega]^\omega$ such that either $[H]^\omega \subset B$ or $[H]^\omega \cap B = \emptyset$. Answering a question of D. Mauldin we prove:

> If $B \subset 2^\omega \times [\omega]^\omega$ is Borel, then there exists $C \subset 2^\omega$ perfect and $X \in [\omega]^\omega$ such that $C \times [X]^\omega \subset B$ or $(C \times [X]^\omega) \cap B = \emptyset$.

Section 3 is concerned with a question of D. Fremlin. We prove that:

> It is consistent that the continuum is $\omega_3$ and for every family F of Borel sets of size $\omega_2$ if F has empty intersection then some subfamily of F of size $\leq \omega_1$ has empty intersection. Equivalently by taking complements, if $R = \cup F$ then for some $G \subset F$ of cardinality $\omega_1$, $R = \cup G$.

The remaining sections are about finding Borel versions of various combinatorial theorems. Probably the first result along this line is a theorem of Sierpiński 1938 [42] that any nontrivial ultrafilter on $\omega$ when considered as a subset of $2^\omega$ cannot be Borel. Talagrand 1980 [45] pursues this. Mathias 1977 [27] shows that a maximal almost disjoint family in $[\omega]^\omega$ cannot be $\Sigma^1_1$. Jones 1942 [18] shows that a Hamel basis cannot be $\Sigma^1_1$. This was done earlier by Sierpiński 1920 [40].

Another Borelized version of a combinatorial theorem is an unpublished result of Galvin (1968) [10]. Namely, for any Borel partition of the two element sets of reals into finitely many pieces there exists a perfect set all of whose two element subsets are in the same piece of the partition. Later Silver noticed that using a result of Mycielski (1964) [35] it can be reduced to clopen partitions and that in fact works for any Baire partition (or measurable partitions Mycielski (1967) [36]). Galvin extended his result to three tuples and Blass (1981) [3] extended it to arbitrary n-tuples.

Friedman and Shelah (see Shelah 1984 [38]) proved that no Borel linear order can be a Souslin line as well as many other results about Borel partial orders ( see also Harrington-Shelah 1982 [15]). Harrington, Marker, and Shelah (1988) [16] have proved a Borel version of Dilworth's Theorem. Martin

1981 [26] proves a theorem of Eggleston about subsets of the plane of positive measure using forcing and absoluteness. Komjáth 1984 [21] proves and improves a result of Laczkovich about sequences of Borel sets using similar ideas. In section 4 we give a proof of a Borel version of the delta system lemma:

> If $B \subset 2^\omega \times 2^\omega$ is any Borel set all of whose cross sections are finite, then there exists a perfect set $C \subset 2^\omega$ with the property that $\{B_t \mid t \in C\}$ is a delta system.

We also consider weak delta systems of families of countable sets.

In section 5 we give some counterexamples to Borel versions of combinatorial theorems about families of strongly almost disjoint families.

In section 6 we show that for any convergent sequence of real numbers it is possible to partition the real line into two pieces so that neither piece contains a sequence which is similar to the given sequence.

In section 7 we show that if V=L then there exists a $\Pi^1_1$ subset of the plane that meets every line in exactly two points. In section 8 we show there is a $\Pi^1_1$ maximal almost disjoint family of subsets of $\omega$. In section 9 we show that if V=L then there is a $\Pi^1_1$ Hamel basis. We also give a category proof of Jones' Theorem that no Hamel basis can be $\Sigma^1_1$.

In section 10 we show that maximal independent families cannot be $\Sigma^1_1$ but if V=L then there are ones which are $\Pi^1_1$.

# 2 Parameterized Galvin Prikry Theorem

The following theorem answers a question of D. Mauldin. It is a sort of parameterized Galvin-Prikry Theorem 1973 [11].

**Theorem 2.1** *If $B \subset 2^\omega \times [\omega]^\omega$ is Borel, then there exists $C \subset 2^\omega$ perfect and $X \in [\omega]^\omega$ such that $C \times [X]^\omega \subset B$ or $(C \times [X]^\omega) \cap B = \emptyset$.*

The proof will broken down into several lemmas.

**Lemma 2.2** *If $U$ is a Ramsey ultrafilter in a model $M$ of $ZFC^*$ and $t$ is Sacks perfect set forcing generic over $M$ and*

$$U^* = \{X \in [\omega]^\omega \cap M[t] \mid \exists Y \in U \;\; Y \subset X\}$$

*then*

$$M[t] \models U^* \text{ is a Ramsey ultrafilter}$$

proof: see Baumgartner-Laver 1979 [1] Theorem 4.4 p 280.
□

If U is a Ramsey ultrafilter then define

$$P_U = \{(s, X) \mid s \in [\omega]^{<\omega}, X \in U, \text{ and max(s)} < \text{min(X)} \}$$

ordered by $(s, X) \leq (t, Y)$ iff $s \supset t, X \subset Y$ and $s \setminus t \subset Y$.

The next lemma says that any statement can be decided without extended the finite part of the condition.

**Lemma 2.3** *If $\theta$ is any sentence and $(s, X) \in P_U$ then there exists $Y \in U \cap [X]^\omega$ such that either $(s, Y) \Vdash \theta$ or $(s, Y) \Vdash \neg\theta$*

proof: see Mathias 1977 [27] Prop 2.9 p 74.
□

For G a $P_U$ filter define $Z_G = \cup\{s \mid \exists X\ (s, X) \in G\}$ . Conversely given $Y \in [\omega]^\omega$ define $G_Y = \{(s, X) \in P_U \mid s \subset Y \subset X \cup s\}$. It is well known that G and Z are definable from each other. Z is called a Mathias real. The next lemma says that every infinite subset of a Mathias real is Mathias.

**Lemma 2.4** *If $G$ is $P_U$ generic over $M$ then for every $Y \in [Z_G]^\omega$ $G_Y$ is $P_U$ generic over M.*

proof: see Mathias 1977 [27] Corollary 2.5 p. 73.
□

**Lemma 2.5** *For any perfect set $p$ and infinite $Y \subset \omega$ in $M$ there exists a perfect set $C \subset p$ and $X \in [Y]^\omega$ such that for every $t \in C$ and $Z \in [X]^\omega$ (t,Z) is $S \times P_U$ generic over M.*

proof: Construct C perfect and $X \in [\omega]^\omega$ such that for every $t \in C$ $(t, X)$ is $S \times P_U$ generic over M. This is easy to do since M is countable. By the product lemma for any $t \in C$ we have that X is $P_U$ generic over $M[t]$. But $P_U$ is a dense subset of $P_{U^*}$ and so X is $P_{U^*}$ generic over $M[t]$. By Lemma

2.2 $U^*$ is a Ramsey ultrafilter in $M[t]$ and by Lemma 2.4 any $Z \in [X]^\omega$ is $P_{U^*}$ generic over $M[t]$. Since $P_U$ is dense in $P_{U^*}$ Z is $P_U$ generic over $M[t]$ and so $(t, Z)$ is $S \times P_U$ generic over M.
□

proof of Theorem 2.1: Let M be a countable standard model of a sufficiently large finite fragment of the theory ZFC + CH and containing a code for the Borel set B. Let S be Sacks perfect set forcing and let t be S generic over M. By Lemma 2.2 and Lemma 2.3 either:

$$M[t] \models \exists A \in U^* \ \ (\emptyset, A) \Vdash Z \in B_t$$

or

$$M[t] \models \exists A \in U^* \ \ (\emptyset, A) \Vdash Z \notin B_t$$

where Z is Mathias over M[t]. Assume the first case since the argument is symmetric. Hence working in M there exists p $\in S$ and $Y \in U$ such that

$$p \Vdash_S \text{“} \ (\emptyset, Y) \Vdash_{P_{U^*}} Z \in B_t \text{”}$$

Note that by Lemma 2.2 $P_U$ is a dense subset of $P_{U^*}$ and hence $S \times P_U$ is a dense subset of the iteration $S * P_{U^*}$. It follows that

$$(p, (\emptyset, Y)) \Vdash_{S \times P_U} Z \in B_t$$

Since B is Borel and hence absolute we have

$$C \times [X]^\omega \subset B$$

where C and X are gotten by applying Lemma 2.5.
□

Since we are only using the absoluteness of Borel predicates the same proof gives a parameterized version of Silver's 1970 Theorem [43] that $\Sigma^1_1$ are Ramsey. Is there a parameterized Ellentuck 1974 Theorem? ([4]). More specifically consider the $\sigma$-algebra of all sets $A \subset 2^\omega \times [\omega]^\omega$ with the property that for every perfect set $C \subset 2^\omega$ and $(s, X)$ there exists a perfect set $D \subset C$ and a $Y \in [X]^\omega$ such that

$$D \times [s, Y] \subset A \ \text{ or } \ D \times [s, Y] \cap A = \emptyset$$

where $[s, Y] = \{Z \in [\omega]^\omega \ \mid \ s \subset Z \subset Y \cup s\}$. Can we characterize this family of sets in terms of the Baire property in the Ramsey topology and perhaps the ideal of $(s_0)$ sets? (A set of reals X has property $(s_0)$ if for every perfect set P there exists a perfect set $Q \subset P$ which is disjoint from X, see A.Miller 1984 section 5 [30] and Marczewski 1935 [25].)

The following corollary which is a result of Mazurkiewicz (1932) [29] was pointed out to me by D.Mauldin and R.Pol.

**Corollary 2.6** *Suppose* $< f_n : 2^\omega \mapsto 2^\omega \mid n \in \omega >$ *is a sequence of Borel functions. Then there exists a perfect set* $P \subset 2^\omega$ *and a subsequence* $< f_{k_n} : 2^\omega \mapsto 2^\omega \mid n \in \omega >$ *which is pointwise convergent on every point in* $P$*.*

proof: Consider

$$B = \{(x, M) \mid < f_n(x) \mid n \in M > \text{converges}\} \subset 2^\omega \times [\omega]^\omega$$

$\Box$

**Examples**

1. Define $B \subset [\omega]^\omega \times [\omega]^\omega$ by $(X, Y) \in B$ iff $min(X) < min(Y)$. Then there cannot be $M, N \in [\omega]^\omega$ such that $[M]^\omega \times [N]^\omega$ is either contained in or disjoint from B.

2. (D.Mauldin) Let

$$B = \{(x, M) \mid x \upharpoonright M \text{ } constant\} \subset 2^\omega \times [\omega]^\omega$$

   It is easy to see that this example shows that the perfect set C in Theorem 2.1 cannot in general be of positive measure. Mauldin also has an example of a uniformly bounded sequence of continuous functions on $[0, 1]$ such that no subsequence converges pointwise on a set of positive measure.

3. It is easy to generalize Theorem 2.1 to the case of Borel subsets of $((2^\omega)^\omega) \times [\omega]^\omega$ by using Laver's [24] infinite version of the Halpern-Läuchli Theorem.

In an earlier version of this paper I remarked that it would be interesting to have a proof of Theorem 2.1 that did not use forcing and absoluteness. Todorčević wrote back that in fact he proved Theorem 2.1 several years ago with the following proof:

Assume first that $B \subset 2^\omega \times [\omega]^\omega$ is clopen. Pick recursively $\{(x_\alpha, A_\alpha) \mid \alpha < \omega_1\}$ such that the $x_\alpha$'s are distinct elements of $2^\omega$, $A_\alpha \in [\omega]^\omega$, $A_\alpha \subset^* A_\beta$ for $\beta < \alpha$, and for all $\alpha < \omega_1$

$$\{x_\alpha\} \times [A_\alpha]^\omega \subset B \text{ or } \{x_\alpha\} \times [A_\alpha]^\omega \cap B = \emptyset$$

Find $X \subset \omega_1$ and $A \in [\omega]^\omega$ such that the order type of $\{x_\alpha \mid \alpha \in X\}$ is the same as the rationals and either for every $\alpha \in X$

$$\{x_\alpha\} \times [A_\alpha]^\omega \subset B$$

or for every $\alpha \in X$

$$\{x_\alpha\} \times [A_\alpha]^\omega \cap B = \emptyset$$

and also

$$A \subset \bigcap_{\alpha \in X} A_\alpha$$

(A and X can be obtained by an argument similar to the proof of Lemma 3.1.1 of [5].) Let P be the closure of $\{x_\alpha \mid \alpha \in X\}$. Then P and A work. The proof when B is closed is almost the same. The obvious induction on the Borel rank reduces the general case to the clopen case.

# 3 Intersections of Borel Sets

In this section we answer a question of D.H.Fremlin. We show that it is consistent that for every family F of Borel sets of size $\omega_2$, if F has empty intersection then some subfamily of F of size $\leq \omega_1$ has empty intersection. Note that the Hausdorff gap 1934 ([14]) shows that there is a family of Borel sets of size $\omega_1$ with empty intersection but every countable subfamily has nonempty intersection. In Fremlin-Jasiński [9] it is shown that MA + not CH implies this is false. They show that the complement of any set of reals of cardinality $\omega_2$ is the union of $\omega_2$ Borel sets. The argument we use is the same as used by Harrington (see Corollary 3.9) in unpublished work.

**Theorem 3.7** *Suppose M is a countable standard model of ZFC + CH and $P = FIN(\omega_3^M)$ ($FIN(\kappa)$ is the usual partial order for adding $\kappa$ Cohen reals* *to M.) Then for any G P-generic over M, in M[G] for every family F of Borel sets of size $\omega_2$, if F has empty intersection then some subfamily of F of size $\leq \omega_1$ has empty intersection. Equivalently by taking complements, if $R = \cup F$ then for some $G \subset F$ of cardinality $\omega_1$, $R = \cup G$.*

The proof will require the following well-known lemma.

**Lemma 3.8** *Suppose M and N are standard models of $ZFC^*$, $j : M \mapsto N$ is an elementary embedding, and $\kappa$ is the first ordinal moved, P a partial order in M, $j(P) = Q$, G is P-generic over N and $H = j^{-1}(G)$. If P has $\kappa - c.c.$ in M then H is Q-generic over M and if $j^* : M[H] \mapsto N[G]$ is defined by $j^*(\tau^H) = (j(\tau))^G$ then $j^*$ is well-defined and an elementary embedding.*

proof: If $A \subset Q$ is a maximal antichain in M, then since Q has the $\kappa - c.c.$, $j(A) = j$“$A$ is a maximal antichain in P which is in N, hence H is Q-generic over M.
Well-defined: Suppose $\tau^H = \sigma^H$. Then there exists $p \in H$ such that

$$M \models \text{“}p \Vdash \tau = \sigma\text{”}$$

but then

$$N \models \text{“}j(p) \Vdash \tau = \sigma\text{”}$$

Elementarity: We use the Tarski-Vaught criterion. Suppose

$$N[G] \models \exists x\; \theta(x, j(\tau)^G)$$

then there exists $p \in H$ such that either

$$M \models\; p \Vdash \exists x\; \theta(x, \tau)$$

or

$$M \models\; p \Vdash \neg\exists x\; \theta(x, \tau)$$

But the latter cannot happen because then

$$N \models\; j(p) \Vdash \neg\exists x\; \theta(x, j(\tau))$$

and $j(p) \in G$. Hence there exist $\sigma \in M$ such that

$$M \models\; p \Vdash\; \theta(\sigma, \tau)$$

so

$$N \models j(p) \Vdash \theta(j(\sigma), j(\tau))$$

and $j(p) \in G$ so

$$N[G] \models \theta(j(\sigma)^G), j(\tau)^G)$$

□

proof of Theorem 3.7: Let $M_\lambda$ be $H_\lambda$ in M for some sufficiently large cardinal $\lambda$. ($H_\lambda$ is set of all sets whose transitive closure has cardinality less than $\lambda$) Let $G : \omega_3 \mapsto 2$ be FIN($\omega_3$)-generic over M and suppose

$$M[G] \models R = \cup\{B_\alpha \mid \alpha < \omega_2\},\ each\ B_\alpha\ Borel$$

Choose $\Sigma \in M$ so that

$$< B_\alpha \mid \alpha < \omega_2 > \in M_\lambda[G \upharpoonright \Sigma]$$

Working in M find a transitive set N and embedding j such that

1. $|N| = \omega_1$.
2. $N^\omega \subset N$.
3. $j : N \mapsto M_\lambda$ is an elementary embedding.
4. $\beta = \omega_2 \cap N$ is an ordinal and hence $j(\beta) = \omega_2$.
5. $\Sigma$ is in the range of j and let $j(\Sigma_0) = \Sigma$.

Note that $j(FIN(\Sigma_0)) = FIN(\Sigma)$ and so if we let $G_0$ be defined by $G_0(\alpha) = G(j(\alpha))$, then by the lemma

$$j^* : N[G_0] \mapsto M_\lambda[G \upharpoonright \Sigma]$$

is an elementary embedding. Since $j^*$ is the identity on the reals $j(B_\alpha) = B_\alpha$. Clearly

$$M_\lambda[G \upharpoonright \Sigma] \models \text{“}1 \Vdash_{FIN(\omega)} R = \cup\{B_\alpha \mid \alpha < \omega_2\}\text{”}$$

So by elementarity

$$N[G_0] \models \text{“}1 \Vdash_{FIN(\omega)} R = \cup\{B_\alpha \mid \alpha < \beta\}\text{”}$$

It suffices to show that for every $x \in R^{M[G]}$ there exists K $FIN(\omega)$-generic over $N[G_0]$ such that $x \in N[G_0][K]$, since then there exists $\alpha < \beta$ such that $N[G_0][K] \models x \in B_\alpha$ and thus by Borel absoluteness $x \in B_\alpha$. This follows from knowing $M \models N^\omega \subset N$. In more detail suppose $\tau \in M$ is a name for x where:

$$\Gamma \in [\omega_2]^\omega \cap M$$

$$y = G \upharpoonright (j``\Sigma_0 \cap \Gamma)$$

$$z = G \upharpoonright (\Gamma \setminus j``\Sigma_0)$$

$$x = \tau(y, z)$$

Working in M we can find a name $\tau^*(u, v) \in N$, $z^*$ $FIN(\omega)$-generic over $N[G_0]$ ( or rather a name for $z^*$ ), such that

$$x = \tau^*(G_0 \upharpoonright j^{-1}(\Gamma), z^*)$$

Note that $j^*$ maps $G_0 \upharpoonright j^{-1}(\Gamma)$ to $G \upharpoonright (j``\Sigma_0 \cap \Gamma)$.
□

Fremlin notes that it is easy to generalize the theorem to adding any number $\kappa \geq \omega_3$ Cohen reals to M.

**Corollary 3.9** *(Harrington[3]) In the Cohen real model there are no mad families of cardinality $\kappa$ where $\omega_1 < \kappa < c$.*

proof: For $X \in [\omega]^\omega$ consider the Borel set

$$B_X = \{Y \in [\omega]^\omega \mid X \cap Y \ infinite\}$$

If M is a mad then $R = \cup\{B_X \mid X \in M\}$.
□

There are mad families of cardinality $\omega_1$ (see Kunen [22] chapter 8 Theorem 2.3 p.256 ) in the Cohen real model. If we start with MA + $c = \omega_2$ and add $\omega_4$ random reals, then we get a model where there are MAD's of size $\omega_2$ and $\omega_4$, but none of size $\omega_1$ or $\omega_3$.

[3]The referee remarks that this result is folklore, and was known when Harrington was in knee pants. I first heard of it from Harrington.

# 4 Delta Systems Lemma

A delta system is a family of sets F such that there is a R ( called the root ) with the property that for every two distinct elements A and B of F, $A \cap B = R$. The classical delta systems lemma says that every uncountable family of finite sets contains an uncountable delta system. The following theorem is a definable version of this.

**Theorem 4.10** *If $B \subset 2^\omega \times 2^\omega$ is any Borel set all of whose cross sections are finite, then there exists a perfect set $C \subset 2^\omega$ with the property that $\{B_t \mid t \in C\}$ is a delta system.*

We will use the following lemma.

**Lemma 4.11** *Suppose $P$ is a partial order in a model $M$ then for any $G$ and $H$ such that $G \times H$ is $P \times P$ generic over $M$*

$$M[G] \cap M[H] = M$$

proof: see Solovay 1970 [44] Lemma 2.5 p.13
□
proof of Theorem 4.10: Let M be a countable standard model of $ZFC^*$ which contains a code for the Borel set B. Let P be the usual partial order for forcing a Cohen real, i. e. $P = 2^{<\omega}$. Let $x \in 2^\omega$ be P-generic over M. Suppose $R = B_x \cap M$ and let $p \in P$ be such that

$$p \Vdash B_x \cap M = R$$

Let C be a perfect set of elements of $2^\omega$ such that for every $x \in C$, $p \subset x$ and for any two distinct elements of $x, y \in C$ we have that (x,y) is $P \times P$ generic over M (hence each $x \in C$ is P generic over M). It follows by absoluteness and Lemma 4.11 that for any two elements x and y of C we have $B_x \cap B_y = R$.
□

Todorčević remarks that Theorem 4.10 can also be proved using Galvin's perfect partition theorem mentioned in the introduction. The proof is done in the same standard way Ramsey's Theorem shows that any infinite $F \subset [\omega]^m$ contains an infinite $\Delta$-system. R.Pol sent me a similar proof.

We now will generalize to the case of families of countable sets. We call a family of countable sets F a weak delta system iff there is a countable set R such that for any two elements X and Y of F $X \cap Y \subset R$.

**Theorem 4.12** *If $B \subset 2^\omega \times 2^\omega$ is any Borel set all of whose cross sections are countable, then there exists a perfect set $C \subset 2^\omega$ with the property that $\{B_t \mid t \in C\}$ is a weak delta system.*

proof: The proof is the same except we just use $R = M \cap 2^\omega$ as our weak root. Note that $B_x \subset M[x]$ since otherwise

$$M[x] \models B_x \textit{ is uncountable}$$

and since it is Borel there would be a perfect set P coded in M[x] such that

$$M[x] \models P \subset B_x$$

and so by $\Pi^1_1$ absoluteness $B_x$ would really contain P.
□

Note that a weak delta system is the best we can hope to obtain since there is a Borel parameterized family of almost disjoint sets, e. g. let

$$\{x_s \mid s \in 2^{<\omega}\} \subset 2^\omega$$

and define $B \subset 2^\omega \times 2^\omega$ by

$$(t, x) \in B \text{ iff } \exists n \in \omega \; x = x_{t\upharpoonright n}$$

The following result about weak delta systems is what we can say if we drop the assumption of definability.

**Theorem 4.13** *Suppose $\kappa$ is any cardinal such that $\omega_1 < \kappa \leq \omega_\omega$ and $F$ is a family of $\kappa$ many countable sets. Then $F$ contains a weak delta system of cardinality $\kappa$. If $2^\omega < \omega_\omega$ then this is false for $\kappa = \omega_{\omega+1}$.*

The following lemma is half of one of the standard proofs of the usual delta lemma for families of countable sets.

**Lemma 4.14** *Let $\kappa > \omega_1$ be a regular cardinal and suppose $F \subset [\kappa]^\omega$ has cardinality $\kappa$. then there exists $G \in [F]^\kappa$ and $R$ of cardinality less than $\kappa$ such that for every two $A, B \in G$ we have $A \cap B \subset R$.*

proof: Let $F = \{A_\alpha \mid \alpha < \kappa\}$ and define

$$f(\alpha) = \sup(A_\alpha \cap \alpha)$$

Since f is pressing down on all $\alpha$ of uncountable cofinality there is a stationary set $S \subset \kappa$ and a $\gamma < \kappa$ such that for all $\alpha \in S$

$$A_\alpha \cap \alpha \subset \gamma$$

Since

$$\{\alpha \mid \forall \beta < \alpha \;\; A_\beta \subset \alpha\}$$

is closed and unbounded in $\kappa$, we may assume for every $\alpha, \beta \in S$ that $\alpha < \beta \to A_\alpha \subset \beta$. Hence $R = \gamma$ and $G = \{A_\alpha \mid \alpha \in S\}$ work.
□

**Lemma 4.15** *Suppose $\kappa > \omega_n$ is a regular cardinal, $1 \leq n < \omega$ and $F \subset [\omega_n]^\omega$ has cardinality $\kappa$, then there exists $C$ countable such that $[C]^\omega \cap F$ has cardinality $\kappa$.*

proof: The proof is by induction on n. Since $\omega_n$ is regular and uncountable there is some $\alpha < \omega_n$ which contains $\kappa$ many elements of F. Now just regard $\alpha$ as $\omega_{n-1}$ and proceed.
□

proof of Theorem 4.13: The proof of the first part of Theorem 4.13 is by induction on $\kappa$. For $\kappa = \omega_n$ first apply Lemma 4.14 to get G and R with R of cardinality $\omega_{n-1}$. Then apply Lemma 4.15 to the family $\{X \cap R \mid X \in G\}$.

To do the case $\kappa = \omega_\omega$ let $F = \cup_{n\in\omega} F_n$ where $|F_n| = \omega_n$. Apply Lemma 4.15 to find $H_n \in [F_n]^{\omega_n}$ and $C_n \in [\cup F_{n-1}]^\omega$ such that for every $A \in H_n$

$$A \cap (\cup F_{n-1}) \subset C_n$$

Next apply the induction case to the $H_n$ to obtain $G_n \in [H_n]^{\omega_n}$ and weak roots $R_n$ and let

$$G = \cup_{n\in\omega} G_n$$

and

$$R = \cup\{R_n \cup C_n \mid n \in \omega\}$$

To prove the second part of Theorem 4.13 let F be a family of $\omega_{\omega+1}$ countable subsets of $\omega_\omega$, which exists by König's Theorem. This family cannot contain a large weak delta system if the continuum is small. For suppose $G \subset F$ is a weak delta system of size $\kappa$ with a countable weak root R. Since

$$\{A \setminus R \mid A \in G\}$$

are disjoint all but at most $\omega_\omega$ must be empty and hence $2^\omega = 2^{|R|} \geq \omega_{\omega+1}$ □

The referee remarks that the second part of Theorem 4.13 holds also under the assumption that $2^\omega = \omega_{\omega+1}$. Just construct the family F inductively avoiding every potential weak root. I don't know what happens when the continuum is larger.

# 5 Strongly Almost Disjoint Families

A family of sets F is strongly almost disjoint if there is an $n \in \omega$ such that any two distinct elements of F meet in a set of cardinality at most n. Now we show that there is no Borel version of E. Miller's Theorem. This Theorem says that for every strongly almost disjoint family F of infinite countable sets there exists a set X such that for every $A \in F$ both $X \cap A$ and $A \setminus X$ are infinite. The family F is said to have property B if such an X exists. This is in honor of Bernstein 1908 [2] who showed that countable families of infinite sets have property B.

**Theorem 5.16** *There is a Borel set $A \subset R \times R$ such that $\{A_x \mid x \in R\}$ is strongly almost disjoint family of countable infinite sets, but there does not exist a Borel set $X \subset R$ such that for every $x \in R$ both $X \cap A_x$ and $A_x \setminus X$ are infinite.*

proof: Define $A_x = \{x + a_n \mid n = 1, 2, \ldots\}$. where $< a_n : n \in \omega >$ is a sequence converging to zero with the property that $a_n - a_m = a_k - a_l$ iff $< n, m > = < k, l >$. For example $a_n = 1/2^n$. Then if $x \neq y$ $A_x \cap A_y$ has size at most one. For suppose there were n, m, l, and k such that

$$x + a_n = y + a_m$$

$$x + a_k = y + a_l$$

Then

$$a_n - a_m = y - x = a_k - a_l$$

Suppose X is Borel. Let $x \in R$ be a Cohen real. Then for some rational interval p=(r,s) with $r < x < s$ we have either

$$p \Vdash x \in X$$

or

$$p \Vdash x \notin X$$

Suppose the first happens. Then for all but finitely many n $r < x + a_n < s$ and for every n, $x + a_n$ is Cohen real and so $A_x \subset^* X$. If the second happens, then $A_x \cap X =^* \emptyset$.

□

Komjáth 1984 [20] showed that if $F = \{A_\alpha \mid \alpha < \kappa\}$ is any strongly almost disjoint family of countable sets then there exists a family of finite sets $\{B_\alpha \mid \alpha < \kappa\}$ such that

$$\{A_\alpha \setminus B_\alpha \mid \alpha < \kappa\}$$

is a disjoint family. This result generalizes E.Miller 1937 [31]. It is easy to see using standard selection theorems that Theorem 5.16 gives a counterexample to a Borel version of Komjáth's theorem. Below we will give an alternative proof of this fact.

**Theorem 5.17** *There is a Borel set $A \subset 2^\omega \times 2^\omega$ such that $\{A_x \mid x \in 2^\omega\}$ is strongly almost disjoint family of countable sets such that there is a Borel set $X$ such that for every $x \in 2^\omega$, $X \cap A_x$ and $A_x \setminus X$ are infinite, but there is no Borel set $B \subset 2^\omega \times 2^\omega$ such that for every $x \in 2^\omega, B_x$ is finite and*

$$\{A_x \setminus B_x \mid x \in 2^\omega\}$$

*is a disjoint family.*

proof: We can regard $[2^\omega]^2$ as being a Borel subset of $2^\omega$, since $2^\omega \times 2^\omega$ is homeomorphic to $2^\omega$ and we can regard $[2^\omega]^2 \subset 2^\omega \times 2^\omega$ by looking only at ordered pairs where the the first coordinate is lexicographically less than the second coordinate. The example is defined by

$$A_x = \{\{x, y\} \mid x \equiv_T y, x \neq y\}$$

where $\equiv_T$ stands for Turing equivalent, i. e. x and y are each recursive in the other. It is easy to see that A is Borel and each $A_x$ is countable. Also $X = \{\{x, y\} \mid x(0) = y(0)\}$ splits every element of the family. Note that for distinct x and y, $A_x \cap A_y$ is empty unless x and y are Turing equivalent and then it contains exactly the pair $\{x, y\}$. So we have a strongly almost disjoint family. Now suppose B were a counterexample to the Theorem and let P be Cohen real forcing and x the name of the Cohen real. There exists $p \in P$ and $k \in \omega$ such that

$$p \Vdash |B_x| = k$$

For any recursive automorphisms $\pi$ of P we have

$$\pi(p) \Vdash |B_{\pi(x)}| = k$$

It is easy to find infinitely many recursive automorphisms of P which fix p but give different $\pi(x)$ with boolean value one. Hence by using a countable standard model of $ZFC^*$ and absoluteness, there must be a Turing degree X such that for infinitely many $x \in X, |B_x| = k$. Let $Y \subset X$ be these x and for each $y \in Y$ let

$$C_y = \{z \mid \{y, z\} \in B_y\}$$

Since each of the $C_y$ has cardinality k we can apply the delta system lemma to find distinct u and v in Y such that $u \notin C_v$ and $v \notin C_u$. It follows then that

$$\{u, v\} \in (A_u \setminus B_u) \cap (A_v \setminus B_v)$$

contradicting their disjointedness.
□

# 6 Similar Sequences

Two sequences of real numbers $< a_n : n \in \omega >$ and $< b_n : n \in \omega >$ are similar iff there are real numbers $q \neq 0$ and r such that for all $n \in \omega$, $b_n = qa_n + r$. It is shown in H.Miller-P.Xenikakis 1980 [32] that given any convergent sequence $< c_n : n \in \omega >$ and set A which has the property of Baire and is not meager, there exists a similar sequence entirely contained in A. The forcing proof of this fact would be as follows. Suppose A is comeager in the interval (a,b). Choose a rational number r so that for some $\epsilon > 0$ and for every $n \in \omega$, $a + \epsilon < rc_n < b - \epsilon$. Let $x \in (-\epsilon, \epsilon)$ be a Cohen real. Then for every $n \in \omega$, $rc_n + x$ is a Cohen real and since each $rc_n + x \in (a, b)$ we have that for every $n \in \omega$, $rc_n + x \in A$.

**Theorem 6.18** *For every sequence of distinct reals $< c_n : n \in \omega >$ there exists a set $X \subset R$ such that neither $X$ nor $R \setminus X$ contain a sequence similar to $< c_n : n \in \omega >$.*

This will be proved using Lowenheim-Skolem arguments. Both Komjáth 1984 [20] and E.Miller 1937 [31] could be proved this way also. In fact for most sequences the result would follow from E.Miller's Theorem.

First note the following lemma.

**Lemma 6.19** *Suppose M is a standard model of $ZFC^*$, $< c_n : n \in \omega > \in M$, and $< b_n : n \in \omega >$ is similar to $< c_n : n \in \omega >$. Then if M contains at least two points of $< b_n : n \in \omega >$, then it contains all of them.*

proof: Suppose for every $n \in \omega$, $b_n = qc_n + r$. Then if two of the $b_n$'s are in M we can solve for q and r, so they are in M and so all the $b_n$'s are in M.
□

proof of the Theorem 6.18: We show by induction on the cardinality of $Y \subset R$ that there exists $X \subset Y$ such that every sequence $< b_n : n \in \omega >$ similar to $< c_n : n \in \omega >$ which meets Y in an infinite set meets both $X$ and $Y \setminus X$ in an infinite set ( in this case we say X splits Y ). If Y is countable, then let M be a countable standard model of a large enough finite fragment of ZFC which contains Y and the sequence $< c_n : n \in \omega >$. Then by the lemma we need only to split countably many infinite sets. But this is easy to do, in fact, it is the classical result of Bernstein 1908 [2] for which property B is named. If Y has cardinality $\kappa$ then find a chain $M_\alpha$ of standard models of $ZFC^*$ such that

1. Y,$< c_n : n \in \omega > \in M_0$
2. cardinality of each $M_\alpha$ is less than $\kappa$

3. the $M_\alpha$'s form a continuous chain, i. e. for $\alpha < \beta$ we have $M_\alpha \subset M_\beta$ and for limit ordinals $\lambda < \kappa$, $M_\lambda = \cup_{\alpha<\lambda} M_\alpha$

4. $Y \subset \cup_{\alpha<\kappa} M_\alpha$

Now let

$$Y_\alpha = Y \cap (M_{\alpha+1} \setminus M_\alpha)$$

By induction we can find $X \subset Y$ which splits each $Y_\alpha$. But such an X must split Y. If $< b_n : n \in \omega >$ is similar to $< c_n : n \in \omega >$ and meets Y in an infinite set, then it meets some $Y_\alpha$ in an infinite set. This follows from the lemma and the continuity of the $M_\alpha$'s since no new sequence can appear at limit stages.

□

Next we improve on a theorem of H.Miller 1979 [33].

**Theorem 6.20** *Suppose the E is a three element set of reals. Then there exists a set of reals X which has full outer measure and is of the second category everywhere but contains no three element subset similar to E.*

proof: Let $\{G_\alpha \mid \alpha < c\}$ be the set of all uncountable Borel sets. Note that each $G_\alpha$ has cardinality c. It is enough to find X which intersects each $G_\alpha$ and contains no three element subset similar to E. Inductively choose $y_\alpha$ for $\alpha < c$ as follows. At stage $\alpha$ suppose we have $\{y_\beta \mid \beta < \alpha\}$. Let $F_\alpha$ be the smallest subfield of R such that

$$\{y_\beta \mid \beta < \alpha\} \cup E \subset F_\alpha$$

Let $y_\alpha$ be any point in $G_\alpha \setminus F_\alpha$. Note that this is possible since $|F_\alpha| = \alpha + \omega < c$. We claim that $X = \{y_\beta \mid \beta < c\}$ contains no three element subset similar to E. For suppose $\{y_\alpha, y_\beta, y_\gamma\}$ was similar to E where $\alpha < \beta < \gamma$. Then for some reals $a \neq 0$ and $b$ we would have

$$y_\alpha = ax_1 + b$$

$$y_\beta = ax_2 + b$$

$$y_\gamma = ax_3 + b$$

where $E = \{x_1, x_2, x_3\}$. Then

$$\frac{y_\gamma - y_\beta}{y_\beta - y_\alpha} = \frac{x_3 - x_2}{x_2 - x_1}$$

and hence

$$y_\gamma = y_\beta + (y_\beta - y_\alpha)(\frac{x_3 - x_2}{x_2 - x_1}) \in F_\gamma$$

contradicting $y_\gamma \notin F_\gamma$.

□

Erdös conjectured that for every convergent sequence there is a set of reals of positive measure which contains no subset similar to the sequence. This still seems to be open. Falconer 1984 [8] has proved this if the sequence does not converge too rapidly. Komjáth 1983 [19] has proved this if we consider only translates of the sequence. H.Miller and P.Xenikakis 1980 [32] have proved that the set of reals cannot have full measure in any interval. It is also easy to see that for every finite set of reals E and positive measure set X, X contains a set similar to E. ( see [33]). It is impossible to partition the reals into two ( or even finitely many ) sets neither of which contains a set similar to a given finite set E. This is the one dimensional case of Gallai's Theorem [13] p.38.

# 7 Hitting every line twice

It is a well-known result of Mazurkiewicz 1914 [28] that there exists a subset of the plane which hits every line in exactly two points. It is not known whether a Borel set can have this property. D.G. Larman 1968 [23] has shown that an $F_\sigma$ set cannot have this property.

Mauldin remarks that if a $\Sigma^1_1$ set S has this property, then it is a Borel set. To see this just note that

$$(x, y) \notin S \Longleftrightarrow \exists u, v\ u \neq v, v \neq y, u \neq v, (x, u) \in S, (x, v) \in S$$

and so the complement of S is $\Sigma^1_1$ and so S is Borel.

**Theorem 7.21** *If V=L then there is a $\Pi^1_1$ subset of the plane that meets every line in exactly two points.*

The result will follow easily from the following lemma.

**Lemma 7.22** *Suppose $z \in 2^\omega$ is arbitrary, $l$ is a line in the plane, and $X$ is a countable subset of the plane which does not contain three collinear points* *and contains at most one point of l. Then there exists a point P on l such that $z \preceq_T P$ and $X \cup \{P\}$ does not contain three collinear points. Furthermore the point P can be found recursively in the given data.*

proof: Note that the noncollinearity condition only rules out countable many points on l. It is easy to have either the x-coordinate encode z and then choose the y-coordinate to put P on l or vice-versa if l is a vertical line.
□

Since V=L implies there is a $\Delta^1_2$ well ordering of the reals, many transfinite constructions of a subset of the reals ( which are sufficiently effective ) will in the context of V=L produce a $\Delta^1_2$ set. When can such a construction be done to get a $\Pi^1_1$ set?

For example, it is easy to show that if V=L, then there exists a $\Delta^1_2$ Luzin set X contained in the reals ( i.e. X is uncountable and for every meager Borel set B, $X \cap B$ is countable ). The usual construction is to just choose the $\alpha^{th}$ element of X so as to avoid the first $\alpha$ many meager Borel sets. However a Luzin set cannot have the property of Baire and so cannot be $\Sigma^1_1$ or $\Pi^1_1$. The reason is that Cohen reals cannot encode information. For example, it is not hard to show that if x is a Cohen real and y is a ground model real recursive in x, then y is recursive.

The general principle is that if a transfinite construction can be done so that at each stage an arbitrary real can be encoded into the real constructed at that stage then the set being constructed will be $\Pi^1_1$. The reason is basically that then each element of the set can encode the entire construction up to that point at which it itself is constructed. This encoding argument will be used in all the $\Pi^1_1$ constructions in this paper. See also Erdös-Mauldin-Kunen 1981 [7] and van Engelen-Miller-Steel 1987 [5].

proof of Theorem 7.21:

The usual transfinite induction would be to list all lines $\{l_\alpha \mid \alpha < c\}$ and inductively pick points in the plane so that at each stage $\alpha$ we would have picked at most $|\alpha|$ points, no three of which are collinear, but $l_\alpha$ containing two of the points. The construction of a $\Delta^1_2$ set in L which meets every line in exactly two places is the same except the ordering on lines is the constructible ordering and at each stage we choose the first constructed points that will do. To see that the set $X \subset R^2$ is $\Delta^1_2$ note that

$$\begin{aligned} x \in X \quad & \text{iff} \quad \exists L_\alpha \models x \in X \\ & \text{iff} \quad \forall L_\alpha\, x \in L_\alpha \rightarrow L_\alpha \models x \in X \end{aligned}$$

The statement $L_\alpha \models x \in X$ refers to the definition of X relativized to $L_\alpha$. The statement "$\exists L_\alpha$" can be replaced by "$\exists$ a well founded relation on $\omega$ which models V=L". Since well-foundedness is a $\Pi^1_1$ relation and $\models$ is a $\Delta^1_1$ relation we see that X is $\Delta^1_2$.

Define $L_\alpha$ to be point definable iff the Skolem-hull of $(L_\alpha, \in)$ under the usual definable Skolem functions of V=L is isomorphic to $(L_\alpha, \in)$. Note that the Skolem-hull is the same as the set of definable elements. It is well known that there are unboundedly many $\alpha < \omega_1^L$ such that $L_\alpha$ is point definable (see for example [5]). Also since L has built in Skolem functions if $L_\alpha$ is point definable then there exists $E \subset \omega \times \omega$ recursive in $Th(L_\alpha, \in)$ (the first-order theory of $(L_\alpha, \in)$) such that $(L_\alpha, \in)$ is isomorphic to $(\omega, E)$. Since the first-order of $(L_\alpha, \in)$ appears in say $L_{\alpha+2}$ we have that the E above appears in $L_{\alpha+3}$

Let

$$\{L_{\beta_\alpha} \mid \alpha < \omega_1\}$$

be the set of all point definable $L_\alpha$'s listed in order. Inductively construct points in the plane $x_{\beta_\alpha}, y_{\beta_\alpha}$ for $\alpha < \omega_1$ as follows. At stage $\alpha$ choose $x_{\beta_\alpha}, y_{\beta_\alpha}$ so that $< x_{\beta_\alpha}, y_{\beta_\alpha} >$ is the least constructed pair of points in the plane such that:

1. $< x_{\beta_\alpha}, y_{\beta_\alpha} > \in L_{\beta_\alpha + \omega}$.

2. No three points of $\{x_{\beta_\gamma}, y_{\beta_\gamma} \mid \gamma \leq \alpha\}$ are collinear.

3. If l is the first constructed line which fails to contain two points from $\{x_{\beta_\gamma}, y_{\beta_\gamma} \mid \gamma < \alpha\}$ then $\{x_{\beta_\gamma}, y_{\beta_\gamma} \mid \gamma \leq \alpha\}$ does contain two points of l.

4. There is a relation $E \subset \omega^2$ such that $E \preceq_T x_{\beta_\gamma}$ and $E \preceq_T y_{\beta_\gamma}$ and $(\omega, E)$ is isomorphic to $L_{\beta_\alpha}$.

Such a pair exists by Lemma 7.22 and note that there exists $E \subset \omega \times \omega$ which is $\Delta^1_1$ in $x_{\beta_\alpha}$ such that $(L_{\beta_\alpha + \omega}, \in)$ is isomorphic to $(\omega, E)$ (similarly for $y_{\beta_\alpha}$).

Now let $X = \{x_{\beta_\gamma}, y_{\beta_\gamma} \mid \gamma < \omega_1\}$. To see that X is $\Pi^1_1$ note that

$$z \in X \text{ iff } \exists L_\alpha \ \Delta^1_1 \text{ in z } L_\alpha \models z \in X$$

□

Is there a Borel subset of the plane which meets every circle in exactly three points? The same proof as above shows that if V=L, then there is such a $\Pi^1_1$ set.

Call a set in the plane a two point set iff every line meets it in exactly two points. Mauldin (unpublished) has shown that any two point set in the plane must be totally disconnected. He asks whether every two point set must be zero-dimensional.

Kunen and I have shown that any $\Sigma^1_1$ subset of the plane which cannot be covered by countably many lines must contain a perfect set P with the property that that no three points of P are collinear. Dougherty, Kechris, and Jackson have shown that the axiom of determinacy and V=L[R] implies that every subset of the plane which cannot be covered by countably many lines must contain a perfect set P with the property that that no three points of P are collinear.

# 8 Maximal Almost Disjoint Families

A maximal almost disjoint (mad) family is a set $F \subset [\omega]^\omega$ such that for every two distinct $A, B \in F$, $A \cap B$ is finite and for every $X \in [\omega]^\omega$ there is an $A \in F$ such that $A \cap X$ is infinite. In Mathias 1977 [27] it is shown that no mad family can be $\Sigma^1_1$. The following theorem was proved jointly with K. Kunen.

**Theorem 8.23** *If V=L then there is a mad family which is* $\Pi^1_1$.

We will need the following lemma.

**Lemma 8.24** *Suppose* $P \subset [\omega]^\omega$ *is a countable family of almost disjoint sets which include an infinite recursive partition of* $\omega$*, and let* $z \in 2^\omega$ *be arbitrary and let* $u$ *be any element of* $[\omega]^\omega$ *which is almost disjoint from every element of* $P$*. Then there exists* $x \in [\omega]^\omega$ *such that* $z \preceq_T x$*,* $x$ *is almost disjoint from every element of* $P$*, and* $u \subset x$*. Furthermore* $x$ *can be found recursively in the given data.*

proof: Let $\{A_n \mid n \in \omega\}$ be the infinite recursive partition of $\omega$ which is contained in P and let

$$\{B_n \mid n \in \omega\} = P \setminus \{A_n \mid n \in \omega\}$$

In order to make $z \preceq_T x$ we will choose x so that for every $n \in \omega$,$z(n) = 0$ iff $x \cap A_n$ has even cardinality. The set x will be $u \cup \bigcup\{F_n \mid n \in \omega\}$ where each $F_n$ is a finite subset of $A_n$, where $F_n$ is disjoint from $B_m$ for each $m \leq n$ and the appropriate cardinal so as to encode z(n).
□

Now this encoding lemma allows us to prove the theorem just as in section 7.

R. Pol has pointed out some connections between mad families and compact sets in the spaces $B_1(E)$ of the first Baire class functions on E, endowed with the pointwise topology. What follows are some excerpts from a letter he wrote to me.

Given an almost disjoint family $F \subset [\omega]^\omega$, let $E = F \cup [\omega]^{<\omega}$, let $f_A : E \mapsto \{0,1\}$ be the characteristic function of the singleton $A \in F$, let $p_n : E \mapsto \{0,1\}$ for $n \in \omega$ be defined by $p_n(S) = 1$ iff $n \in S$, and let $f_\infty \equiv 0$ on E. The space

$$\Phi = \{p_n \mid n \in \omega\} \cup \{f_A \mid A \in F\} \cup \{f_\infty\}$$

considered as a subspace of $B_1(E)$ is compact; in fact $\Phi$ is homeomorphic to the compact space associated in a standard way with the almost-disjoint family F, see Gillman-Jerrison, (1960) [12] , 5I ($f_\infty$ is the point at infinity). The space $\Phi$ is not Fréchet iff F is mad. Now, if F is analytic, so is E, and $\Phi$, being a compact subspace of $B_1(E)$, is Fréchet, by Rosenthal's theorem (see S. Negrepontis (1984) [37] section 1), therefore F is not mad.

Theorem 8.23 provides (under V=L) a compact subspace $\Phi$ of $B_1(E)$ with E being a $\Pi^1_1$ set, which is not Fréchet.

# 9 Hamel Basis

Sierpiński 1920 [40] and F. Burton Jones 1942 ([18]) showed that it is impossible to have a Hamel basis for the real line R considered as a vector space over the rationals Q which is Borel, or even in fact $\Sigma^1_1$. This result is also proved in Erdös 1950 [6]. These proofs use the measurability of $\Sigma^1_1$ sets and Steinhaus's Theorem that the difference set of a set of positive measure contains an interval. Here we give a proof using the property of Baire instead of measure.

**Theorem 9.25** *There does not exist a $\Sigma^1_1$ Hamel basis for $R$ over $Q$.*

proof: Suppose H was such a Hamel basis. We can assume without loss of generality that $1 \in H$, since there must be for some n,

$$\text{nonzero } r_1, r_2, \ldots, r_n \in Q \text{ and } x_1, x_2, \ldots, x_n \in H$$

such that

$$1 = r_1x_1 + r_2x_2 + \cdots + r_nx_n$$

so then $(H \setminus \{x_1\}) \cup \{1\}$ is a $\Sigma^1_1$ Hamel basis. Let P be the partial order for forcing a Cohen real in R (i.e. P is the set of open intervals with rational end points ordered by inclusion). Since the statement "H is a Hamel basis" is $\Pi^1_2$, it is absolute. Hence for any $x \in R$ P-generic over V there exist $n \in \omega$ and $r_0, r_1, r_2, \ldots, r_n \in Q$ such that

$$p \Vdash \exists x_1, x_2, \ldots, x_n \in H \; x = r_0 + r_1x_1 + r_2x_2 + \cdots + r_nx_n$$

where $r_0$ the coefficient of 1 may be zero. Let $\epsilon$ be a small positive rational number such that $r < x + \epsilon < s$ where $p = (r, s)$. Then since $x + \epsilon$ is a Cohen real too

$$\exists y_1, y_2, \ldots, y_n \in H \; x + \epsilon = r_0 + r_1y_1 + r_2y_2 + \cdots + r_ny_n$$

But then

$$\epsilon = (r_1y_1 + r_2y_2 + \cdots + r_ny_n) - (r_1x_1 + r_2x_2 + \cdots + r_nx_n)$$

which is a contradiction since none of the $y_i$'s or $x_i$'s are rational but all are from H.
□

This proof gives a little bit more than the measure theory proof since Shelah 1984 [39] has shown that Con(ZF) implies Con(ZF+DC+BP), where BP is the statement that every set has the property of Baire. Hence BP implies there is no Hamel basis. Shelah's result also shows that it is relatively consistent with ZFC that no Hamel basis is definable. The analogous statement for Lebesgue measure requires the existence of an inaccessible cardinal.

Sierpiński 1935 [41] gives a proof that a Hamel basis with the property of Baire must be meager. But this does not give the above result.

**Theorem 9.26** *If V=L then there is a $\Pi^1_1$ Hamel basis for R over Q.*

Let Q[X] for $X \subset R$ be the smallest vector space over Q containing X.

**Lemma 9.27** *Suppose $X \subset R$ is countable, $z \in R \setminus Q[X]$, and $w \in 2^\omega$. Then there exists $y_1, y_2 \in R$ such that $w \preceq_T y_1$, $w \preceq_T y_2$, $y_1 \notin Q[X]$, $y_2 \notin Q[X \cup \{y_1\}]$, and $z \in Q[X \cup \{y_1, y_2\}]$. Furthermore $y_1, y_2$ can be found recursively in the given data.*

proof: First without loss of generality we may assume that w is not recursive in any finite join of elements from $X \cup \{z\}$, since it can always be replaced by something more complicated. Let

$$v = .w(0)w(0)w(1)w(1)w(2)w(2)\cdots \in R$$

and note that $0 < v < \frac{1}{9}$. Find $r \in Q$ such that $\frac{1}{9} < rz < 1$ and let $u = rz - v$, so $0 < u < 1$, and write

$$u = .u(0)u(1)u(2)\cdots$$

Define

$$y_1 = .w(0)u(1)w(1)u(3)w(2)u(5)\cdots$$
$$y_2 = .u(0)w(0)u(2)w(1)u(4)w(2)\cdots$$

and note that $y_1 + y_2 = rz$, so $z \in Q[X \cup \{y_1, y_2\}]$. It is clear that $w \preceq_T y_1$ and $w \preceq_T y_2$. We also have that $y_1 \notin Q[X]$ since otherwise w is recursive in some finite join of elements of X. The last thing to check is that $y_2 \notin Q[X \cup \{y_1\}]$. This is true since otherwise $z \in Q[X \cup \{y_1\}]$ but since $z \notin Q[X]$ then $y_1 \in Q[X \cup \{z\}]$ but this would imply that w is recursive in a finite join from $X \cup \{z\}$.
□

The lemma allows us to choose inductively a Hamel basis so that at each stage the reals we choose recursively code up the whole construction, and hence we get a $\Pi^1_1$ set just as in section 7.

R. Pol has proven the following generalization of Theorem 9.25.

> Let X be a complete separable linear metric space over a field K which is an analytic set and let E be an analytic linear subspace of X. If the codimension of E in X is infinite, then it is $2^{\aleph_0}$ (in fact, there is a Cantor set $C \subset X$ linearly independent over E).

This gives Theorem 9.25 for $X = R$ and $K = Q$. Pol asks if V=L then does there always exist a linear $\Pi^1_1$ subspace E of a Banach separable space X with codimension $\aleph_0$?

# 10 Maximal Independent Families

A set $I \subset [\omega]^\omega$ is called an independent family iff for every $F \in [I]^{<\omega}$ and disjoint $G \in [I]^{<\omega}$ the set

$$\left(\bigcap_{A\in F} A\right) \cap \left(\bigcap_{B\in G} \omega \setminus B\right)$$

is infinite.

**Theorem 10.28** *There does not exist a $\Sigma^1_1$ maximal independent family.*

We will use the following lemma in the proof.

**Lemma 10.29** *Suppose that $M$ is a countable standard model, then there exists a perfect set $P$ of Cohen reals over $M$ such that every pair from $P$ is almost disjoint.*

proof: Inductively construct an increasing sequence

$$< n_k \mid k \in \omega > \in \omega^\omega$$

and a nested sequence of trees

$$< T_k \subset 2^{\leq n_k} \mid k \in \omega >$$

with the following properties:

1. Every branch of $T_k$ has length $n_k$.
2. All but at most one $s \in T_{k+1} \cap 2^{n_{k+1}}$ is identically zero on $[n_k, n_{k+1})$.

3. For every $k \in \omega$ and $s \in T_k$ there exists $l > k$ such that $T_l$ contains two incomparable extensions of s.

4. For every the dense open subset $D \subset 2^{<\omega}$ in M there exists $k \in \omega$ such that $T_k \cap 2^{n_k} \subset D$.

The details of this construction will be left to the reader. The perfect set P is just the set of infinite branches through the tree $\cup\{T_k \mid k \in \omega\}$.
□
proof of Theorem 10.28: Define

$$\sigma(F,G) = \left(\bigcap_{A\in F} A\right) \cap \left(\bigcap_{B\in G} \omega \setminus B\right)$$

$$\begin{aligned} H &= \{X \in [\omega]^\omega \mid \exists F \in [I]^{<\omega}\ \exists G \in [I \setminus F]^{<\omega}\ \sigma(F,G) \subset^* X\} \\ K &= \{X \in [\omega]^\omega \mid \exists F \in [I]^{<\omega}\ \exists G \in [I \setminus F]^{<\omega}\ \sigma(F,G) \cap X =^* \emptyset\} \end{aligned}$$

By the maximality of I, $[\omega]^\omega = H \cup K$ and since I is $\Sigma^1_1$ so are both H and K. Hence they have the property of Baire and so one must be nonmeager, say H. It follows easily from the lemma that there exists a perfect set $P \subset H$ of almost disjoint sets. For each $x \in P$ let $F_x$ and $G_x$ witness that x is in H. By applying the delta systems lemma we can find distinct x and y in P such that

$$(F_x \cup F_y) \cap (G_x \cup G_y) = \emptyset$$

But then

$$\left(\bigcap_{A\in F_x\cup F_y} A\right) \cap \left(\bigcap_{B\in G_x\cup G_y} \omega \setminus B\right) \subset^* X \cap Y =^* \emptyset$$

which contradicts the independence of I. A similar proof can be given if K is nonmeager.
□

A family of functions $F \in \omega^\omega$ is independent iff for every $n \in \omega$, distinct $f_0, f_1, \ldots, f_{n-1} \subset F$, and $s \in \omega^n$ the set

$$\{m \in \omega \mid \forall i < n\ f_i(m) = s(i)\}$$

is infinite.

**Theorem 10.30** *There does not exist a maximal independent family $F$ of functions in $\omega^\omega$ which is $\Sigma^1_1$.*

proof: The proof is similar to the last one so we only sketch it. For some $n \in \omega$, $s \in \omega^n$, and $k \in \omega$ the set H of all $g \in \omega^\omega$ such that

$$\exists f_0, f_1, \ldots, f_{n-1} \subset F \ \ |\{m \in \omega \ \mid \ g(m) = k, \forall i < n \ f_i(m) = s(i)\}| < \omega$$

is a nonmeager $\Sigma^1_1$ set. By an argument similar to Lemma 10.29, we can find a perfect set $P \subset H$ with the property that for every two distinct g and h in P for all but finitely many $m \in \omega \ \ g(m) = k$ or $h(m) = k$. Letting $< f_0^g, f_1^g, \ldots, f_{n-1}^g >$ witness that $g \in H$, apply the delta systems lemma to get $g, h \in H$ such that for all $i \in \omega$ either $f_i^g = f_i^h$ or else both $f_i^g$ is distinct from all $f_j^h$'s and also $f_i^h$ is distinct from all $f_j^g$'s. But then the set

$$\{m \in \omega \ \mid \ \forall i < n \ f_i^g(m) = s(i)\} \cap \{m \in \omega \ \mid \ \forall i < n \ f_i^h(m) = s(i)\}$$

is finite, contradicting the independence of F.
□

**Theorem 10.31** *If V=L then there exists a $\Pi^1_1$ maximal independent family of functions $F \subset \omega^\omega$.*

We need the following coding lemma.

**Lemma 10.32** *Suppose $F \cup \{g\} \subset \omega^\omega$ is a countable independent family and $z \in 2^\omega$ is arbitrary. Then there exists $f \in \omega^\omega$ such that $z \preceq_T f$, $F \cup \{f\}$ is independent, but $F \cup \{f, g\}$ is not.*

proof: We will construct an increasing sequence $< n_k \ \mid \ k \in \omega > \in \omega^\omega$ and then define $f \in \omega^\omega$ by

$$f(m) = \begin{cases} g(m) & \text{if } m \neq n_k \text{ all k} \\ p(z(k), n_{k+1}) & \text{if } m = n_k \end{cases}$$

where $p(\ ,\ )$ is a recursive pairing function. Note that this makes $z \preceq_T f$. We will also pick the $n_k$'s so that for all k , $g(n_k) \neq 0$ and this guarantees that f and g cannot belong to the same independent family. The only thing left is to pick the $n_k$'s thin enough so as to ensure that $F \cup \{f\}$ is an independent

family. But since F is countable and $F \cup \{g\}$ is independent, there is a countable family $C \subset [\omega]^\omega$ such that it is enough to make sure that for all $X \in C$ we have $X \setminus \{n_k \mid n \in \omega\}$ is infinite.
□

This coding lemma allows us to get a $\Pi^1_1$ maximal independent family from V=L the same as in section 7.

**Theorem 10.33** *If V=L, then there exists a $\Pi^1_1$ maximal independent family in $[\omega]^\omega$.*

The proof will follow from the following encoding lemma.

**Lemma 10.34** *Suppose $I \subset [\omega]^\omega$ is countable containing an infinite recursive subset, $z \in [\omega]^\omega$ is such that $I \cup \{z\}$ is an independent family, and $w \in 2^\omega$ is arbitrary. Then there exists $u \in [\omega]^\omega$ such that $w \preceq_T u$ and $I \cup \{u\}$ is independent but $I \cup \{u, z\}$ is not. Furthermore u can be found recursively in the given data.*

proof: Suppose $< X_n \mid n \in \omega >$ is the infinite recursive subset of I. Define

$$P_n = X_0 \cap X_1 \cap \cdots \cap X_{n-1} \cap (\omega \setminus X_n)$$

$$H = \{ \left( \bigcap_{A \in F} A \right) \cap \left( \bigcap_{B \in G} \omega \setminus B \right) \mid F \in [I]^{<\omega} \ G \in [I \setminus F]^{<\omega} \}$$

Note that for every $X \in H$ there exist $n \in \omega$ such that $X \cap Z \cap P_n$ is infinite. Construct $u \subset \omega$ such that

1. $u \in [z]^\omega$ , so for every $x \in H$, $x \cap (\omega \setminus u)$ is infinite.
2. for every $x \in H$, $x \cap u$ is infinite.
3. for every $n \in \omega$ ( $w(n) = 0$ iff $\min(P_{2n} \cap u) < \min(P_{2n+1} \cap u)$).

The last condition ensures $w \preceq_T u$ and the first two ensure that $I \cup \{u\}$ is an independent family while $I \cup \{u, z\}$ is not.
□

It is consistent with ZFC that the continuum is arbitrarily large but there is a maximal independent family of size $\omega_1$ ( see Kunen [22] chapter 8 ex. A13 p.289 ). Note that if every $\Sigma^1_2$ set of reals has the property of Baire, then the arguments above shows that there is no $\Pi^1_1$ maximal independent family of either type. Similarly, if every set of reals has the Baire property, then there are no maximal independent families.

In ZF (no choice) does the existence of a maximal independent family in $[\omega]^\omega$ imply the existence of a maximal independent family in $\omega^\omega$? What about the converse?

# 11 Notation

For general background see Kunen [22], Jech [17], and Moschovakis [34].

1. R denotes the real line.
2. $[X]^{<\omega}$ is the set of all finite subsets of X.
3. $[X]^\omega$ is the set of countably infinite subsets of X.
4. $f \upharpoonright A$ is the restriction of the function f to the domain A.
5. $X \preceq_T Y$ means that X is Turing reducible to Y.
6. $2^\omega$, the Cantor space is the set of functions from $\omega$ into $2 = \{0, 1\}$. This is given the product topology where 2 is given the discrete topology.
7. $2^{<\omega}$ is the partial order of functions whose domain is some $n \in \omega$ and range is 2. The order is just inclusion.
8. FIN(X) is the partial order of functions whose domain is some finite subset of X and whose range is 2.
9. $ZFC^*$ stands for a sufficiently large finite fragment of ZFC Zermelo-Fraenkel set including the axiom of choice. By sufficiently large we mean whatever it takes to get the argument to work.
10. Standard models of $ZFC^*$ are transitive sets which model $ZFC^*$ when $\in$ interprets itself. For any $ZFC^*$ there exist a countable transitive model of it. This follows from the reflection theorem and the Mostowski collapse ( see Kunen [22] ). Also generic extensions of countable standard model of $ZFC^*$ are also models of $ZFC^*$ , although of a smaller fragment.

11. $\Sigma^1_1$ sets are the projection of Borel sets , i.e. analytic sets . $\Pi^1_1$ sets are the complements of $\Sigma^1_1$ sets, i.e. coanalytic sets. For more on descriptive set theory see Moschovakis [34]. We use here the absoluteness of $\Sigma^1_1$ predicates in models of $ZFC^*$. All of the positive results, e.g. if V=L then there exists a $\Pi^1_1$ maximal independent family, actually give a light-face $\Pi^1_1$ set. Similarly all of the negative results, e.g. there is no $\Sigma^1_1$ maximal independent family, actually show there is no bold-face $\Sigma^1_1$ set.

12. An ultrafilter U on $\omega$ is Ramsey iff for every partition $f : [\omega]^2 \mapsto 2$ there exists $X \in U$ such that $f \upharpoonright [X]^2$ is constant.

# References


[1] J.Baumgartner and R.Laver, Iterated perfect set forcing, Annals Math. Logic, 17(1979), 271-288.

[2] F.Bernstein, Zur Theorie der trigonometrischen Reihen, Berichte über die Verhandlungen der Königl. Sächsischen Gesellschaft der Wissenschaften zu Leipzig. Math.-phs. Klasse, 60(1908),325-338.

[3] A.Blass, A partition theorem for perfect sets, Proceedings American Mathematical Society, 82(1981), 271-277.

[4] E.Ellentuck, A new proof that analytic sets are Ramsey, J. Symbolic Logic, 39(1974),163-165.

[5] F.van Engelen, A.Miller, and J.Steel, Rigid Borel sets and better quasiorder theory, in **Logic and Combinatorics, Proceedings of a Summer Reasearch Conference held August 4-10, 1985,** edited by S. Simpson, Contemporary Math Series AMS, 65(1987),199-222.

[6] E.Erdös, Some remarks on set theory, Proceedings American Mathematical Society, 1(1950), 127-141.

[7] E.Erdös, D.Mauldin, and K.Kunen, Some additive properties of sets of real numbers, Fundamenta Mathematicae, 93(1981), 187-199.

[8] K.Falconer, On a problem of Erdös on sequences and measurable sets, Proceedings American Mathematical Society, 90(1984), 77-78.

[9] D.H.Fremlin and Jasiński, $G_\delta$-covers and large thin sets of reals, to appear.

[10] F.Galvin, Partition theorem for the real line, Notices AMS, 15(1968), 666; erratum Notices AMS, 16(1969), 1095.

[11] F.Galvin and K.Prikry, Borel sets and Ramsey's theorem, J. Symbolic Logic, (1973)38, 193-198.

[12] L.Gillman and M.Jerison, **Rings of Continuous Functions**, Van Nostrand, (1960).

[13] R.Graham, B.Rothschild, and J.Spencer, **Ramsey Theory** , John Wiley and Sons, (1980).

[14] F.Hausdorff, Summen von $\aleph_1$ Mengen, Fundamenta Mathematicae, 26(1934), 241-255.

[15] L.Harrington and S.Shelah, Counting equivalence classes for co-$\kappa$-Souslin equivalence relations, in **Logic Colloquim 80**, North-Holland, 1982.

[16] L.Harrington, D.Marker, and S.Shelah, Borel Orderings, to appear in Transactions of American Mathematical Society (1988).

[17] T.Jech, **Set Theory** , Academic Press, 1978.

[18] F.B.Jones, Measure and other properties of a Hamel basis, Bulletin American Mathematical Society, 48(1942), 472-481.

[19] P.Komjáth, Large sets not containing images of a given sequence, Canadian Mathematical Bulletin, 26(1983), 41-43.

[20] P.Komjáth, Families close to disjoint ones, Acta. Math. Hungary, 43(1984), 199-207.

[21] P.Komjáth, On the limit superior of analytic sets, Analysis Mathematica, 10(1984), 283-293.

[22] K.Kunen, **Set Theory** , North-Holland, 1980.

[23] D.G.Larman, A problem of incidence, J. London Mathematical Society, 43(1968), 407-409.

[24] R.Laver, Products of infinitely many perfect trees, to appear.

[25] E.Marczewski, Sur une classe de fonctions de M. Sierpiński et la classe correspondant d'ensembles, Fundamenta Mathematicae, 24(1935), 17-34.

[26] D.Martin, The use of set-theoretic hypotheses in the study of measure and topology, in **General Topology and Modern Analysis** , Academic Press, 1981, 417-429.

[27] A.R.D.Mathias, Happy families, Annals of Math Logic, 12(1977), 59-111.

[28] S.Mazurkiewicz, C. R. Sc. et Letres de Varsovie, 7(1914), 322-383.

[29] S.Mazurkiewicz, Sur les suites de fonctions continues, Fundamenta Mathematicae, 18(1932), 114-117.

[30] A.Miller, Special subsets of the real line, in **Handbook of Set Theoretic Topology**, ed. by K.Kunen and J.Vaughan, North-Holland, 1984, 201-234.

[31] E.Miller, On a property of families of sets, Comptes rendus Varsovie, 30(1937), 31-38.

[32] H.Miller and P.Xenikakis Some results connected with a problem of Erdös, Akademija Nauka i Umjetnosti Bosne i Hercegovine Radovi Odjeljenje Prirodnih i Matematičkih Nauka, 19(1980), 71-75.

[33] H.Miller, Some results connected with a problem of Erdös. II, Proceedings American Mathematical Society, 75(1979), 265-268.

[34] Y.Moschovakis, **Descriptive Set Theory** , North-Holland, 1980.

[35] J.Mycielski, Independent sets in topological algebras, Fundamenta Mathematicae, 55(1964), 139-147.

[36] J.Mycielski, Algebraic independence and measure, Fundamenta Mathematicae, 61(1967), 165-169.

[37] S.Negrepontis, Banach spaces and topology, in **Handbook of Set-Theoretic Topology**, North-Holland (1984), 1045-1142.

[38] S.Shelah, On co-$\kappa$ Souslin relations, Israel J. Math, 47(1984), 139-153.

[39] S.Shelah, Can you take Solovay's inaccessible away?, Israel J. Math, 48(1984), 1-47.

[40] W.Sierpiński, Sur la question de la mesurabilit/'e de la base de M. Hamel, Fundamenta Mathematicae, 1(1920), 105-111.

[41] W.Sierpiński, La Base de M. Hamel et al propriété de Baire, Publ. Math. Univ. Belgrade, (1935), 220-224.

[42] W.Sierpiński, Fonctions additives non-complètement additives et fonctions nonmeasurables, Fundamenta Mathematicae, 30(1938), 96-99.

[43] J.Silver, Every analytic set is Ramsey, J. Symbolic Logic, 35(1970), 60-64.

[44] R.Solovay, A model of set-theory in which every set of reals is Lebesgue measurable, Annals of Mathematics, 92(1970), 1-56.

[45] M.Talagrand, Compacts de fonctions mesurables et filtres non mesurables, Studia Mathematica, 67(1980), 13-43.