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DESCRIPTIVE SET THEORY OVER HYPERFINITE SETS

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Abstract. The separation, uniformization, and other properties of the Borel and projective hierarchies over hyperfinite sets are investigated and compared to the corresponding properties in classical descriptive set theory. The techniques used in this investigation also provide some results about countably determined sets and functions, as well as an improvement of an earlier theorem of Kunen and Miller.

§0. Introduction. In this paper the separation, uniformization, and other properties of the Borel and projective hierarchies over hyperfinite sets are investigated and compared to the corresponding properties in classical descriptive set theory. The techniques used in this investigation also provide some results about countably determined sets and functions, as well as an improvement of an earlier theorem of Kunen and Miller [KM].

Let S be an infinite hyperfinite set in an ω_1-saturated nonstandard universe. The Borel and projective hierarchies over S are defined as in the classical case, except that countable intersections of internal subsets of S play the role of the closed sets, and projections are from $S \times T$, for any hyperfinite set T, onto S. Of particular interest is the case in which $S = 2^H$ for some hyperfinite natural number H, for then we may define the standard part map st: $S \to 2^N$ by

$$\mathrm{st}(s) = \langle s(n) \colon n \in N \rangle.$$

A fundamental result which ties together the classical and the hyperfinite settings was given by Kunen and Miller [KM], who showed that all levels in both hierarchies are preserved under the inverse standard part map.

In §1, after reviewing a few basic facts about the Borel and projective hierarchies over a hyperfinite set, we prove a theorem which strengthens the result of Kunen and Miller and gives a way of reducing questions about prewellordering, reduction, and separation to the classical case.

In §2 we consider countably determined sets and functions. This notion, due to Henson [H2], is without a classical analogue. The countably determined sets contain all the sets in the projective (and thus also the Borel) hierarchy, as well as considerably more. Among the results in this section are a finite Ramsey theorem for

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countably determined partitions and a proof that, in the definition of the projective hierarchy, one can use a fixed hyperfinite set U and take only projections from $S \times U$ onto S. Also in this section we show that, for hyperfinite sets S and T, $\#(S)/\#(T)$ is finite if and only if there is a countably determined bijection between them. This complements a recent result of Henson and Ross [HR], who show that $\#(S)/\#(T)$ is in the monad of 1 if and only if there is a Borel bijection from S to T.

In §3 it is shown that, except for the first Borel level, the questions about prewellordering, reduction and separation for the Borel and projective classes all have the same answers as in classical descriptive set theory.

In §4 the uniformization properties of the hierarchy levels are considered and it is shown that, in marked contrast to the results of §3, the answers for the hyperfinite case are quite different than for the classical case. Perhaps the most striking example is that there is a Π^0_2 relation which does not have a countably determined uniformization (and thus does not have a uniformization at any projective level). On the other hand, we have the positive result that every Σ^0_2 relation has a Σ^0_2 uniformization. Other positive results for the hyperfinite case which differ from the classical case are that the preimage of any Σ^0_α, Borel, Σ^1_n, or Π^1_n function is of the same class, and that any total function is in the dual class.

Finally, in §5 we present a few open problems.

We refer to the book of Stroyan and Bayod [SB] for background in nonstandard analysis, and to the article of Martin [M] for background in classical descriptive set theory.

§1. The Borel and projective hierarchies. Assume throughout this paper that S is an infinite hyperfinite set in an ω_1-saturated nonstandard universe. We begin with a review of some basic notions and known results in hyperfinite descriptive set theory.

DEFINITION. The *Borel hierarchy* over S is defined as follows. $\Sigma^0_0(S) = \Pi^0_0(S) = {}^*P(S)$, the set of all internal subsets of S. For each ordinal $0 < \alpha < \omega_1$, $\Sigma^0_\alpha(S)$ is the set of countable unions of sets in $\bigcup\{\Pi^0_\beta(S)\colon \beta < \alpha\}$, and $\Pi^0_\alpha(S)$ is the set of complements of sets in $\Sigma^0_\alpha(S)$. Moreover, $\mathrm{Borel}(S) = \bigcup\{\Sigma^0_\alpha(S)\colon \alpha < \omega_1\}$, and $\Delta^0_\alpha(S) = \Sigma^0_\alpha(S) \cap \Pi^0_\alpha(S)$.

The *projective hierarchy* over S is defined as follows. The set $\Sigma^1_1(S)$ of analytic sets over S is defined as the set of projections of sets which are Borel sets over $S \times T$ for some hyperfinite T. $\Sigma^1_{n+1}(S)$ is the set of sets B such that for some hyperfinite T, B is the projection on S of a set $A \in \Pi^1_n(S \times T)$. $\Pi^1_n(S)$ is the set of complements of sets in $\Sigma^1_n(S)$, and $\Delta^1_n(S) = \Sigma^1_n(S) \cap \Pi^1_n(S)$.

We shall see in §2 that a fixed hyperfinite set can be taken for T.

The following proposition lists two simple consequences of ω_1-saturation.

PROPOSITION 1.1. (a) *If each A_n is internal, A_n is decreasing, and F is an internal function, then $F(\bigcap_n A_n) = \bigcap_n F(A_n)$.*

(b) $\Delta^0_1(S) = \Sigma^0_0(S)$.

DEFINITION. Let N^∞ be the set of all finite sequences of natural numbers. The *Suslin operation* on a family of sets A_s, $s \in N^\infty$, is defined by:

$$\mathrm{Suslin}(A_s\colon s \in N^\infty) = \bigcup_{f \in N^N} \bigcap_{n \in N} A_{f|n}.$$

PROPOSITION 1.2. *The following are equivalent*:

(a) *A is analytic over S.*

(b) *A is obtained from internal subsets of S using the Suslin operation.*

(c) *For some hyperfinite T, A is the image of an analytic set over T by an internal function from S into T.*

(d) *For any hyperfinite T, A is the projection of a Π_2^0 set over $S \times T$.*

Proposition 1.2 is essentially in the papers [H1] and [H2] of Henson. The proof is similar to the proof of the corresponding results in classical descriptive set theory.

The next proposition lists some consequences of known classical results which concern hyperfinite sets.

PROPOSITION 1.3. (a) *For each α, $\Sigma_\alpha^0(S)$ and $\Pi_\alpha^0(S)$ are closed under finite unions and intersections and preimages by internal functions.*

(b) *For each n, $\Sigma_n^1(S)$ and $\Pi_n^1(S)$ are closed under finite unions and intersections and preimages by internal functions.*

(c) *Analytic sets over S are Loeb measurable with respect to any internal weight function.*

PROPOSITION 1.4. (a) *$\Pi_n^0(S)$ is the class of all sets definable with number quantifiers of type Π_n^0 over some $S \times T^n$.*

(b) *For $n \geq 1$, $\Pi_n^1(S)$ is the class of all sets definable with quantifiers of type Π_n^1 over some $S \times T^n$, absorbing number quantifiers.*

(c) *$\Pi_n^1(S)$ is closed under countable unions and intersections.*

The above proposition also holds for Σ.

PROPOSITION 1.5. $\mathrm{Borel}(S) = \Delta_1^1(S)$.

(See Henson [H1], who observed that this is a special case of a more general classical result concerning pavings.)

If T is of the form 2^H for some hyperfinite H, then the standard part map st: $T \to 2^N$ is defined in the natural way.

PROPOSITION 1.6 (HENSON [H1]). *Let $T = 2^H$.*

(a) *If each $A_n \subseteq T$ is internal, then $\mathrm{st}(\bigcap_n A_n) = \bigcap_n \mathrm{st}(A_n)$.*

(b) *If $A \in \Pi_1^0(T)$ then $\mathrm{st}(A)$ is closed (and hence compact) in 2^N.*

(c) *If $A \in \Sigma_2^0(T)$ then $\mathrm{st}(A)$ is Σ_2^0 over 2^N.*

(d) *If $A \subseteq T$ is analytic, then $\mathrm{st}(A)$ is analytic over 2^N.*

(e) *If B is analytic over 2^N, then there is a set $A \in \Pi_2^0(T)$ such that $B = \mathrm{st}(A)$.*

PROPOSITION 1.7. *Let F be an internal function from S onto T. Then for each set $B \subseteq T$, $F^{-1}(B)$ is at the same level as B in both the Borel and projective hierarchies. That is,*

(a) *For each countable α, $B \in \Sigma_\alpha^0(T)$ if and only if $F^{-1}(B) \in \Sigma_\alpha^0(S)$, and similarly for Π.*

(b) *For each n, $B \in \Sigma_n^1(T)$ iff $F^{-1}(B) \in \Sigma_n^1(S)$, and similarly for Π and Δ.*

PROOF. By transfer there is an internal function $G: T \to S$ such that for all $t \in T$, $F(G(t)) = t$. It is easy to check in turn that the sets B, $G(B) = F^{-1}(B) \cap G(B)$, and $F^{-1}(B)$ are at the same level in either hierarchy. □

The analogue of Proposition 1.7 for the standard part function was proved in Kunen and Miller [KM]. We give a simpler proof of a stronger result here. The arguments are patterned after arguments used for different purposes in

Saint-Raymond [S] and Jayne and Rogers [JR]. For the remainder of this section suppose that $T = 2^H$ for some infinite hyperinteger H.

LEMMA 1.8. *Suppose Q and A are Π_1^0 over T and $k \in N$. Then there is a sequence Q_n, $n \in N$, of Π_1^0 sets over T that satisfy the following conditions*:

(a) *The Q_n are disjoint subsets of Q.*

(b) *The sets* $\operatorname{st}(Q_n)$, *$n \in N$, form a disjoint partition of* $\operatorname{st}(Q)$.

(c) *For each n, Q_n is either included in A or disjoint from A.*

(d) *For each $n \in N$ and $u, v \in Q_n$, $u | k = v | k$.*

PROOF. Let $R = Q \cap A$. By 1.6, $\operatorname{st}(Q)$ and $\operatorname{st}(R)$ are compact in 2^N. Let U and V be the sets of all finite sequences of elements of 2 which can be extended to elements of $\operatorname{st}(Q)$ and $\operatorname{st}(R)$ respectively. Then $f \in \operatorname{st}(Q)$ iff every initial segment of f belongs to U, and similarly for $\operatorname{st}(R)$ and V. Let $\{t_n\}$ be a finite or countable enumeration of all elements u of $U \setminus V$ which are minimal in the sense that each proper initial segment of u belongs to V. For each n let $R_n = \{x \in Q: t_n \subseteq x\}$. The sequence R, R_n satisfies conditions (a), (b), and (c). For each $s \in 2^k$ let B_s be the internal set $B_s = \{x \in T: s \subseteq x\}$. Let $\{Q_n: n \in N\}$ be the collection of all intersections of the sets R or R_n with the sets B_s. The family $\{Q_n\}$ has the required properties (a)–(d). □

LEMMA 1.9. *Suppose $\langle A_n: n \in N \rangle$ is a family of Π_1^0 sets over T. Then there exists a family $\langle Q_s: s \in N^\infty \rangle$ of Π_1^0 sets over T that satisfy the following conditions*:

(a) $Q_\varnothing = T$.

(b) *For each s, the sets $Q_{s^\wedge\langle n \rangle}$, $n \in N$, are disjoint subsets of Q_s.*

(c) *For each s, the sets* $\operatorname{st}(Q_{s^\wedge\langle n \rangle})$, *$n \in N$, form a disjoint partition of* $\operatorname{st}(Q_s)$.

(d) *For each $s \in N^{n+1}$, Q_s is either included in or disjoint from A_n.*

(e) *For each $s \in N^{n+1}$ and all $u, v \in Q_s$, $u | n = v | n$.*

PROOF. Iterate Lemma 1.8. □

LEMMA 1.10. *Given a sequence $\langle A_n: n \in N \rangle$ of Π_1^0 sets over T, there is a function $f: 2^N \to T$ such that for each $x \in 2^N$, $\operatorname{st}(f(x)) = x$ and for each n, $f^{-1}(A_n)$ is Σ_2^0 over 2^N.*

PROOF. Arbitrarily choose $f(x) \in \bigcap\{Q_s: x \in \operatorname{st}(Q_s)\}$. This is possible because for each n there is a unique $s \in N^n$ such that $x \in \operatorname{st}(Q_s)$, and the sets Q_s form a decreasing chain, so by saturation their intersection is nonempty.

We claim that $\operatorname{st}(f(x)) = x$. To see this, observe that if $x \in \operatorname{st}(Q_s)$ where s has length n, then for all $u \in Q_s$, $u | n = x | n$, so $f(x) | n = x | n$.

We now claim that $f^{-1}(A_n) \in \Sigma_2^0(2^N)$. To prove this claim let $B = \{s \in N^{n+1}: Q_s \subseteq A_n\}$. If $s \in N^{n+1} \setminus B$ then Q_s is disjoint from A_n. The range of f is included in the set $\bigcup\{Q_s: s \in N^{n+1}\}$, and the sets

$$\bigcup\{Q_s: s \in B\}, \qquad \bigcup\{Q_s: s \in N^{n+1} \setminus B\}$$

are disjoint. Moreover,

$$A_n \cap \operatorname{range}(f) \subseteq \bigcup_{s \in B} Q_s,$$

and

$$f^{-1}(A_n) = f^{-1}\left(\bigcup_{s \in B} Q_s\right) = \bigcup_{s \in B} f^{-1}(Q_s).$$

But $f^{-1}(Q_s) = \mathrm{st}(Q_s)$, because

$$\begin{aligned} x \in f^{-1}(Q_s) \quad &\text{iff} \quad f(x) \in Q_s \\ &\text{iff} \quad f(x) \in \bigcap\{Q_s : x \in \mathrm{st}(Q_s)\} \\ &\text{iff} \quad x \in \mathrm{st}(Q_s). \quad \square \end{aligned}$$

REMARK. By taking complements we see that Lemma 1.10 also holds for Σ_1^0 and Π_2^0 in place of Π_1^0 and Σ_2^0.

THEOREM 1.11. (a) *For each* $\alpha > 1$ *and each set* $A \in \Sigma_\alpha^0(T)$ *there is a set* $B \in \Sigma_\alpha^0(2^N)$ *such that* $B \subseteq \mathrm{st}(A)$ *and* $2^N \setminus B \subseteq \mathrm{st}(T \setminus A)$. *Similarly for* Π_α^0 *and* Δ_α^0.

(b) *For each* $n > 0$ *and each set* $A \in \Sigma_n^1(T)$ *there is a set* $B \in \Sigma_n^1(2^N)$ *such that* $B \subseteq \mathrm{st}(A)$ *and* $2^N \setminus B \subseteq \mathrm{st}(T \setminus A)$. *Similarly for* Π_n^1 *and* Δ_n^1.

PROOF. (a) Let β be the ordinal such that $\alpha = 1 + \beta$. Suppose first that α is even. Then A is a Σ_β^0 combination of sets A_n which are Π_1^0 over T. Let f be as in Lemma 1.10 and let $B = f^{-1}(A)$. B is a Σ_β^0 combination of the Σ_2^0 sets $f^{-1}(A_n)$ over 2^N. Since $\alpha > 1$ and countable unions of countable unions are countable unions, B is Σ_α^0 over 2^N. Finally, since $\mathrm{st}(f(x)) = x$, we have $B \subseteq \mathrm{st}(A)$ and $2^N \setminus B \subseteq \mathrm{st}(T \setminus A)$ as required.

(b) This follows from (a) and the observation that if $U = 2^K$, then the standard part maps from $T \times U$ to $2^N \times 2^N$ and from T to 2^N commute with the projection maps from $T \times U$ to T and from $2^N \times 2^N$ to 2^N. $\square$

COROLLARY 1.12 (KUNEN AND MILLER [KM]). (a) *For each countable* α, *and* $B \subseteq 2^N$, $B \in \Sigma_\alpha^0(2^N)$ *if and only if* $\mathrm{st}^{-1}(B) \in \Sigma_\alpha^0(T)$, *and similarly for* Π and Δ.

(b) *For each* $n > 0$ *and* $B \subseteq 2^N$, $B \in \Sigma_n^1(2^N)$ *if and only if* $\mathrm{st}^{-1}(B) \in \Sigma_n^1(T)$, *and similarly for* Π *and* Δ.

PROOF. Apply Theorem 1.11 with $A = \mathrm{st}^{-1}(B)$. $\square$

We shall give another application of Theorem 1.11 in the next section.

COROLLARY 1.13. *The classes* $\Sigma_\alpha^0(S)$ *and* $\Pi_\alpha^0(S)$ *are strictly increasing in* α *for all countable* α. *The classes* $\Sigma_n^1(S)$ *and* $\Pi_n^1(S)$ *are strictly increasing for all finite* n.

§2. Countably determined sets.

DEFINITION. A set $B \subseteq S$ is *countably determined* if there is a countable sequence A_n, $n \in N$, of internal sets such that for some $I \subseteq 2^N$,

$$B = \bigcup_{i \in I} \bigcap_{n \in N} A_n^{i(n)}.$$

Here $A^1 = A$ and $A^0 = S \setminus A$. We shall write $B = \Phi_I(A_0, A_1, \ldots)$, and say that B *is determined on* $A_0, A_1, \ldots$.

Countably determined sets were introduced by Henson [H2].

LEMMA 2.1. (a) *The set of countably determined subsets of* S *is a* σ-*algebra.*

(b) *For each countable sequence of internal sets* $A_0, A_1, \ldots$, *the set of all sets determined on* $A_0, A_1, \ldots$ *is closed under complements and arbitrary unions and intersections.*

(c) *Projections of countably determined subsets of* $S \times T$ *are countably determined subsets of* S.

(d) *All projective sets over* S, *that is, sets in* $\Sigma_n^1(S)$ *for some finite* n, *are countably determined.*

(e) *The class of countably determined subsets of S is closed under the Suslin operation.*

PROOF. (b) The existential quantifier commutes with arbitrary unions and with countable intersections of internal sets. □

LEMMA 2.2. *For each countable sequence of internal subsets B_n, $n \in N$, of S and each hyperfinite $T = 2^H$ there is an internal function $F: S \to T$ with the following properties*:

(a) *For each $I \subseteq 2^N$*,

$$\Phi_I(B_0, B_1, \ldots) = F^{-1}(\mathrm{st}^{-1}(I)).$$

(b) *For each set $I \subseteq \mathrm{st}(F(S))$, $F^{-1}(\mathrm{st}^{-1}(I))$ is at the same level in the Borel and projective hierarchies as I.*

PROOF. (a) For each $n \in N$, let C_n be the clopen set $\{x \in 2^N: x(n) = 1\}$. Let B_n, $n \in N$, be a sequence of internal subsets of S. For each k there is an internal function $F_k: S \to T$ such that for each $n < k$, $B_n = F_k^{-1}(\mathrm{st}^{-1}(C_n))$. By saturation there is an internal function $F: S \to T$ such that for each $n \in N$, $B_n = F^{-1}(\mathrm{st}^{-1}(C_n))$.

(b) This follows from 1.7 and 1.8. □

THEOREM 2.3. *Let S and U be hyperfinite. Each projective set $B \in \Sigma^1_{n+1}(S)$ is the projection on S of a set $A \in \Pi^1_n(S \times U)$.*

PROOF. By the proof of Lemma 2.2 there is a set $I \in \Sigma^1_{n+1}(2^N)$, a sequence of internal sets B_n, a hyperfinite $T = 2^H$, and an internal $F: S \to T$ such that

$$B = \Phi_I(B_0, B_1, \ldots) = F^{-1}(\mathrm{st}^{-1}(I)).$$

I is the projection of a set $J \in \Pi^1_n(2^N \times N^N)$. Let K be an infinite hyperinteger such that $K^K \leq U$, and let $G: S \times U \to T \times U$ be the product of F and the identity function on U. Then B is the projection on S of the set

$$G^{-1}(\mathrm{st}^{-1}(J)) \in \Pi^1_n(S \times U). \quad \square$$

EXAMPLE. There is a countably determined free ultrafilter in the Boolean algebra $^*P(S)$ of internal subsets of S.

To see this, assume without loss of generality that $N \subseteq S$, and let U be any free ultrafilter over N, $U \subseteq P(N)$. Then the set

$$V = \{A \in {}^*P(S): A \cap N \in U\}$$

is an ultrafilter in $^*P(S)$, and V is countably determined because

$$V = \bigcup_{i \in U} \left[\bigcap_{n \in i} \{A \in {}^*P(S): n \in A\} \right].$$

If we identify $^*P(S)$ with 2^S and $P(N)$ with 2^N in the natural way, then $V = \mathrm{st}^{-1}(U)$, so that V is at the same level of the projective hierarchy as U.

This example complements the result of Panetta [P] in 1978 that no free ultrafilter in $^*P(S)$ is Loeb measurable with respect to the counting measure, and hence no such ultrafilter is analytic. See §5 for a related open question.

Panetta showed that the existence of a Loeb measurable free ultrafilter in $^*P(S)$ is equivalent to the following finite combinatorial statement:

(*) For every real $\varepsilon > 0$ and $n \in N$ there exists $m \in N$ and a collection of subsets U of $\{0, \ldots, m-1\}$ such that $\#(U)/2^m > 1/2 - \varepsilon$ but the intersection of any $n+1$ elements of U has size $\geq n$.

However, Frankl [F] in 1976 showed that (*) fails when $n = 2$ and $\varepsilon = 1/4$. See also Mills [M].

REMARK. Consider an internal set H. We make some observations on the possible kinds of free ultrafilters U in the Boolean algebra $*P(H)$ of internal subsets of H. Call U *countably complete* iff whenever $\{A_n : n \in N\}$ is a sequence of elements of U, there is a $B \in U$ such that $B \subset A_n$ for each n. Assuming the continuum hypothesis, there is a nonstandard universe in which every hyperfinite set H has $\aleph_1$ internal subsets, and in such a universe it is easy to construct a countably complete free ultrafilter in $*P(H)$ by transfinite recursion. Also, for any infinite internal set H, there exist *internal* free ultrafilters in $*P(H)$, and any internal free ultrafilter in $*P(H)$ is countably complete. On the other hand, it is easy to see that the countably determined free ultrafilter constructed in the preceding example is not countably complete. The following result shows more.

PROPOSITION 2.4. *Let H be a hyperfinite set and let U be a countably determined free ultrafilter in $*P(H)$. Then U is not countably complete.*

PROOF. Let Σ be the set of all finite $\{0, 1\}$-valued sequences. It is immediate from the definition that we may write U as

$$U = \bigcup_{f \in S} \bigcap_{n \in N} A_{f|n},$$

where S is some subset of $\{0, 1\}^N$ and A_s is an internal subset of $*P(H)$ for each $s \in \Sigma$. Let $B_s = \bigcap_{t \subseteq s} A_t$ and let C_s be the set of all internal $X \subseteq H$ such that $Y \subseteq X$ for some $Y \in B_s$. We see that we can replace the A_s by B_s (trivially), and then by C_s (by ω_1-saturation), so that we have

$$U = \bigcup_{f \in S} \bigcap_{n \in N} C_{f|n}.$$

Each $C_s \subseteq *P(H)$ is internal and upward closed in the sense that if $X \subseteq Y \subseteq H$, where X, Y are internal and $X \in C_s$, then $Y \in C_s$. Whenever $s \subseteq t$ we have $C_t \subseteq C_s$.

Now define a subtree $T \subseteq \Sigma$ and choose X_s for $s \in T$ as follows; once we have T, we can let W denote the antichain of minimal sequences not in T. Let $\varnothing$ be the empty sequence, put $\varnothing$ in T, and let $X_\varnothing \in C_\varnothing \setminus U$. Such an $X_\varnothing$ exists because U is a subset of $C_\varnothing$, and is a proper subset since U is external and $C_\varnothing$ is internal. Now suppose we have $s \in T$ and X_s. For each $i \in \{0, 1\}$, choose $X_{si} \in C_{si} \setminus U$ if possible, so that $X_s \subseteq X_{si}$, and place si in T. If this is not possible, then si will be in W.

Assuming U is countably complete, we may fix an internal set Y such that $Y \notin U$ but $X_s \subseteq Y$ for all $s \in T$. Let $R = U \cap \{Z : Y \subseteq Z\}$ and let $D_s = C_s \cap \{Z : Y \subseteq Z\}$. If $t \in W$, then $D_t \subseteq R$ (otherwise t would have been put in T, not W). We claim that $R = \bigcup_{t \in W} D_t$. If not, we may fix a $Z \in R$ which is not in any such D_t, and then fix $f \in S$ such that $Z \in C_{f|n}$ for all n; then f must be a path through T, which would imply that $Y \in U$ since the C_s are upward closed. Thus R, and hence U, would be Σ_1^0 over the internal sets, which is shown to be impossible by Panetta [P]. □

PROPOSITION 2.5. *Every uncountable countably determined set has an infinite internal subset.*

PROOF. This result is a special case of Theorem 1 of Henson [H2], obtained by taking μ to be the internal counting measure. Here is a direct proof. Suppose B is countably determined on A_n, $n \in N$, and B is uncountable. By a *type over* A_n we mean an intersection of the form $\bigcap_n A_n^{i(n)}$. If each type over A_n contained in B is finite, then by saturation each type contained in B is a finite Boolean combination of A_n's, and thus B is countable. Therefore there is an infinite type contained in B. By saturation, this infinite type contains an infinite internal set. □

COROLLARY 2.6. *Let* $S = 2^H$. *Every countably determined subset of* S *which intersects each monad* $\mathrm{st}^{-1}\{x\}$ *in a countable set is countable.*

PROOF. Suppose A is an uncountable countably determined subset of S. By Proposition 2.5, A has an infinite internal subset B. There is an $x \in 2^N$ such that for all $n \in N$, $A \cap \{s \in S: x \mid n \subseteq s\}$ is infinite. By saturation, $A \cap \mathrm{st}^{-1}\{x\}$ has an infinite internal subset, and therefore $A \cap \mathrm{st}^{-1}\{x\}$ is uncountable. □

EXAMPLES. Corollary 2.6 can be used to give some examples of sets which are not countably determined. Let H be a positive infinite hyperinteger and let $S = \{n \in {}^*N: n < H\}$. Then the set $A = \{x \in S$: there is a real between x/H and $(x + 1)/H\}$ is not countably determined by 2.6. Moreover, the set $B = \{x \in S: \mathrm{st}(x) \leq x\}$ is not countably determined because $B \cap \{x + 1/H: x \in S\backslash B\} = A$.

COROLLARY 2.7. *Every countably determined wellordering is countable.*

PROOF. Let A be a countably determined subset of $S \times S$ which is a wellordering of a subset B of S. Then B is uncountable, and since B is the projection of A on S, B is countably determined. By 2.5, B has an infinite internal subset C. For some infinite hyperinteger H, there is an internal function F mapping B onto 2^H. Let D be the set of all $t \in 2^H$ such that

$$(\exists c \in C)[F(c) = t \wedge (\forall b \in C)[F(b) \approx F(c) \rightarrow (b, c) \in A]].$$

D is a countably determined subset of 2^H and contains exactly one point of each monad, contradicting 2.6. □

The above corollary improves the result of Panetta [P] that no infinite hyperfinite set has an analytic wellordering.

We now prove a version of Ramsey's theorem for countably determined partitions of hyperfinite sets. For finite n, $U^{[n]}$ is the set of all subsets of U of size n. Let T be a hyperfinite set containing N, so that a relation on $S \times N$ is also a relation on the hyperfinite set $S \times T$.

THEOREM 2.8. *Let* $n \in N$. *Any function* $F: S^{[n]} \rightarrow N$ *whose graph is countably determined has an infinite internal homogeneous set, that is, there is an infinite internal set* $A \subseteq S$ *such that* F *is constant on* $A^{[n]}$.

PROOF. Let $F = \bigcup_{i \in I} \bigcap_{k \in N} A_k^{i(k)}$. Call a relation $B \subseteq S^{[n]}$ *small* if there is no infinite internal set $C \subseteq S$ such that $C^{[n]} \subseteq B$, and *large* otherwise. We wish to show that for some $l \in N$, $F^{-1}\{l\}$ is large.

By the finite Ramsey theorem and transfer, the union of finitely many small internal sets is small, and the complement of a small internal set is large. By saturation, the intersection of a countable chain of large internal sets is large.

Claim. There exist $l \in N$ and $i \in I$ such that for all $k \in N$, the internal set

$$\left\{x \in S^{[n]}: (x, l) \in \bigcap_{m<k} A_k^{i(k)}\right\} = D_k$$

is large.

It follows from this claim that the intersection $\bigcap_k D_k$ is large, and hence $F^{-1}\{l\}$ is large.

To prove the claim, suppose to the contrary that for all $l \in N$ and $i \in I$ there exists $k(l,i) \in N$ such that $D_{k(l,i)}$ is small. There are only countably many different sets $D_{k(l,i)}$. Since F has domain $S^{[n]}$, $S^{[n]}$ is covered by a countable family of small internal sets. By saturation, $S^{[n]}$ is a finite union of small internal sets, so $S^{[n]}$ is small, a contradiction. □

Let $S^{[\text{fin}]}$ denote the set of all finite subsets of S. Thus $S^{[\text{fin}]}$ is a Σ_1^0 subset of the internal set $^*P(S)$.

COROLLARY 2.9. *Any function $F: S^{[\text{fin}]} \to N$ with a countably determined graph has an infinite internal homogeneous set A, that is, there is an infinite internal set $A \subseteq S$ such that for each $n \in N$, F is constant on $A^{[n]}$.*

PROOF. By Theorem 2.8, there is a decreasing sequence A_n, $n \in N$, of infinite internal subsets of S such that for each n, F is constant on $A_n^{[n]}$. By saturation there is an infinite internal set $A \subseteq \bigcap_n A_n$. A is homogeneous for F, as required. □

DEFINITION. Γ is said to be a *hyperfinite point class* if for each hyperfinite set S, $\Gamma(S)$ is a set of subsets of S such that:

(a) each $A \in \Gamma(S)$ is countably determined;

(b) $\Gamma(S)$ is closed under finite unions and intersections; and

(c) if $A \in \Gamma(T)$ and $F: S \to T$ is internal, then $F^{-1}(A) \in \Gamma(S)$.

EXAMPLES. By Proposition 1.3, each level of the Borel and projective hierarchies is a hyperfinite point class. The class of all countably determined sets is a hyperfinite point class.

LEMMA 2.10. *Let Γ be a hyperfinite point class.*

(a) *For any partial function F whose graph is in $\Gamma(S \times T)$ and any internal function $G \subseteq S \times T$, the set A of all $x \in S$ such that $G(x) = F(x)$ belongs to $\Gamma(S)$.*

(b) *Any function $F \in \Gamma(S \times T)$ is a union of countably many restrictions of internal functions to sets in $\Gamma(S)$.*

PROOF. (a) Let H be the internal function $H(x) = (x, G(x))$. Then $A = H^{-1}(F) \in \Gamma(S)$.

(b) Let $F(x) = y$ be a function determined on the internal relations A_n, $n \in N$. Then for each $i \in I$, the intersection of the $A_n^{i(n)}$ is a function. By saturation, some finite initial part of this intersection is an internal function. F is the union of the restrictions of these internal functions to the sets where they agree with F. □

THEOREM 2.11. *Let S and T be hyperfinite sets.*

(a) *The following are equivalent*:

(i) *$\#(T)/\#(S)$ is finite.*

(ii) *There is a countably determined function F mapping S onto T.*

(iii) *There is a countably determined injection G mapping T into S.*

(b) *There is a countably determined bijection F mapping S onto T if and only if $\#(T)/\#(S)$ is finite but not infinitesimal.*

PROOF. (a) First suppose that (i) fails, so that $\#(T)/\#(S)$ is infinite. We show that (ii) fails. Let F be a countably determined function mapping S into T. By 2.10, F is the union of restrictions of internal functions F_n, $n \in N$. For each $k \in N$, the union of the ranges of F_n, $n < k$, has internal size at most $k \cdot \#(S)$ and is therefore a proper subset of T. By saturation, the union of the ranges of F_n, $n \in N$, is a proper subset of T, and therefore the range of F is a proper subset of T. Thus (ii) fails. We have shown that (ii) implies (i).

We next show that (iii) implies (ii). Let G be a countably determined injection mapping T into S. Let $t \in T$, and define $F = G^{-1} \cup (S \times \{t\})$. Then F is a countably determined function mapping S onto T, so (ii) holds.

When $\#(T)/\#(S) \leq 1$, it follows by transfer that there is an internal function mapping S onto T. When $\#(T)/\#(S)$ is finite but greater than one, part (b) will show that there is a countably determined bijection of S onto T, and it will then follow that (i) implies (iii).

(b) The implication from left to right follows from the fact that (iii) implies (i) in (a). Assume that $\#(T)/\#(S)$ is finite but not infinitesimal. Let $t = \mathrm{st}(\#(T)/\#(S))$. Then there are an infinite hyperinteger H with $\Delta t = 1/H!$ and sets S' and T' of the form

$$S' = \{K\,\Delta t\colon K \in {}^*N \text{ and } 0 < K\,\Delta t \leq 1\},$$

$$T' = \{K\,\Delta t\colon K \in {}^*N \text{ and } 0 < K\,\Delta t \leq t\}$$

such that

$$\#(S')/\#(S) \approx \#(T')/\#(T) \approx 1.$$

By the results of Henson and Ross [HR], there are Borel bijections from S onto S' and from T onto T'. This uses the easy direction of their result. The idea is that if H and K are hyperintegers and K/H is infinitesimal, then the function f from $\{1,\ldots,H\}$ to $\{1,\ldots,H+K\}$ defined by $f(M) = M$ if M/K is finite, and $f(M) = M + K$ if M/K is infinite, is a Borel bijection. We shall prove that there is a countably determined bijection from S' onto T'. It will then follow by composition of functions that there is a countably determined bijection from S onto T.

We first claim that there is a bijection h from $[0,1]$ onto $[0,t]$ such that for all x, $h(x) - x$ is rational. Suppose first that t is rational. To find such an h, let $x \equiv y$ mean that $x - y$ is rational. Then $[0,1]$ and $[0,t]$ have $\mathbf{c}$ classes under $\equiv$, and each class is countable. Choose a bijection from the set of classes of $[0,1]$ onto the set of classes of $[0,t]$, and form h by choosing a bijection of each class of $[0,1]$ onto the corresponding class of $[0,t]$. To make the endpoints behave, we choose h so that $h(0) = 0$ and $h(1) = t$. For each triple of rational numbers q, r, s, let G_{qrs} be the internal partial function which translates the set $S \cap (q,r)$ by the distance s. Define F to be the countably determined function

$$F = \bigcup_{i \in [0,1]} \bigcap_{i \in (q,r),\, h(i) = i + s} G_{qrs}.$$

(F is countably determined by Lemma 2.1(b).) Then F is a bijection from S' onto T'. In the case that t is irrational the argument is the same except that we add $t - 1$ to the set of rationals so that we can still make $h(1) = t$. $\square$

PROPOSITION 2.12. *$\#(T)/\#(2^S) \approx 0$ if and only if there is no countably determined relation over $T \times S$ which is universal for $\Sigma_0^0(S)$.*

PROOF. Suppose that $\#(T)/\#(2^S) \approx 0$ and $A \subseteq T \times S$ is countably determined. Let F be the partial function from T into 2^S such that $F(t) = B$ iff B is internal and is the section of A at t. Then F is countably determined. By 2.10(b), F is the union of countably many restrictions of internal functions. Since 2^S is infinitely large compared to T, the union of the ranges of finitely many internal functions is a proper subset of 2^S. By saturation, the range of F is a proper subset of 2^S. Thus A is not universal for internal(S). The converse follows from Theorem 2.11(c). □

REMARK. It follows from 2.1(c) that the domain and range of any countably determined function are countably determined sets. Moreover, the image and preimage of any countably determined set by a countably determined function are countably determined sets. The following example shows that there are functions F which are not countably determined but have the property that all images and preimages of countably determined sets under F are countably determined.

EXAMPLE. Henson [H3] defined a nonstandard universe to have the *ω-isomorphism property* if any two internal models for a finite language which are elementarily equivalent are isomorphic. He showed that there exist nonstandard universes with the ω-isomorphism property, and that in a nonstandard universe with the ω-isomorphism property, the Boolean algebras $^*P(S)$ and $^*P(T)$ are isomorphic whenever S and T are (infinite) hyperfinite sets. However, if $\#(T)/\#(S)$ is infinite or infinitesimal, then the isomorphism cannot be countably determined in view of Theorem 2.11.

§3. **Prewellordering, reduction, and separation.** In this section we use saturation to settle the prewellordering, reduction, and separation problems for hyperfinite sets at the first Borel level. We then prove transfer results which show that beyond the first Borel level these problems are equivalent to the corresponding problems over 2^N.

PROPOSITION 3.1. *For any hyperfinite point class Γ:*

(a) *Prewellordering(Γ) implies reduction(Γ).*

(b) *Uniformization(Γ) implies reduction(Γ).*

(c) *Reduction(Γ) implies separation(dual(Γ)).*

The proof is the same as the corresponding proof in classical descriptive set theory.

The following positive result corresponds to a negative result in the classical case.

PROPOSITION 3.2. (a) *The internal subsets of S have the prewellordering, reduction, and separation properties.*

(b) *Reduction(Π^0_1)(S) and separation(Σ^0_1)(S).*

PROOF. By saturation, $\Sigma^0_1(S)$ has the separation property. Reduction for $\Pi^0_1(S)$ now follows from the reduction and separation properties for $\Sigma^0_1(S)$. □

THEOREM 3.3. *For any projective or Borel class Γ, if prewellordering, reduction, or separation holds for Γ over 2^N, then it holds for Γ over a hyperfinite set S.*

PROOF. This follows from Lemma 2.2. As an illustration we give the proof for separation. Suppose separation holds for a projective or Borel class Γ over 2^N. Let A and B be disjoint sets in $\Gamma(S)$. By 2.2(a), there is an internal function F from S into T and sets I and J in $\Gamma(2^N)$ such that

$$A = F^{-1}(\mathrm{st}^{-1}(I)), \qquad B = F^{-1}(\mathrm{st}^{-1}(J)).$$

Then I and J are disjoint, and by separation for $\Gamma(2^N)$ there is a set K such that both K and $2^N \setminus K$ belong to $\Gamma(2^N)$, and K separates I and J; that is, $I \subseteq K$ and J is disjoint from K. Then the set $C = F^{-1}(\mathrm{st}^{-1}(K))$ is such that both C and $S \setminus C$ belong to $\Gamma(S)$ and C separates A and B. □

COROLLARY 3.4. (a) *For each countable* α, *prewellordering*$(\Sigma^0_\alpha)(S)$, *reduction* $(\Sigma^0_\alpha)(S)$, *and separation*$(\Pi^0_\alpha)(S)$.

(b) *Prewellordering*$(\Pi^1_1)(S)$, *reduction*$(\Pi^1_1)(S)$, *and separation*$(\Sigma^1_1)(S)$.

(c) *Prewellordering*$(\Sigma^1_2)(S)$, *reduction*$(\Sigma^1_2)(S)$, *and separation*$(\Pi^1_2)(S)$.

THEOREM 3.5. *Let* Γ *be a Borel or projective class beyond the first Borel level. If separation, reduction, or prewellordering fails for* Γ *over* 2^N *then it fails for* Γ *over* S.

PROOF. We may assume without loss of generality that S is of the form $T = 2^H$. Again we illustrate the method with the case of separation. Suppose A and B are disjoint sets of class Γ over 2^N. Then $A' = \mathrm{st}^{-1}(A)$ and $B' = \mathrm{st}^{-1}(B)$ are of class Γ over T. Suppose there is a set D which separates A' and B' such that both B and $T \setminus B$ belong to $\Gamma(T)$. By Theorem 1.11, there is a set $C \subseteq 2^N$ such that $C \subseteq \mathrm{st}(D)$ and $2^N \setminus C \subseteq \mathrm{st}(T \setminus D)$, and both C and $2^N \setminus C$ are of class Γ over 2^N. Then C separates A from B, because if $a \in A$, then $\mathrm{st}^{-1}(a)$ is contained in A' and disjoint from $T \setminus D$, so $a \notin \mathrm{st}(T \setminus D)$ and $a \in C$. On the other hand, if $b \in B$, then $\mathrm{st}^{-1}(b)$ is contained in B' and disjoint from D, so $b \notin \mathrm{st}(D)$, and thus $b \in 2^N \setminus C$. □

COROLLARY 3.6. (a) *For each ordinal* $1 < \alpha < \omega_1$, *reduction*$(\Pi^0_\alpha)(S)$ *fails and separation*$(\Sigma^0_\alpha)(S)$ *fails*.

(b) *Reduction*$(\Sigma^1_1)(S)$ *and separation*$(\Pi^1_1)(S)$ *fail*.

(c) *Reduction*$(\Pi^1_2)(S)$ *and separation*$(\Sigma^1_2)(S)$ *fail*.

PROOF. Use the corresponding results for 2^N. □

§4. Functions and uniformization. In the preceding section we saw that except for the first Borel level, sets in the hyperfinite hierarchies behave much like sets in the classical hierarchies. In this section we shall see that, in contrast, functions with graphs in the hyperfinite hierarchies behave very differently from functions with graphs in the classical hierarchies.

REMARK. Every Σ^1_n total function is Δ^1_n over $S \times T$, and every $\Sigma^1_n(S \times T)$ partial function has a $\Sigma^1_n(S)$ domain, as in the classical case.

DEFINITION. Given a hyperfinite point class Γ, let $U\Gamma$ be the hyperfinite point class such that $U\Gamma(S)$ is the set of all countable unions of sets in $\Gamma(S)$.

PROPOSITION 4.1. *If* F *is a partial function whose graph belongs to* $\Pi^0_1(S \times T)$, *then the domain of* F *belongs to* $\Pi^0_1(S)$.

PROOF. By Lemma 1.1(a). □

THEOREM 4.2. *If* Γ *is a hyperfinite point class and* F *is a partial function whose graph belongs to* $U\Gamma(S \times T)$, *then the domain of* F *belongs to* $U\Gamma(S)$. *In particular, the preimage of any internal*, Σ^0_α, *Borel*, Π^1_n, *or* Σ^1_n *function is of the same class*.

PROOF. By Lemma 2.10. □

PROPOSITION 4.3. *Every total function whose graph belongs to* $\Pi^0_1(S \times T)$ *is internal*.

PROOF. By ω_1-saturation. □

THEOREM 4.4. *If* Γ *is a hyperfinite point class and* F *is a total function whose graph belongs to* $U\Gamma(S \times T)$, *then the complement of the graph of* F *belongs to* $U\Gamma(S \times T)$.

PROOF. By Lemma 2.10, F is the union of restrictions of countably many internal functions G_k to sets C_k in $\Gamma(S)$. For each k, the complement of G_k restricted to $C_k \times T$

is in $\Gamma(S \times T)$. Since F is total, the union of these complements is equal to the complement of F and belongs to $U\Gamma(S \times T)$. □

COROLLARY 4.5. (a) *For each* α, *every* $\Sigma_\alpha^0(S \times T)$ *total function is* $\Delta_\alpha^0(S \times T)$.

(b) *Every* $\Pi_1^1(S \times T)$ *total function* F *is Borel*$(S \times T)$.

(c) $\Pi_n^1(S \times T)$ *total function is* $\Delta_n^1(S \times T)$.

THEOREM 4.6. (a) *For each countable* $\alpha > 1$ *there is a* $\Pi_\alpha^0(S \times T)$ *partial function whose preimage is not* $\Pi_\alpha^0(S \times T)$.

(b) *For each countable* $\alpha > 1$ *there is a total* $\Pi_\alpha^0(S \times T)$ *function which is not* $\Sigma_\alpha^0(S \times T)$.

PROOF. We may assume without loss of generality that $N \subseteq T$. Let A be a Π_α^0 subset of S which is not Σ_α^0. Then $S \backslash A$ is the union of a disjoint sequence B_n, $0 < n \in N$, of Δ_α^0 sets. Put $B_0 = A$. Let F be the function such that for each n, $F(B_n) = n$. Then F is not Σ_α^0 because $F^{-1}\{0\} = A$. Moreover, the preimage of the restriction of F to the complement of A is not $\Pi_\alpha^0(S)$.

We show that F is Π_α^0, so (b) holds. Moreover, the restriction of F to the complement of A is equal to the $F \cap [S \times (T\backslash\{0\})]$, which is again Π_α^0. Thus (a) also follows. For each n, the set

$$D_n = [S \times (T\backslash\{n\})] \cup [B_n \times \{n\}]$$

is Π_α^0. Since $\alpha > 1$, the Σ_1^0 set $S \times N$ is also Π_α^0. However,

$$F = [S \times N] \cap \left[\bigcap_n D_n\right],$$

so F is Π_α^0. □

PROPOSITION 4.7. *Uniformization holds for* $\Sigma_1^0(S \times T)$, $\Sigma_2^0(S \times T)$, *and* $\Pi_1^0(S \times T)$.

PROOF. For each $\Sigma_1^0(S \times T)$ or $\Pi_1^0(S \times T)$ relation A, use saturation to get an internal function F such that $F \cap A$ is a choice function for A. The functions for countably many increasing Π_1^0 relations may be pieced together to get a choice set for a Σ_2^0 relation. □

THEOREM 4.8. *There is a* Π_2^0 *relation which does not have a countably determined uniformization.*

PROOF. Take a Π_2^0 relation $A(x, y, z)$ whose projection on z is the set $B(x, y)$ such that $y \cap Q$ is a subset of $x \cap Q$ of order type ω^*. B is Borel. Suppose B has a countably determined uniformization F. By Lemma 2.10, F is the union of restrictions of internal functions G_n, $n \in N$, to countably determined sets. For each n, the set of x such that $(x, G_n(x)) \in B$ is Borel, and so is the union C of these sets over n. C is equal to the set of x such that $\exists y B(x, y)$, which is the set of all x such that $x \cap Q$ is not wellordered. However, this set is $\mathrm{st}^{-1}(D)$, where D is the set of all nonwellordered subsets of Q. D and hence C is not Π_1^1, contradicting that C is Borel. □

REMARK. One can readily carry out an alternative development of this paper in which the hyperfinite set S is replaced by the set $P = {}^*(2^N)$ of all internal functions from $^*N$ into 2. Define $\Delta_0^0(P)$ to be the internal algebra of sets generated by all sets of the form $\{f\colon f(n) = 0\}$ where $n \in {}^*N$, which is the same thing as the internal algebra of *-clopen subsets of P. The Borel and projective hierarchies over P and the countably determined subsets of P are then defined in the natural way. The standard part mapping st: $P \to N$ is defined by $\mathrm{st}(f) = f \mid N$. Almost all of the proofs and results in this paper can be modified to apply to the case where the hyperfinite set S is replaced by P, and a hyperfinite product $S \times T$ is replaced by $P \times P$. Two results which do not carry over to this case are 2.11 and 2.12, which depend on the internal cardinality of S.

§5. Problems.

5.1. How far must one go to uniformize $\Pi_2^0(S \times T)$, or $\Pi_1^1(S \times T)$?

5.2. How far must one go to wellorder S?

5.3. In the Boolean algebra $^*P(S)$, can the filter of internal subsets $A \subseteq S$ such that $|A|/|S| \approx 1$ be extended to a countably determined free ultrafilter? (Recall from [P] and [F] that no free ultrafilter can be analytic.)

5.4. Does every set $A \in \Sigma_{n+1}^1(S)$ belong to the least class containing $\Pi_n^1(S)$ and closed under finite intersections, countable unions, and images under internal functions on S?

5.5. Which reals t and functions f: $[0,1] \to [0,t]$ have the property that there is a countably determined function $F: S \to T$ with

$$S = \{K\,\Delta t: K \in {}^*N \text{ and } 0 < K\,\Delta t \le 1\},$$

$$T = \{K\,\Delta t: K \in {}^*N \text{ and } 0 < K\,\Delta t \le t\},$$

and for all $s \in S$, ${}^0F(s) = f({}^0s)$?

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