On some properties of Hurewicz, Menger, and Rothberger

by

Arnold W. Miller * (Madison, Wis.) and **David H. Fremlin** (Essex)

Abstract. Our main result is that property C' and C'' of Rothberger are not the same. We also discuss property M of Menger and a generalization of it due to Hurewicz.

We assume throughout this paper that our topological spaces are metrizable and separable.

DEFINITION. A metric space X has *property C* iff for every sequence $\langle \varepsilon_n : n < \omega \rangle$ of positive real numbers X can be covered with a sequence of sets $\langle I_n : n < \omega \rangle$ such that each I_n has diameter less than ε_n.

Property C is also known as strong measure zero. It was introduced by Borel (1919), who conjectured that every set of reals with property C must be countable. Sierpiński (1928) showed that under CH there is an uncountable set of reals with property C (see also Miller (1984) § 2 and § 3). Laver (1976) showed that it is consistent with ZFC that every set of reals with property C is countable. In his model it is easy to see that every metric space with property C is countable. In fact, answering a question of Galvin, Carlson (unpublished) has shown that if there is an uncountable metric space with property C then there is an uncountable set of reals with property C. (It seems to be unknown whether or not there must be a set of reals of cardinality the continuum with property C if there is a metric space of cardinality the continuum with property C.)

Clearly a space with property C must be separable and Szpilrajn–Marczewski (1937) (see also Kuratowski (1966) p. 528) showed that it must be zero-dimensional, and hence homeomorphic to a set of reals. However, this does not give Carlson's result since property C depends on the metric.

It is easy to see that a uniformly continuous image of a set with property C has property C. Sierpiński (1935, 1938) asked if every continuous real image of a set with property C has property C. This was answered in the negative by Rothberger (1941) who showed that assuming CH there is a set of reals with property C that can

* Research partially supported by NSF grant MCS-8401711.

icm©

be continuously mapped onto the whole real line. He also defined property C' in Rothberger (1938) as below. Rothberger showed that this is equivalent to the property that every real continuous image has property C.

DEFINITION. A space X has *property* C' iff for every sequence $\langle \mathcal{G}_n : n<\omega\rangle$ such that each $\mathcal{G}_n$ is a finite open cover of X there exist a sequence $\langle U_n : n\in\omega\rangle$ with $U_n \in \mathcal{G}_n$ such that $\{U_n : n\in\omega\}$ covers X.

Rothberger also introduced property C'' by dropping the requirement of "finite".

DEFINITION. A space X has *property* C'' (called "*Rothberger's property*" in Fremlin (1984)) iff for every sequence $\langle \mathcal{G}_n : n\in\omega\rangle$ of open covers of X there exists an open cover of X, $\{U_n : n<\omega\}$, with each $U_n \in \mathcal{G}_n$.

Note that $C'' \to C'$ and $C' \to C$ for metrics which are totally bounded, i.e. for every $\varepsilon>0$ the space can be covered by finitely many ε balls.

He asked if $C' = C''$. We will show that assuming CH the answer is no. First we give a Rothberger-like characterization of C''.

DEFINITION. The usual *metric* on the Baire space ω^ω, the set of infinite sequences of natural numbers, is defined by

$$\varrho(x,y) = \frac{1}{n+1} \quad \text{where } x\restriction n = y\restriction n \text{ and } x(n)\neq y(n)\,.$$

THEOREM 1. *The following are equivalent for any metric space* X:

(a) *X has property C'';*

(b) *X has property C with respect to every metric which gives the same topology; and*

(c) *X is zero-dimensional and every continuous image of X in ω^ω has property C in the usual metric.*

Proof. (a) ⇒ (b). Suppose δ is any metric on X and $\langle \varepsilon_n : n\in\omega\rangle$ is any sequence of positive real numbers. For each $n<\omega$ let

$$\mathcal{G}_n = \{U\subseteq X \mid U \text{ is an open set of } \delta\text{-diameter less than } \varepsilon_n\}\,.$$

By C'' there exist $U_n\in\mathcal{G}_n$ such that $\{U_n : n\in\omega\}$ covers X. Hence X has property C with respect to δ.

(a) ⇒ (c). It is enough to note that the continuous image of a space with property C'' has property C''.

(b) ⇒ (a). Let $\langle \mathcal{G}_n : n\in\omega\rangle$ be a sequence of open covers of X. Since X is a separable zero dimensional metric space, we can find $\langle \mathcal{G}_n^* : n\in\omega\rangle$ such that:

(1) $\mathcal{G}_n^*$ is a clopen disjoint cover refining $\mathcal{G}_n$;
(2) $U\in\mathcal{G}_n^*$ implies that the diameter of U is less than $1/n$; and
(3) $\mathcal{G}_{n+1}^*$ refines $\mathcal{G}_n^*$.

Let ϱ be an arbitrary metric on X. First replace $\mathcal{G}_n$ by

$$\left\{U \mid U \text{ clopen, } \varrho\text{-diam}(U)<\frac{1}{n}, \text{ and } \exists V\in\mathcal{G}_n\ U\subseteq V\right\}.$$

Since X is separable, we can replace this by a countable subcover $\{U_m : m<\omega\}$. Since they are clopen, we can make the cover disjoint. Finally obtain (3) by further intersections.

Now define a metric β on X by $\beta(x,y) = \frac{1}{n+1}$ where n is the least such that there exist $U\in\mathcal{G}_n^*$ with $x\in U$ and $y\notin U$. Since X has property C with respect to β, there exist I_n such that the β-diameter of I_n is less than $\frac{1}{n+1}$ and X is covered by $\{I_n : n\in\omega\}$. Hence for each n there exists $U_n^*\in\mathcal{G}_n^*$ and $U_n\in\mathcal{G}_n$ such that

$$I_n\subseteq U_n^*\subseteq U_n\,.$$

So X is covered by $\{U_n : n\in\omega\}$.

(c) ⇒ (a). This is the same as the above proof. So let $\langle \mathcal{G}_n^* : n\in\omega\rangle$ be as above, and let $\mathcal{G}^* = \bigcup\{\mathcal{G}_n^* : n\in\omega\}$. Let $\omega^{<\omega}$ be the set of finite sequences of natural numbers. Since the $\mathcal{G}_n^*$ satisfy (1), (2), and (3), it is easy to find $\sigma: \mathcal{G}^*\to\omega^{<\omega}$ such that for each n

$$\sigma''\mathcal{G}_n^*\subseteq\omega^n$$

and for all $U, V\in\mathcal{G}^*$

$$U\subsetneq V \quad \text{iff} \quad \sigma(V)\subsetneq\sigma(U)$$

where $\subsetneq$ on $\omega^{<\omega}$ means strict initial segment. Use σ to define a map

$$f: X\to\omega^\omega$$

by

$$f(x) = \bigcup\{\sigma(U) : x\in U\in\mathcal{G}^*\}\,.$$

Then f is continuous and so by assumption $Y = f''X$ has property C in the usual metric on ω^ω. Suppose $\{I_n : n\in\omega\}$ cover Y and diameter $(I_n)<\frac{1}{n+1}$. Then there is a unique $U_n\in\mathcal{G}_n^*$ such that $f''(U_n\cap I_n)\neq\emptyset$. So X is covered by $\{U_n : n\in\omega\}$ since if $f(x)\in I_n$ then $x\in U_n$. ■

Property M was introduced by Menger (1924) who called it *property E*.

DEFINITION. A space X has *property* M iff for every sequence $\langle \mathcal{G}_n : n\in\omega\rangle$ of open covers of X there exists $\langle \mathcal{F}_n : n\in\omega\rangle$ where $\mathcal{F}_n\in[\mathcal{G}_n]^{<\omega}$ and

$$X = \bigcup_{n<\omega}\bigcup\mathcal{F}_n\,.$$

Observe that M, like C' and C'' is a topological property, and also $C''\to M$. Our next result shows that our definition is equivalent to the one used by Menger.

THEOREM 2. *Let (X, ϱ) be any metric space. Then X has property M iff for every open basis $\mathcal{G}$ for X there exists $\{U_n: n \in \omega\} \subseteq \mathcal{G}$ such that*

$$\lim_{n\to\infty} \varrho\text{-diameter}(U_n) = 0$$

and X is covered by $\{U_n: n \in \omega\}$.

Proof. ($\Rightarrow$) This implication is due to Hurewicz (1927). Suppose X has property M and let $\mathcal{G}_n = \left\{U \in \mathcal{G} \mid \varrho\text{-diameter}(U) < \frac{1}{n+1}\right\}$. Then let $\mathcal{F}_n \in [\mathcal{G}_n]^{<\omega}$ be such that $X = \bigcup_{n<\omega} \bigcup \mathcal{F}_n$. If $\bigcup_{n<\omega} \bigcup \mathcal{F}_n = \{U_n: n \in \omega\}$, then clearly

$$\lim_{n\to\infty} \varrho\text{-diameter}(U_n) = 0 .$$

($\Leftarrow$) Let $\langle \mathcal{G}_n: n \in \omega \rangle$ be a sequence of open covers of X. Assume without loss of generality that:

$$U \text{ open} \subseteq V \in \mathcal{G}_n \Rightarrow U \in \mathcal{G}_n .$$

Let

$$\mathcal{G} = \left\{U \cup V \mid \exists n\ U, V \in \mathcal{G}_n \text{ and } \varrho\text{-diameter}(U \cup V) > \frac{1}{n}\right\}.$$

We claim that $\mathcal{G}$ is a basis for the non-isolated points of X. For suppose $x \in W^{\text{open}}$. Suppose $y \in W$ and $\varrho(x, y) > \frac{1}{n}$. Then there exists $U, V \in \mathcal{G}_n$ such that $x \in U$, $y \in V$, and $U \cup V \subseteq W$.

By assumption there exists $W_n \in \mathcal{G} \cup \{\{x\}: x \text{ is isolated}\}$ such that

$$\lim_{n\to\infty} \varrho\text{-diameter}(W_n) = 0$$

and X is covered by $\{W_n: n \in \omega\}$. If $W_n = U_n \cup V_n$ and $\varrho\text{-diameter}(W_n) < \frac{1}{m}$ then $U_n, V_n \in \mathcal{G}_k$ for some $k > m$. Hence $\mathcal{G}_k$ is used by only finitely many W_n. So if we also take care to cover the nth isolated point using $\mathcal{F}_n$ we can find $\mathcal{F}_n \in [\mathcal{G}_n]^{<\omega}$ such that

$$X = \bigcup_{n<\omega} \bigcup \mathcal{F}_n . \ \blacksquare$$

DEFINITION. $\mathcal{K}_\sigma$ is the class of all countable unions of compact spaces.

DEFINITION. A space X is *concentrated on* Y iff for every open $U \supseteq Y$ $X \setminus U$ is countable. X is *concentrated* iff X is concentrated on some countable $Y \subseteq X$.

Since it is easy to show that every $\mathcal{K}_\sigma$ has property M, Menger asked if property M implies $\mathcal{K}_\sigma$. Hurewicz (1925) showed that any analytic set (i.e. $\underset{\sim}{\Sigma}{}^1_1$ set) which has property M is $\mathcal{K}_\sigma$. Sierpiński (1926) noted that a Luzin set has property M but is not $\mathcal{K}_\sigma$. Note that Luzin set implies concentrated implies C''. If $V = L$ there is an uncountable $\underset{\sim}{\Pi}{}^1_1$ set concentrated on the rationals (Erdös, Kunen, Mauldin (1981)). Hence it is possible to have a $\underset{\sim}{\Pi}{}^1_1$ set with property M but not $\mathcal{K}_\sigma$. On the other hand if we have that every $\underset{\sim}{\Pi}{}^1_1$ game is determined, then there can be no such $\underset{\sim}{\Pi}{}^1_1$ set. This is so because using Wadge games it is possible to show that every $\underset{\sim}{\Pi}{}^1_1$ set which is not F_σ contains a closed homeomorphic copy of ω^ω and property M is inherited by closed subspaces (see Kunen, Miller (1983)). The fact that every non F_σ $\underset{\sim}{\Sigma}{}^1_1$ set contains a closed copy of ω^ω is due to Hurewicz (1928). In Menger (1924), Hurewicz (1925, 1927) and Sierpiński (1926), property M is referred to as property E. Our definition corresponds to property E^* of Hurewicz (1925, 1927). Sierpiński (1934) refers to it as property M (in honor of Menger) as does Rothberger (1938).

THEOREM 3. *For any cardinal $\varkappa$ the following are equivalent:*

(a) *Every separable metric space X of cardinality less than $\varkappa$ has property M.*

(b) *For every $X \subseteq \omega^\omega$ of cardinality less than $\varkappa$ there exists $g \in \omega^\omega$ such that for every $f \in X$ there is an $n < \omega$ such that $f(n) < g(n)$. (In the language of van Douwen, $\varkappa \leqslant \mathfrak{b}$.)*

Proof. (a) $\Rightarrow$ (b). Let $X \subseteq \omega^\omega$ be of cardinality less than $\varkappa$. Let

$$\mathcal{G}_n = \{[s]: s \in \omega^{n+1}\}$$

where $[s] = \{f \in \omega^\omega \mid s \subseteq f\}$. Since $\langle \mathcal{G}_n: n \in \omega \rangle$ is a sequence of open covers of X and X has property M, there exists $\mathcal{F}_n \in [\mathcal{G}_n]^{<\omega}$ such that X is covered by $\bigcup_{n<\omega} \mathcal{F}_n$. Define $g \in \omega^\omega$ by

$$g(n) = 1 + \max\{s(n) \mid s \in \mathcal{F}_n\} .$$

Then $f \in \bigcup \mathcal{F}_n$ implies $f(n) < g(n)$.

(b) $\Rightarrow$ (a). Let X be a separable metric space of cardinality less than $\varkappa$. Let $\langle \mathcal{G}_n: n \in \omega \rangle$ be a sequence of open covers. We can assume without loss of generality that each $\mathcal{G}_n$ is countable, say $\mathcal{G}_n = \{U_n^m: m < \omega\}$. For each $x \in X$ let $f_x \in \omega^\omega$ be defined as follows:

$$f_x(n) = \text{least } m \text{ such that } x \in U_n^m .$$

Suppose $g \in \omega^\omega$ and for every $x \in X$ there exists n such that $f_x(n) < g(n)$. Then X is covered by

$$\bigcup_{n<\omega} \bigcup_{m<g(n)} U_n^m . \ \blacksquare$$

THEOREM 4. *There exists a set $X \subseteq R$ of cardinality ω_1 with property M.*

Proof. If Theorem 3(b) is true for $\varkappa = \omega_2$, then every set $X \subseteq \omega^\omega$ of size ω_1 has property M. Otherwise there must exist an ω_1 scale, i.e. $\{f_\alpha: \alpha < \omega_1\} \subseteq \omega^\omega$ such that for all $g \in \omega^\omega$ there exists α such that for all but finitely many n $g(n) < f_\alpha(n)$ and for all $\alpha < \beta$ for all but finitely many n $f_\alpha(n) < f_\beta(n)$. But an ω_1 scale is concentrated on the rationals when we identify ω^ω with the irrationals (this is due to Rothberger, see also Miller (1984) p. 216). So the union of this set with the rationals has property C'' hence property M. $\blacksquare$

icm©

It follows that without any set theoretic hypotheses there is always a set with property M which is not $\mathscr{K}_\sigma$. The set constructed in Theorem 4 is not hereditarily of property M since an ω_1-scale cannot have property M (see the proof of Theorem 3). For the same reason Rothberger's example of set which has property C but not C' also fails to have property M (see Rothberger (1941) or Miller (1982) Thm 9.4).

QUESTION. *Without extra set theoretic assumption can one show there is a set $X \subseteq R$ of cardinality ω_1 such that all subsets of X have property M?*

DEFINITION. $\exists_n^\infty$ means "there exists infinitely many n"; and

$\forall_n^\infty$ means "for all but finitely many n".

THEOREM 5. *The following are equivalent for a cardinal $\varkappa$.*

(a) *Every separable metric space of cardinality less than $\varkappa$ has property C''.*

(b) *Every $X \subseteq \omega^\omega$ of cardinality less than $\varkappa$ has property C in the usual metric.*

(c) $\forall \mathscr{F} \in [\omega^\omega]^{<\varkappa}\ \exists g \in \omega^\omega\ \forall f \in \mathscr{F}\ \exists_n^\infty\ f(n) = g(n)$.

(d) $\forall \mathscr{F} \in [\omega^\omega]^{<\varkappa}\ \forall \mathscr{G} \in [[\omega]^\omega]^{<\varkappa}\ \exists g \in \omega^\omega$

$$\forall f \in \mathscr{F}\ \forall X \in \mathscr{G}\ \exists_n^\infty \in X\ f(n) = g(n).$$

(e) *R is not the union of less than $\varkappa$ many meager sets.*

Proof. The equivalence of (a) and (b) follows immediately from Theorem 1. The equivalence of (a) and (c) is due to Rothberger [1941] who also proved (e) implies (a). The equivalence of (d) and (e) is due to Miller (1982). The equivalence of (c) and (d) is due to Bartoszyński (1984). We give a shorter proof of (c) $\Rightarrow$ (d).

LEMMA 5.1. *Suppose $M \models$ ZFC* (a large finite fragment of ZFC) and*

$$\exists f \in \omega^\omega\ \forall g \in \omega^\omega \cap M\ \exists_n^\infty\ f(n) = g(n).$$

Then

$$\exists f \in \omega^\omega\ \forall g \in \omega^\omega \cap M\ \forall A \in [\omega]^\omega \cap M\quad \exists_n^\infty\ n \in A\ f(n) = g(n).$$

Proof. Let

$$Q = \{\langle u_n : n \in \omega\rangle \mid \forall n\ u_n : F_n \to \omega \wedge F_n \in [\omega]^{n+1}\}.$$

Clearly

$$\exists u \in Q\ \forall v \in Q \cap M\ \exists_n^\infty\ u_n = v_n.$$

Choose $x_n \in \mathrm{dom}(u_n) \setminus \{x_0, x_1, \ldots, x_{n-1}\}$. Define $f \in \omega^\omega$ so that $f(x_n) = u_n(x_n)$. Now suppose g, A are from M. Let v_n be g restricted to the first $n+1$ elements of A. Then $u_n = v_n$ implies $f(x_n) = g(x_n)$ and $x_n \in A$. ■

Lemma 5.1 shows that (c) $\Rightarrow$ (d) by the reflection principle. This ends the proof of Theorem 5.

Clearly $[R]^{<\varkappa} \subseteq C$ is equivalent to $[R]^{<\varkappa} \subseteq C'$, however these are not necessarily equivalent to $[R]^{<\varkappa} \subseteq C''$. In the infinitely often equal reals model (see Miller (1981) § 7) it is easy to see $[R]^{\omega_1} \subseteq C$, however in that model R is the union of ω_1 meager sets.

Clearly $C'' \to C'$, $C'' \to C$, and $C'' \to M$. Also $C' \to C$ with respect to any totally bounded metric (i.e. for every $\varepsilon > 0$ there is a covering by finitely many ε balls.) Our remaining results show that under CH no other implications hold.

THEOREM 6. *Assume* CH, *then there exists $X \subseteq R$ of each of the following kinds*:

(a) *property C' but not property M;*

(b) *property C' and property M but not property C''; and*

(c) *property C and property M but not property C'.*

Proof. Our first example will need the Galvin–Prikry theorem (1973).

Regard $[\omega]^\omega$ as a subspace of 2^ω, the Cantor space. Then the Galvin–Prikry theorem implies that for any finite Borel partition of $[\omega]^\omega$,

$$[\omega]^\omega = B_1 \cup B_2 \cup \ldots \cup B_n,$$

there exists $X \in [\omega]^\omega$ and k such that

$$[X]^\omega \subseteq B_k.$$

DEFINITION. For $X \in [\omega]^\omega$ define $[X]^{*\omega}$ to be the set of infinite $Y \subseteq \omega$ which are almost contained in X (i.e. $[X]^{*\omega} = \{Y \in [\omega]^\omega \mid Y \setminus X \text{ finite}\}$).

LEMMA 6a. *Suppose $\langle \mathscr{G}_n : n \in \omega\rangle$ is a sequence of finite open covers of $[X]^{*\omega}$. Then there exist $Y \in [X]^\omega$ and $\langle U_n : n \in \omega\rangle$ with $U_n \in \mathscr{G}_n$ such that*

$$[Y]^{*\omega} \subseteq \bigcup_{n<\omega} U_n.$$

Proof. Construct l_i, n_i, Y_i, U_i so that at stage $k < \omega$ we have:

(1) $U_i \in \mathscr{G}_i$ for $i < l_k < \omega$;

(2) $\{n_i : i < k\} \cup Y_k \subseteq X$;

(3) $Y_k \in [Y_{k-1}]^\omega$;

(4) $n_0 < n_1 < \ldots n_{k-1} < \min(Y_k)$; and

(5) $\forall F \subseteq n_{k-1}+1\ \exists i\ l_{k-1} \leqslant i < l_k$

$$\forall Y \in [Y_k]^\omega\ F \cup Y \in U_i.$$

To do stage $k+1$ let $n_k = \min(Y_k)$ and let

$$\{F_i : l_k < i < l_{k+1}\} = P(n_k+1)$$

(so $l_{k+1} = l_k + 2^{n_k+1}$).

Set $Z_{l_k} = Y_k \setminus \{n_k\}$ and use the Galvin–Prikry Theorem to find

$$Z_{l_k} \supseteq Z_{l_k+1} \supseteq Z_{l_k+2} \supseteq \ldots \supseteq Z_{l_{k+1}} = Y_{k+1}$$

as follows. Given Z_i, find $U_i \in \mathscr{G}_i$ and $Z_{i+1} \in [Z_i]^\omega$ so that for all $W \in [Z_{i+1}]^\omega$ $F_i \cup W \in U_i$. This ends the construction. Let $Y = \{n_i : i < \omega\}$. Suppose $Q \subseteq (n_k+1) \cup Y$, say $F = Q \cap (n_k+1)$. Then $Y \setminus (n_k+1) \subseteq Y_{k+1}$, so $\exists i\ l_k \leqslant i < l_{k+1}$ $Q \in U_i$. ■

Using the Lemma and CH it is easy to construct $\{X_\alpha\colon \alpha<\omega_1\}\subseteq[\omega]^\omega$ such that

(1) $\alpha<\beta \Rightarrow X_\beta \subseteq^* X_\alpha$;

(2) if $f_\alpha\colon \omega\to\omega$ enumerates X_α in increasing order then for all $g\in\omega^\omega$ there is an $\alpha<\omega_1$ such that for all $n<\omega$ $g(n)<f_\alpha(n)$; and

(3) for $\langle \mathscr{G}_n\colon n\in\omega\rangle$ a sequence of finite families of open sets there exists $\alpha<\omega_1$ such that either there exists n such that $X_{\alpha+1}\notin\bigcup\mathscr{G}_n$ or there exists $\langle U_n\colon n\in\omega\rangle$ with $U_n\in\mathscr{G}_n$ and $[X_{\alpha+1}]^{*\omega}\subseteq\bigcup\limits_{n<\omega}U_n$.

Now let $\Gamma=\{X_\alpha\colon \alpha<\omega_1\}$. Then Γ has property C' by (3), since for any $\alpha<\omega_1$ $\Gamma\setminus[X_{\alpha+1}]^{*\omega}$ is countable (so reserve half of the covers to take care of these countably many points). On the other hand Γ cannot have property M by (2). Let

$$U_n^m=\{X\in[\omega]^\omega|\ m \text{ is the } n\text{th element of } X\}.$$

Suppose for contradiction that

$$\Gamma\subseteq\bigcup_{n<\omega}\ \bigcup_{m<g(n)} U_n^m.$$

But if $g(n)<f_\alpha(n)$ then $X_\alpha\notin\bigcup\limits_{m<g(n)}U_n^m$. This concludes the proof of Theorem 6a.

Next we want to give an example of a space which has property C' and M but not C''. We will make use of eventually different forcing (see Miller [1981] § 5). Let

$$P=\{(s,F)|\ s\in\omega^{<\omega} \text{ and } F\in[\omega^\omega]^{<\omega}\}$$

where

$$(s,F)\leqslant(t,H)\quad\text{iff}\quad s\supseteq t,\ F\supseteq H,\quad\text{and}$$
$$\forall i\in\operatorname{dom}(s)\setminus\operatorname{dom}(t)\ \forall g\in H\ s(i)\neq g(i).$$

If G is a P-generic filter, then

$$f=\bigcup\{s|\ \exists F\ (s,F)\in G\}$$

is the eventually different real and

$$G=\{(s,F)\in P|\ s\subseteq f \text{ and } \forall i\in\omega\setminus\operatorname{dom}(s)\ \forall g\in F\ f(i)\neq g(i)\}.$$

Lemma 6b. 1. *Suppose $g\in\omega^\omega$ then* $\Vdash$ "$\forall_n^\infty g(n)\neq f(n)$". ($\forall_n^\infty$ *is for all but finitely many n).*

Proof. Suppose $p=(s,F)$ is arbitrary then $q=(s,F\cup\{g\})\leqslant p$ and

$$q\Vdash \text{"}\forall n>\operatorname{dom}(s)\ f(n)\neq g(n)\text{"}.\ ■$$

Lemma 6b. 2. *Suppose $j<\omega$, σ a term such that* $\Vdash$ "$\sigma<j$", *and $s\in\omega^{<\omega}$, then there exists $n<j$ such that for all p in P of the form (s,F) there exists $q\leqslant p$ such that* $q\Vdash$ "$\sigma=n$".

Proof. Suppose not. Then for every $n<j$ there exists $p_n=(s,F_n)$ such that $p_n\Vdash$ "$\sigma\neq n$". But then $(s,\bigcup\limits_{n<j}F_n)\Vdash$ "$\neg(\sigma<j)$". ■

Lemma 6b 3. *Suppose τ is a term and $k\in\omega^\omega$ such that* $\Vdash$ "$\forall n\ \tau(n)<k(n)$". *Then there exists $l\in\omega^\omega$ such that* $\Vdash$ "$\exists_n^\infty\ \tau(n)=l(n)$".

Proof. Let $\{s_n\colon n<\omega\}$ list $\omega^{<\omega}$ with infinite repetitions. Using Lemma 6b. 2 obtain $l\in\omega^\omega$ so that for all n and for all $p=(s_n,F)$ there exists $q\leqslant p$ such that $q\Vdash$ "$\tau(n)=l(n)$". Then it is true that $\Vdash$ "$\exists_n^\infty\ \tau(n)=l(n)$". For if not there is a $p\in P$ and $N<\omega$ so that

$$p\Vdash\text{"}\forall n>N\ \tau(n)\neq l(n)\text{"}.$$

But $p=(s_k,F)$ for some $k>N$ and so there exists $q\leqslant p$ such that $q\Vdash$ "$\tau(k)=l(k)$". ■

Lemma 6b. 4. *Suppose $s\in\omega^{<\omega}$, $n<\omega$, and σ is a term such that* $\Vdash$ "$\sigma\in\omega$". *Then there exists $m\in\omega$ such that for all $p=(s,F)\in P$ with $F\in[\omega^\omega]^n$ there exists $q\leqslant p$* $q\Vdash$ "$\sigma<m$".

Proof. This is Lemma 5.1 of Miller (1981), but here we give an alternative proof. Suppose no such m exists, then there exists $p_m=(s,F_m)$ with $F_m\in[\omega^\omega]^n$ such that $p_m\Vdash$ "$\sigma\geqslant m$". Let $\mathscr{U}$ be a nonprincipal ultrafilter on ω. Suppose $F_m=\{f_1^m,f_2^m,\ldots,f_n^m\}$ and let $g_k=\lim\limits_{\mathscr{U}}f_k^m$, i.e. for any $i\in\omega$ if there exists $X\in\mathscr{U}$ and $j\in\omega$ such that for all $m\in X$ $f_k^m(i)=j$, then $g_k(i)=j$; otherwise let $g_k(i)=0$ (or anything else). Suppose $q=(t,G)\leqslant(s,\{g_1,\ldots,g_n\})$ and $q\Vdash$ "$\sigma=j$". To get a contradiction it suffices to show q is compatible with some p_m with $m>j$. There exists $X\in\mathscr{U}$ such that for all $i\in\operatorname{dom}(t)\setminus\operatorname{dom}(s)$ and k with $l\leqslant k\leqslant n$, either $f_k^m(i)=g_k(i)$ for all $m\in X$ or (since $\{f_k^m(i)\colon m<\omega\}$ is not constant mod $\mathscr{U}$) $f_k^m(i)\neq t(i)$ for all $m\in X$. Since $(t,G)\leqslant(s,\{g_1,\ldots,g_n\})$ we know $g_k(i)\neq t(i)$ for all $i\in\operatorname{dom}(t)\setminus\operatorname{dom}(s)$ and $l\leqslant k\leqslant n$, so for any $m\in X$ $f_k^m(i)\neq t(i)$. Thus for any $m\in X$, p_m and (t,G) are compatible. ■

Lemma 6b. 5. *Suppose τ is a term such that* $\Vdash$ "$\tau\in\omega^\omega$". *Then there exists $h\in\omega^\omega$ such that* $\Vdash$ "$\exists_n^\infty\tau(n)<h(n)$".

Proof. Let $\{(s_k,n_k)\colon k\leqslant\omega\}$ list $\omega^{<\omega}\times\omega$ with infinitely many repetitions. Apply Lemma 6b. 4 to obtain $h\in\omega^\omega$ so that for all k and for all $p=(s_k,F)$ with $F\in[\omega^\omega]^{n_k}$ there exists $q\leqslant p$ such that $q\Vdash$ "$\tau(k)<h(k)$". We claim that

$$\Vdash\text{"}\exists_n^\infty\ \tau(n)<h(n)\text{"}.$$

For suppose for contradiction there is a $p\in P$ such that

$$p\Vdash\text{"}\forall n>N\ h(n)\leqslant\tau(n)\text{"}.$$

Then $p=(s_k F)$ for some $F\in[\omega^\omega]^{n_k}$ and $k>N$. But then there exists $q\leqslant p$ such that

$$q\Vdash\text{"}\tau(k)<h(k)\text{"}.\ ■$$

Lemma 6b. 6. *Suppose $M\subseteq N$ are transitive models of* ZFC* *(a sufficiently large finite fragment of* ZFC*). Then if G is P^N-generic over N, then $G\cap P^M$ is P^M-generic over M.*

Proof. It suffices to note that if $M \models$ "$A \subseteq P$ *is a maximal antichain*", then

$$N \models \text{"}A \subseteq P \text{ is a maximal antichain"}.$$

But it is easy to see that

$$\text{"}A \subseteq P \text{ is a maximal antichain"}$$

is a Π_1^1 statement, hence absolute. ■

Now we give our example of a space Σ with property C' and M but not C'' (i.e. Theorem 6b). So let $\langle M_\alpha: \alpha < \omega_1 \rangle$ be an ω_1 sequence of countable transitive models of ZFC* such that $\alpha < \beta$ implies $M_\alpha \subseteq M_\beta$ and $R = \bigcup_{\alpha<\omega_1} R^{M_\alpha}$. This is possible since we assume CH. Let $\langle G_\alpha: \alpha < \omega_1 \rangle$ be a sequence such that for each $\alpha < \omega_1$, G_α is P^{M_α}-generic over M, and for each $p \in P$ there are unboundedly many $\alpha < \omega_1$ with $p \in G_\alpha$. If $f_\alpha \in \omega^\omega$ is the eventually different real associated with G_α then we claim $\Sigma = \{f_\alpha: \alpha < \omega_1\}$ works.

CLAIM. *Σ has property M.*

Proof. Suppose $\langle \mathscr{G}_n: n \in \omega \rangle$ is a sequence of open covers of Σ. Without loss we may assume each $\mathscr{G}_n$ is countable. Since the entire thing can be coded by a real there is an $\alpha < \omega_1$ such that $\langle \mathscr{G}_n: n \in \omega \rangle$ is coded in M_α. Now work in M_α, and let $\mathscr{G}_n = \{U_n^m: m < \omega\}$ and f be the name for the eventually different generic real. Since all f_β for $\beta > \alpha$ are eventually different generic reals over M_α (by Lemma 6b. 6) and every $p \in P$ is in unboundly many G_β, it must be that

$$M_\alpha \models \Vdash \text{"}\forall n\ \exists m\ f \in U_n^m\text{"}.$$

So define a term τ in M_α such that

$$M_\alpha \models \Vdash \text{"}\forall n\ f \in U_n^{\tau(n)}\text{"}.$$

By Lemma 6b.5 there exists $h \in \omega^\omega \cap M_\alpha$ such that

$$M_\alpha \models \Vdash \text{"}\exists_n^\infty\ \tau(n) < h(n)\text{"}.$$

Hence if $\mathscr{F}_n = \{U_n^m: m < h(n)\}$ then

$$M_\alpha \models \Vdash \text{"}f \in \bigcup_n \bigcup \mathscr{F}_n\text{"}.$$

Since all but boundedly many f_β are eventually different generic over M_α, it follows that $\bigcup_n \bigcup \mathscr{F}_n$ covers all but countably much of Σ and this suffices.

CLAIM. *Σ has property C'.*

Proof. This is similar to above. So suppose $\langle \mathscr{G}_n: n \in \omega \rangle$ are finite open covers of Σ with $\mathscr{G}_n = \{U_n^m: m < k(n)\}$ and $\langle \mathscr{G}_n: n \in \omega \rangle$ coded in M_α. So again

$$M_\alpha \models \Vdash \text{"}\forall n\ \exists m < k(n)\quad f \in U_n^m\text{"}.$$

By Lemma 6b.3 we can find $l \in \omega^\omega \cap M$ such that

$$M_\alpha \models \Vdash \text{"}\exists_n^\infty\ f \in U_n^{l(n)}\text{"}.$$

It follows that

$$f_\beta \in \bigcup_{n<\omega} U_n^{l(n)}$$

for all $\beta > \alpha$ and this suffices.

CLAIM. *Σ does not have property C''.*

Proof. Let $U_n^m = \{h \in \omega^\omega \mid h(n) = m\}$ and $\mathscr{G}_n = \{U_n^m: m < \omega\}$. Then there does not exist $g \in \omega^\omega$ such that

$$\Sigma \subseteq \bigcup_{n\in\omega} U_n^{g(n)}.$$

This is because

$$\bigcup_{n\in\omega} U_n^{g(n)} = \{f \mid \exists n\ f(n) = g(n)\}.$$

But for unboundedly many α $(\emptyset, \{g\}) \in G_\alpha$ and hence for such α $\forall n\ f_\alpha(n) \neq g(n)$. This concludes the proof of Theorem 6b.

Next we wish to find a space Φ which has property C and M but not C'. Basically we plan to apply the Rothberger trick to a Luzin set. So from now on let $P = \omega^{<\omega}$ (Cohen forcing) and x a name for the Cohen real.

DEFINITION. For $x \in \omega^\omega$ and $y \in 2^\omega$ let $2x+y \in \omega^\omega$ be defined by $(2x+y)(n) = 2x(n)+y(n)$.

LEMMA 6c. *Let $\langle \mathscr{G}_n: n \in \omega \rangle$ be sequence of families of open subsets of ω^ω. Then either there exists $p \in P$ and $n \in \omega$ such that*

$$p \Vdash \text{"}\exists y \in 2^\omega (2x+y) \notin \bigcup \mathscr{G}_n\text{"}$$

or there exists $\langle \mathscr{F}_n: n \in \omega \rangle$ with $\mathscr{F}_n \in [\mathscr{G}_n]^{<\omega}$ such that

$$\Vdash \text{"}2x+2^\omega \subseteq \bigcup_{n<\omega} \bigcup \mathscr{F}_n\text{"}.$$

Proof. Suppose there is no such $p \in P$. Let $P = \{p_n: n \in \omega\}$. Since for each $n < \omega$

$$\Vdash \text{"}2x+2^\omega \subseteq \bigcup \mathscr{G}_n\text{"}$$

and $2x+2^\omega$ is compact we can find $q \leqslant p_n$ and $\mathscr{F}_n \in [\mathscr{G}_n]^{<\omega}$ so that

$$q \Vdash \text{"}2x+2^\omega \subseteq \bigcup \mathscr{F}_n\text{"}.$$

We claim

$$\Vdash \text{"}2x+2^\omega \subseteq \bigcup_{n<\omega} \bigcup \mathscr{F}_n\text{"}.$$

But if not there exists $p \in P$ such that

$$p \Vdash \text{"}\exists y \in 2^\omega\ \forall n\ 2x+y \notin \bigcup \mathscr{F}_n\text{"}.$$

If $p = p_n$ then $\exists q \leqslant p_n$

$$q \Vdash \text{“}2x+2^\omega \subseteq \bigcup \mathscr{F}_n\text{”}.$$

This is a contradiction. ■

As in the proof of Theorem 6b let $\langle M_\alpha\colon \alpha<\omega_1\rangle$ be models of ZFC*. We will construct an ω_1 sequence $\langle x_\alpha\colon \alpha<\omega_1\rangle$ such that x_α is $\boldsymbol{P}$-generic over M_α and an ω_1 sequence $\langle y_\alpha\colon \alpha<\omega_1\rangle$ with $y_\alpha \in 2^\omega \cap M_\alpha[x_\alpha]$ such that

$$\Phi = \{2x_\alpha+y_\alpha \mid \alpha<\omega_1\}$$

has property C and M but not C'. Let $\langle \mathscr{G}_\alpha\colon \alpha<\omega_1;\ \alpha \text{ even}\rangle$ list all countable sequences of countable families of open subsets of ω^ω, and let $\langle z_\alpha\colon \alpha<\omega_1;\ \alpha \text{ odd}\rangle$ list 2^ω, and assume $\mathscr{G}_\alpha, z_\alpha \in M_\alpha$.

Construction of x_α, y_α:

α odd. Let x_α be $\boldsymbol{P}$-generic over M_α and $y_\alpha = z_\alpha$.

α even. Work in M_α and let $\mathscr{G}_\alpha = \langle \mathscr{G}_n\colon n\in\omega\rangle$ and apply Lemma 6c. If there exists $p\in \boldsymbol{P}$ and $n<\omega$ such that

$$p \Vdash \text{“}\exists y\in 2^\omega (2x+y)\notin \bigcup \mathscr{G}_n\text{”},$$

then choose x_α Cohen generic over M_α and $y_\alpha \in M_\alpha[x_\alpha]\cap 2^\omega$ so that

$$M_\alpha[x_\alpha] \Vdash \text{“}(2x_\alpha+y_\alpha)\notin \bigcup \mathscr{G}_n\text{”}.$$

Otherwise let x_α be any real $\boldsymbol{P}$-generic over M_α and y_α any element of $M_\alpha[x_\alpha]$.

CLAIM. *Φ does not have property C'.*

Proof. Define $\pi\colon \omega^\omega \to 2^\omega$ by

$$\pi(z)(n) = z(n) \bmod 2.$$

Then the odd stages of the construction guarantees that π is onto.

CLAIM. *Φ has property C with respect to some metric.*

Proof. Since the x_α are more and more $\boldsymbol{P}$-generic and $x_\alpha(n)<2x_\alpha(n)+y_\alpha(n)$, Φ has the property that for all $g\in\omega^\omega$ $\{f\in\Phi \mid \forall n\ f(n)<g(n)\}$ is countable. If we view ω^ω as homeomorphic to $[\omega]^\omega \subseteq P(\omega) = 2^\omega$, then Φ is concentrated on $[\omega]^{<\omega}$. That is to say, for any open set U containing $[\omega]^{<\omega}$. Φ is contained in U except for countably many points. This is because the compliment of U in 2^ω is closed hence a compact subset of “ω^ω”, so bounded by some $g\in\omega^\omega$. (This is the Rothberger trick, see Miller (1984) p. 225.) Hence Φ has property C with respect to any metric which can be extended to a metric on 2^ω.

CLAIM. *Φ has property M.*

Proof. Suppose $\mathscr{G}^\alpha = \langle \mathscr{G}_n^\alpha\colon n\in\omega\rangle$ is a sequence of countable open covers of Φ. Then by construction either there exists n such that $M_\alpha[x_\alpha] \models 2x_\alpha+y_\alpha \notin \bigcup \mathscr{G}_n$” or there exists $\langle \mathscr{F}_n\colon n\in\omega\rangle \in M_\alpha$ with $\mathscr{F}_n \in [\mathscr{G}_n]^{<\omega}$ and

$$M_\alpha \models \Vdash \text{“}2x+2^\omega \subseteq \bigcup_n \bigcup \mathscr{F}_n\text{”}.$$

The first cannot happen because

$$2x_\alpha+y_\alpha \notin \bigcup \mathscr{G}_n \text{ is } \underset{\sim}{\Pi}{}^0_1$$

hence absolute and then $\mathscr{G}_n$ does not cover Φ. If the second happens, then we claim for all $\beta\geqslant\alpha$

$$2x_\beta+y_\beta \in \bigcup_n \bigcup \mathscr{F}_n.$$

This is true because x_β is $\boldsymbol{P}$-generic over M_α, so

$$M_\alpha[x_\beta] \models \text{“}2x_\beta+2^\omega \subseteq \bigcup_n \bigcup \mathscr{F}_n\text{”}$$

and so by absoluteness.

$$2x_\beta+2^\omega \subseteq \bigcup_n \bigcup \mathscr{F}_n.$$

Thus $\bigcup_n \bigcup \mathscr{F}_n$ covers all but countably many elements of Φ and thus Φ has property M.

This concludes the proof of Theorem 6c. ■

Theorem 6 is also true if only MA is assumed, in fact we need only MA for σ-centered partially ordered sets. Theorem 6b and 6c can be proved without using the terminology of forcing, in fact, the original proof of 6b did not use it.

The following property was introduced in Hurewicz (1927) where it is called property E^{**}.

DEFINITION. A space X has *property H* iff for all sequences $\langle \mathscr{G}_n\colon n\in\omega\rangle$ of open covers of X there exists $\langle \mathscr{F}_n\colon n\in\omega\rangle$ with $\mathscr{F}_n\in[\mathscr{G}_n]^{<\omega}$ and

$$X \subseteq \bigcup_{n<\omega} \bigcap_{m>n} \bigcup \mathscr{F}_n.$$

Hurewicz proved the analogue of Theorem 3 for property H. Namely every space of cardinality less than $\varkappa$ has property H iff

$$\forall \mathscr{F} \in [\omega^\omega]^{<\varkappa}\ \exists g\in\omega^\omega\ \forall f\in\mathscr{F}\ \forall_n^\infty\ f(n)<g(n).$$

(In the language of van Douwen (1984), $\varkappa \leqslant \mathfrak{b}$.)

A set of reals X is called a *Sierpiński set* iff it is uncountable and meets every measure zero set in a countable set. Such a set can be constructed analogously to a Luzin set using CH (see Miller (1984) or Fremlin (1984)).

THEOREM 7. *If X is a Sierpiński set, then X has property H.*

Proof. Assume without loss of generality that $X\subseteq[0,1]$ and X has outer measure one. Choose for each $n<\omega$ $\mathscr{F}_n\in[\mathscr{G}_n]^{<\omega}$ such that the measure of $\bigcup \mathscr{F}_n$ is at least $1-2^{-n}$. Then

$$\bigcup_{n<\omega} \bigcap_{m>n} \bigcup \mathscr{F}_n$$

icm©

has measure one and so contains all but countably many points of X, say $\{x_n: n \in \omega\}$. Now expand $\mathscr{F}_n$ to $\mathscr{F}_n^* \in [\mathscr{G}_n]^{<\omega}$ so that $n < m$ implies $x_n \in \bigcup \mathscr{F}_m^*$. Then

$$X \subseteq \bigcup_{n<\omega} \bigcap_{m>n} \mathscr{F}_m^* . \ \blacksquare$$

Since property H implies property M a Sierpiński set has property M. A set with property H is the union of a $\mathscr{K}_\sigma$ with a set of first category (Hurewicz (1927)), so for example a Luzin set which has property C'' hence M, cannot have property H.

QUESTION. *Is it consistent to suppose that a set has property H iff it is $\mathscr{K}_\sigma$?*

The next theorem partially answers a question of Hurewicz [1927] the other half being the question above.

THEOREM 8. *A set X with properties H and C has property C''.*

Proof. Assume that X has H and C. Let ϱ be the metric of X and for $x \in X$, $\varepsilon > 0$ set $U(x, \varepsilon) = \{y: \varrho(y, x) < \varepsilon\}$. Let $\langle \mathscr{G}_n: n < \omega \rangle$ be a sequence of open covers of X and for each $n < \omega$ set

$$\mathscr{H}_n = \{U(x, \varepsilon): x \in X,\ \varepsilon > 0,\ \exists G \in \mathscr{G}_n,\ U(x, 3\varepsilon) \subseteq G\} .$$

Then $\mathscr{H}_n$ is an open cover of X. Because X has property H, there is a sequence $\langle \mathscr{F}_n: n < \omega \rangle$ such that $\mathscr{F}_n \in [\mathscr{H}_n]^{<\omega}$ for each $n < \omega$ and

$$X = \bigcup_{n<\omega} \bigcap_{m>n} \bigcup \mathscr{F}_m .$$

We can suppose that no $\mathscr{F}_m$ is empty. Express $\mathscr{F}_m$ as $\{U(x_{mi}, \varepsilon_{mi}): i \in I_m\}$, where I_m is finite, for each $m < \omega$. Set

$$\varepsilon_m = \min\{\varepsilon_{mi}: i \in I_m\} > 0$$

for $m < \omega$. Let $\langle J(k): k < \omega \rangle$ be a partition of ω into infinite sets. Because X has property C, there is for each $k < \omega$ a sequence $\langle y_m: m \in J(k) \rangle$ in X such that $X = \bigcup_{m \in J(k)} U(y_m, \varepsilon_m)$.

For each $m < \omega$ choose $G_m \in \mathscr{G}_m$ such that $U(y_m, \varepsilon_m) \subseteq G_m$, if this is possible; otherwise take any $G_m \in \mathscr{G}_m$. Now let x be any member of X. Let $n < \omega$ be such that $x \in \bigcup \mathscr{F}_m$ for every $m > n$. Let $k < \omega$ be such that $m > n$ for every $m \in J(k)$. Let $m \in J(k)$ be such that $x \in U(y_m, \varepsilon_m)$. Because $m > n$, there is an $i \in I(m)$ such that $x \in U(x_{mi}, \varepsilon_{mi})$. Now $\varrho(y_m, x_{mi}) < \varepsilon_m + \varepsilon_{mi} \leqslant 2\varepsilon_{mi}$ so $U(y_m, \varepsilon_m) \subseteq U(x_{mi}, 3\varepsilon_{mi})$. But by the definition of $\mathscr{H}_m$ there is a $G \in \mathscr{G}_m$ such that $U(x_{mi}, 3\varepsilon_{mi}) \subseteq G$ and $U(y_m, \varepsilon_m) \subseteq G$. So $U(y_m, \varepsilon_m) \subseteq G_m$ and $x \in G_m$.

As x is arbitrary, $X \subseteq \bigcup_{n<\omega} G_n$. As $\langle \mathscr{G}_n: n < \omega \rangle$ is arbitrary, X has property C''. $\blacksquare$

DEFINITION. A space X is a *σ-set* iff every G_δ is an F_σ.

THEOREM 9. *Every set $X \subseteq \omega^\omega$, all of whose subsets have property H, is a σ-set.*

Proof. Suppose $X \subseteq \omega^\omega$ and G is a G_δ subset of ω^ω. Then G is homeomorphic to a closed $C \subseteq \omega^\omega$, say by $f: C \to G$ (see Kuratowski Cor. 2a, p. 441). Let $Y = f^{-1}(X)$. Then since Y is homeomorphic to a subspace of X, it has property H. Letting $\langle \mathscr{G}_n: n \in \omega \rangle$ be defined by

$$\mathscr{G}_n = \{[s] |\ s \in \omega^n\}$$

we see that there exists $h \in \omega^\omega$ such that for all $g \in Y$ for all but finitely many $n < \omega$ $g(n) < h(n)$. But

$$Q = \{g \in \omega^\omega |\ \forall_n^\infty\ g(n) < h(n)\} \cap C$$

is $\mathscr{K}_\sigma$. Since f is a homeomorphism $f''Q$ is $\mathscr{K}_\sigma$, but $f''Q \cap X = G \cap X$, so G is a relative F_σ. $\blacksquare$

Let us remark that hereditarily property M does not imply σ-set, since a Luzin set is hereditarily M but not a σ-set. Also σ-set does not imply property M since the example of Theorem 6a can be made into a σ-set (see Miller (1984) Theorem 5.7, p. 216). In Miller (1979) Theorem 22 it is shown that is relatively consistent with ZFC that every σ-set is countable.

Rothberger (1938) also introduced property M'. A metric space X with metric δ has property M' iff for any open basis $\mathscr{G}$ of X and any sequence of positive real numbers $\langle \varepsilon_n: n \in \omega \rangle$ there exists a sequence $\langle U_n: n \in \omega \rangle$ from $\mathscr{G}$ which covers X such that δ-diameter $(U_n) < \varepsilon_n$ for all $n < \omega$. He shows that C'' implies M'. However M' is equivalent to C''. It is enough to see that X has property C with respect to an arbitrary metric ϱ. So let $\langle \varepsilon_n: n \in \omega \rangle$ and ϱ be given (with ε_n's decreasing) and define

$$\mathscr{G} = \{U \cup V |\ U,\ V \text{ open and } \max(\varrho\text{-diam}(U),\ \varrho\text{-diam}(V)) < \delta\text{-diam}(U \cup V)\} .$$

It is easy to check that $\mathscr{G}$ is a basis and if $\langle U_n \cup V_n: n \in \omega \rangle$ are from $\mathscr{G}$ with

$$\delta\text{-diam}(U_n \cup V_n) < \varepsilon_{2n+1}$$

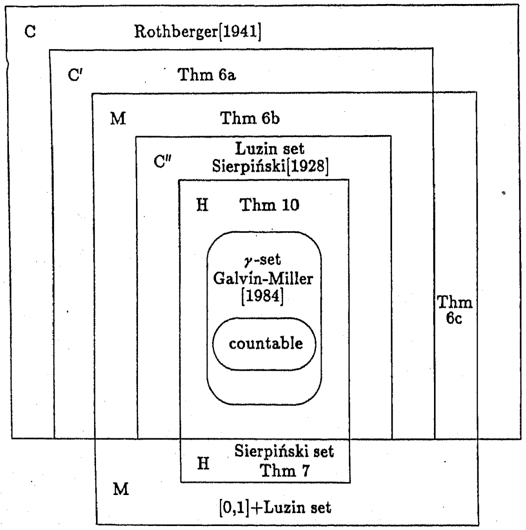
then letting $W_{2n} = U_n$ and $W_{2n+1} = V_n$ gives a sequence with ϱ-diam$(W_n) < \varepsilon_n$.

DEFINITION. X is a *γ-set* iff for any $\mathscr{G}$ an ω-cover of X there exists $\langle U_n: n \in \omega \rangle$ with $U_n \in \mathscr{G}$ such that $X \subseteq \bigcup_{n<\omega} \bigcap_{m>n} U_m$. ($\mathscr{G}$ is an *ω-cover* of X iff $\mathscr{G}$ is a family of open subsets of X such that every finite subset of X is contained in some element of $\mathscr{G}$.)

This property was introduced by Gerlits and Nagy (1982) and studied in Galvin and Miller (1984) and Gerlits (1983). It is easy to show that $\gamma \to H$ and it is shown in Gerlits and Nagy (1982) that $\gamma \to C''$.

THEOREM 10. (CH or MA) *There is a set X which has properties C'' and H but is not a γ-set.*

Proof. It is an unpublished result of Todorčević that MA implies there are γ-sets X and Y such that $X \cup Y$ is not a γ-set. (His proof from $\diamondsuit_{\omega_1}$ is given in Galvin and Miller [1984]). However it is easy to see that H and C'' are closed under countable unions. ∎



References

T. Bartoszyński, *Combinatorial aspects of measure and category*, 1984, to appear.

E. Borel, *Sur la classification des ensembles de mesure nulle*, Bull. Soc. Math. France 47 (1919), 97–125.

E. K. van Douwen, *The integers and topology*, in *Handbook of Set-Theoretic Topology*, ed. by K. Kunen and J. E. Vaughan, Elsevier Science Publishers, 1924, 203–233.

P. Erdös, K. Kunen and D. Mauldin, *Some additive properties of sets of real numbers*, Fund. Math. 93 (1981), 187–199.

D. H. Fremlin, *Consequences of Martin's Axiom*, Cambridge Univ. Press, 1984.

F. Galvin and A. W. Miller, *γ-sets and other singular sets of real numbers*, Topology Appl., 17 (1984), 145–155.

F. Galvin and K. Prikry, *Borel sets and Ramsey's Theorem*, J. Symbolic Logic, 38, (1973), 193–198.

J. Gerlits, *Some properties of $C(X)$*, II, Topology Appl. (1983), 255–262.

J. Gerlits and Zs. Nagy, *Some properties of $C(X)$*, I, Topology Appl. 14 (1982), 151–161.

W. Hurewicz, *Perfekte Teile von Punktmengen*, Fund. Math. 12 (1928), 78–109.

— *Über Folgen stetiger Funktionen*, Fund. Math. 9 (1927), 193–204.

— *Über eine Verallgemeinerung des Borelschen Theorems*, Math. Zeit. 24 (1925), 401–421.

K. Kunen and A. W. Miller, *Borel and projective sets from the point of view of compact sets*, Math. Proc. Comb. Phil. Soc., 94 (1983), 399–409.

C. Kuratowski, *Topology*, vol. I Academic Press, New York 1966.

R. Laver, *On the consistency of Borel's conjecture*, Acta Math. 137 (1976), 151–169.

M. K. Menger, *Einige Überdeckungssätze der Punktmengenlehre*, Sitzungsberichte der Wiener Akademie, 133 (1924), 421–444.

A. W. Miller, *Special Subsets of the Real line*, in *Handbook of Set-Theoretic Topology*, ed. by K. Kunen and J. E. Vaughan, Elsevier Science Publishers, 1984, 203–233.

— *A characterization of the least cardinal for which the Baire category theorem fails*, Proc. Amer. Math. Soc. 86 (1982), 498–502.

— *Some properties of measure and category*, Trans. Amer. Math. Soc. 266 (1981), 93–114, Corrections, Trans. Amer. Math. Soc. 271, 347–348.

— *On the length of Borel hierarchies*, Ann. Math. Logic 16 (1979), 233–267.

F. Rothberger, *On the property C and a problem of Hausdorff*, Canad. J. Math. 4 (1952), 111–116.

— *Sur les familles indénombrables des suites de nombres naturels et les problèmes concernant la propriété C*, Proc. Camb. Phil. Soc. 37 (1941), 109–126.

— *Eine Verschärfung der Eigenschaft C*, Fund. Math. 30 (1938), 50–55.

W. Sierpiński, *Remarque sur le problème de l'invariance topologique de la propriété (C)*, Fund. Math. 30 (1938), 56–68.

— *Problème 67*, Fund. Math. 25 (1935), 579.

— *Hypothèse du Continu*, Monografje Matematyczne, Warszawa-Lwów 1934.

— *Sur un ensemble non dénombrable, donc toute image continue est de mesure nulle*, Fund. Math. 11 (1928), 301–304.

— *Sur un problème de M. Menger*, Fund. Math. 8 (1926), 223–224.

DEPT. OF MATH.
UNIV. of WISC.
Madison WI 53706
USA

DEPT. OF MATH.
UNIV. OF ESSEX
Colchester, Essex C043SQ
GB

Received 4 November 1985;
in revised form 4 June 1986