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ON THE LENGTH OF BOREL HIERARCHIES

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0. Introduction

For any separable metric space X and α with $1 \leq \alpha \leq \omega_1$ define the Borel classes $\boldsymbol{\Sigma}^0_\alpha$ and $\boldsymbol{\Pi}^0_\alpha$. Let $\boldsymbol{\Sigma}^0_1$ be the class of open sets and for $\alpha > 1$ $\boldsymbol{\Sigma}^0_\alpha$ is the class of countable unions of elements of $\bigcup \{\boldsymbol{\Pi}^0_\beta : \beta < \alpha\}$ where $\boldsymbol{\Pi}^0_\beta = \{X - A : A \in \boldsymbol{\Sigma}^0_\beta\}$. Hence $\boldsymbol{\Sigma}^0_1 = \text{open} = G$, $\boldsymbol{\Pi}^0_1 = \text{closed} = F$, $\boldsymbol{\Sigma}^0_2 = F_\sigma$, $\boldsymbol{\Pi}^0_2 = G_\delta$, etc. Note that $\boldsymbol{\Sigma}^0_{\omega_1} = \boldsymbol{\Pi}^0_{\omega_1} =$ set of all Borel in X subsets of X. The Baire order of X (ord (X)) is the least $\alpha \leq \omega_1$ such that every Borel in X subset of X is $\boldsymbol{\Sigma}^0_\alpha$ in X. Since the Borel subsets of X are closed under complementation we could equally well have defined ord (X) in terms of $\boldsymbol{\Pi}^0_\alpha$ in X or $\boldsymbol{\Delta}^0_\alpha = \boldsymbol{\Pi}^0_\alpha \cap \boldsymbol{\Sigma}^0_\alpha$ in X. Note also that for $X \subseteq \mathbb{R}$ (the real numbers) ord (X) is the least α such that for every Borel set A in $\mathbb{R}$ there is a $\boldsymbol{\Sigma}^0_\alpha$ in $\mathbb{R}$ set B such that $A \cap X = B \cap X$. Also note that ord $(X) = 1$ iff X is discrete, ord $(Q) = 2$ where Q is the space of rationals, and in general for X a countable metric space ord $(X) \leq 2$ since every subset of X is $\boldsymbol{\Sigma}^0_2(F_\sigma)$ in X.

It is a classical theorem of Lebesgue (see [11]) that for any uncountable Polish (separable and completely metrizable) space ord $(X) = \omega_1$. The same is true for any uncountable analytic ($\boldsymbol{\Sigma}^1_1$) space X since X has a perfect subspace (see [11]) and Borel hierarchies relativize.

The Baire order problem of Mazurkiewicz (see [19]) is: for what ordinals α does there exist $X \subseteq \mathbb{R}$ such that ord $(X) = \alpha$. Banach conjectured (see [29]) that for any uncountable $X \subseteq \mathbb{R}$ the Baire order of X is ω_1. In Section 3 we review the classically known results of Sierpinski, Szpilrajn, and Poprougenko. We show that it is consistent with ZFC that for each $\alpha \leq \omega_1$ there is an $X \subseteq \mathbb{R}$ with ord $(X) = \alpha$. In fact, we prove a theorem of Kunen's that CH implies this. We also show that Banach's conjecture is consistent with ZFC.

Given a set X and R a family of subsets of X ($R \subseteq P(X)$) define for every $\alpha \leq \omega_1$ $R_\alpha \subseteq P(X)$ as follows. Let $R_0 = R$ and for each $\alpha > 0$ if α is even (odd) let R_α be the family of countable intersections (unions) of elements of $\bigcup \{R_\beta : \beta < \alpha\}$. Generalizing Mazurkiewicz's question Kolmogorov (see [8]) asked: for what ordinals α does there exist X and $R \subseteq P(X)$ such that α is the least such

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that $R_\alpha = R_{\omega_1}$. Kolmogorov's question can be generalized by replacing $P(X)$ by an arbitrary σ-algebra (a countably complete boolean algebra). In Section 2 we prove that for any $\alpha \leq \omega_1$ there is a complete boolean algebra with the countable chain condition which is countably generated in exactly α steps. This answers a question of Tarski who had noticed that the boolean algebras Borel(2^ω) modulo the ideal of meager sets and Borel(2^ω) modulo the ideal of measure zero sets are countably generated in exactly one and two steps respectively (see [4]). Theorem 12 which is due to Kunen shows that the same answer to Kolmogorov's problem (every $\alpha \leq \omega_1$) follows from the solution of Tarski's problem.

Let $R = \{A \times B : A, B \subseteq 2^\omega\}$. In Section 4 we show that for any α, $2 \leq \alpha < \omega_1$, it is consistent with ZFC that α is the least ordinal such that R_α is the set of all subsets of $2^\omega \times 2^\omega$. This answers a question of Mauldin [1].

For $\alpha \leq \omega_1$ a set $X \subseteq 2^\omega$ is a Q_α set iff every subset of X is Borel in X and ord $(X) = \alpha$. It is shown that it is consistent with ZFC that for every $\alpha < \omega_1$ there is a Q_α set. In Section 4 we also show that there are no Q_{ω_1} sets. However, we do show that it is consistent with ZFC that there is an $X \subseteq 2^\omega$ with ord $(X) = \omega_1$ and every X-projective set is Borel in X. This answers a question of Ulam [31, p.10].

Also in Section 4 we show that it is relatively consistent with ZFC that the universal $\mathbf{\Sigma}^1_1$ set is not in R_{ω_1} confirming a conjecture of Mansfield [13] who had shown that the universal $\mathbf{\Sigma}^1_1$ set is never in the σ-algebra generated by the rectangles with $\mathbf{\Sigma}^1_1$ sides.

Given $R \subseteq P(X)$ let $K(R)$ (the Kolmogorov number of R) be the least α such that $R_\alpha = R_{\omega_1}$. It is an exercise to show that for $\alpha = 0, 1$, or 2 there is an $R \subseteq P(\{0, 1\})$ with $K(R) = \alpha$.

Proposition 1. *Given* $R \subseteq P(X)$ *then* (a) *if* R *is finite or* X *is countable, then* $K(R) \leq 2$, *and* (b) *there exists* $S \subseteq P(Y)$ *such that cardinality of* S *and* Y *is* $\leq 2^{\aleph_0}$ *and* $K(R) = K(S)$.

Proof. (a) Note

$$\bigcup_{\alpha < \alpha_0} \bigcap_{\beta < \beta_0} \bigcup_{\gamma < \gamma_0} A_{\alpha,\beta,\gamma} = \bigcap_{f:\alpha_0 \to \beta_0} \bigcup_{\alpha < \alpha_0} \bigcup_{\gamma < \gamma_0} A_{\alpha, f(\alpha), \gamma}$$

If R is finite or X countable, then $\bigcap_{f:\alpha_0 \to \beta_0}$ can always be taken to be a countable intersection.

(b) Let V_α be the sets of rank less than α. Choose α a limit ordinal of uncountable cofinality so that $R, X \in V_\alpha$. Let (M, ε) be an elementary substructure of (V_α, ε) containing R and X such that $M^\omega \subseteq M$ and $|M| \leq 2^{\aleph_0}$. Now let $Y = X \cap M$ and $S = \{A \cap Y : A \in R \cap M\}$.

Mazurkiewicz's problem is equivalent to Kolmogorov's problem for R a countable field of sets (that is closed under finite intersection and complementation).

Proposition 2. (Sierpinski [23] also in [30]). *Given $R \subseteq P(X)$ a countable field of sets there exists $Y \subseteq 2^\omega$ such that $K(R) = \text{ord}(Y)$. (That is we may reduce to considering subsets Y of 2^ω and relativizing the usual Borel hierarchy on 2^ω to Y.)*

Proof. Let $R = \{A_n : n \in \omega\}$ and define $F : X \to 2^\omega$ by $F(x)(n) = 1$ iff $x \in A_n$. Put $Y = F''X$.

Define $K = \{\beta : 2 \leq \beta < \omega_1$ and there is $X \subseteq \omega^\omega$ uncountable with $\text{ord}(X) = \beta\}$. What can K be?

Proposition 3. *K is a closed subset of ω_1.*

Proof. Given $A \subseteq \omega^\omega$ and $n \in \omega$ define $nA = \{x \in \omega^\omega : x(0) = n$ and $\exists y \in A\ \forall n\ (x(n+1) = y(n))$. If $X = \bigcup_{n<\omega} nX_n$, then it is readily seen that $\text{ord}(X) = \sup\{\text{ord}(X_n) : n \in \omega\}$.

Note that K is the same set of ordinals if we replace ω^ω by $\mathbb{R}$ the real numbers or 2^ω. This is true for $\mathbb{R}$ because if $X \subseteq \mathbb{R}$ and $\mathbb{R} - X$ is not dense, then X contains a nonempty interval, hence $\text{ord}(X) = \omega_1$; but $\mathbb{R} - X$ dense means we may as well assume $X \subseteq \text{irrationals} \cong \omega^\omega$.

In the definition of $K(R) = \omega$ for $R \subseteq P(X)$ we ignored the possibility that the hierarchy on R might have exactly ω levels, i.e. $R_{\omega_1} = \bigcup\{R_n : n < \omega\}$ but for all $n < \omega$ $R_n \neq R_{\omega_1}$. In fact a Borel hierarchy of length less than ω_1 must have a top level.

Proposition 4. *If $R \subseteq P(X)$ is a field of sets, λ is a countable limit ordinal, and $R_{\omega_1} = \bigcup\{R_\alpha : \alpha < \lambda\}$, then there is $\alpha < \lambda$ such that $R_\alpha = R_{\omega_1}$.*

Proof. Using the proof of Proposition 2 we can assume $X \subseteq 2^\kappa$ for some κ and $R = \{[s] \cap X : \exists D \in [\kappa]^{<\omega}(s \in 2^D)\}$ where $[s] = \{(f \in 2^\kappa : f$ extends $s\}$. For each A in R_{ω_1} there is $T \subseteq \kappa$ countable such that for any f and g in X if $f \upharpoonright T = g \upharpoonright T$, then $f \in A$ iff $g \in A$. In this case we say T supports A. Choose $T \subseteq \kappa$ countable so that for any $D \subseteq T$ finite and $s : D \to 2$ if $\text{ord}(X \cap [s]) = \lambda$, then for any $\alpha < \lambda$ there is an $A \subseteq [s]$ in $R_{\alpha+1} - R_\alpha$ such that T supports A. By taking an autohomeomorphism of 2^κ we may assume $T = \omega$. Define L to be $\{s \in 2^{<\omega} : \text{ord}([s] \cap X) = \lambda\}$.

Claim. *For any s in L there are t and $\hat{t}$ in L incompatible extensions of s.*

Proof. Without loss of generality assume $s = \emptyset$ and there is $f \in 2^\omega$ such that for every $s \in L$ $s \subseteq f$. For each $n < \omega$ define t_n in 2^{n+1} by $t_n(m) = f(m)$ for $m < n$ and $t_n(n) = 1 - f(n)$. Then $[f] \cup \bigcup\{[t_n] : n < \omega\}$ is a disjoint union covering 2^κ. If there is a $\beta_0 < \lambda$ such that for all $n < \omega$ $\text{ord}([t_n] \cap X) < \beta_0$, then for all A in R_{ω_1} supported by ω A is in R_{β_0+1}. This is because $A \cap [f] = \emptyset$ or $X \cap [f] \subseteq A$. But this contradicts the choice of ω.

On the other hand, if there is no such bound β_0, choose $Z_n \subseteq [t_n]$ with $Z_n \in R_{\omega_1}$ so that for every $\beta < \lambda$ there is $n < \omega$ with $Z_n \notin R_\beta$. But then $\bigcup \{Z_n : n < \omega\}$ is not in $\bigcup \{R_\beta : \beta < \lambda\}$. This proves the claim and this last argument also proves the proposition from the claim.

Remark. If $R \subseteq P(X)$ and $R_{\omega_1} = \bigcup \{R_n : n < \omega\}$ and there is $n_0 < \omega$ such that $\{X - A : A \in R\} \subseteq R_{n_0}$, then there is $n_1 < \omega$ such that $R_{n_1} = R_{\omega_1}$. Willard [32] shows that for any $\alpha < \omega_1$ there are R and X with $R \subseteq P(X)$ such that α is the least ordinal such that $\{X - A : A \in R\} \subseteq R_\alpha$.

1. Some basic definitions and lemmas

For $T \subseteq \omega^{<\omega}$ T is a well-founded tree iff T is a tree (if $t \supseteq s \in T$, then $t \in T$) and is well-founded (for any $f \in \omega^\omega$ there is an $n < \omega$ such that $f \upharpoonright n \notin T$). For $s \in T$ define $|s|_T$ (the rank of s in T) by $|s|_T = \sup\{|t|_T + 1 : s \subseteq t \in T\}$. Often we drop T and let $|s| = |s|_T$. T is normal of rank α means that:

(a) T is a well-founded tree;

(b) $|\emptyset| = \alpha$ ($\emptyset$ is the empty sequence);

(c) ($s \in T$ and $|s| > 0$) $\to$ ($\forall i < \omega$ ($s^\frown i \in T$));

(d) ($s \in T$ and $|s| = \beta + 1$) $\to$ ($\forall i < \omega$ ($|s^\frown i| = \beta$));

(e) ($s \in T$ and $|s| = \lambda$ where λ is a limit ordinal) $\to$ ($\forall \beta < \lambda$ $\{i : |s^\frown i| < \beta\}$ is finite and $\forall i < \omega$ $|s^\frown i| \geq 2$).

Note that for any $n < \omega$ the tree $\omega^{\leq n}$ is normal of rank n. If α_n for $n < \omega$ are strictly increasing to α (or $\alpha_n = \beta$ where $\alpha = \beta + 1$) and for each $n < \omega$ T_n is normal of rank $\alpha_n \geq 2$, then $T = \{\emptyset\} \cup \{n^\frown s : n < \omega$ and $s \in T_n\}$ is normal of rank α. We often use T_α to denote some fixed normal tree of rank α. Let M be the ground model of ZFC. Working in M for any $\alpha < \omega_1$ and $Y \subseteq X \subseteq \omega^\omega$ define the partial order $\mathbb{P}_\alpha(Y, X)$ (the order is given by inclusion). Fix some T normal of rank α. $p \in \mathbb{P}_\alpha(Y, X)$ iff $p \subseteq (T - \{\emptyset\}) \times (X \cup \omega^{<\omega})$ and (1) through (5) hold.

(1) p is finite.

(2) $|s| = 0$ implies that if $(s, x) \in p$, then $x \in \omega^{<\omega}$ and if $(s, y) \in p$, then $x = y$. (So if $T^* = \{s \in T : |s| = 0\}$, then $p \upharpoonright (T^* \times (X \cup \omega^{<\omega}))$ is a function from a finite subset of T^* into $\omega^{<\omega}$.)

(3) If $|s| > 0$ and $(s, x) \in p$, then $x \in X$.

(4) If s and $s^\frown i \in T$ and $x \in X$, then not both (s, x) and $(s^\frown i, x)$ are in p, or if $|s^\frown i| = 0$, there is no $k \in \omega$ such that both (s, x) and $(s^\frown i, x \upharpoonright k)$ are in p.

(5) If s of length one and $(s, x) \in p$, then x is not in Y.

Let G be $\mathbb{P}_\alpha(Y, X)$-generic over M. Working in $M[G]$ define for each $s \in T$, $G_s \subseteq \omega^\omega$. For $|s| = 0$, let

$$G_s = \{x \in \omega^\omega : \exists t \in \omega^{<\omega}\ t \subseteq x \text{ and } \{(s, t)\} \in G\}.$$

For $|s|>0$, let $G_s = \bigcap \{\omega^\omega - G_{s^\frown i} : i < \omega\}$. Note that for each $s \in T$, $G_s \in \mathbf{\Pi}^0_{|s|}$.

Lemma 5. *For each x in X and s in $T-\{\emptyset\}$ with $|s|>0$ $[x \in G_s$ iff $\{(s,x)\} \in G]$.*

Proof. *Case* 1. $|s|=1$. (This is the argument from almost-disjoint-sets forcing.)

If $x \in G_s$, then $x \notin G_{s^\frown i}$ for all $i \in \omega$. Hence for all k and i in ω $(s^\frown i, x \restriction k) \notin G$. Let $D = \{p : (s,x) \in p$ or there exist k and i such that $(s^\frown i, x\restriction k) \in p\}$. D is dense since if $(s,x) \notin p$ if we let $\{x_1, x_2, \ldots, x_n\} \subseteq X$ be all the elements of ω^ω mentioned in p other than x, we can choose k sufficiently large so that $x \restriction k \neq x_i \restriction k$ for all $i \leq n$. Also we can choose j sufficiently large so that $(s^\frown j)$ is not mentioned in p and then $p \cup \{(s^\frown j, x\restriction k)\} \in (\mathbb{P}_\alpha(Y,X) \cap D)$. Since $G \cap D$ is non-empty and $x \notin G_{s^\frown i}$ all i; we conclude that $(s,x) \in G$.

If $x \notin G_s$, then $x \in G_{s^\frown i}$ for some i. Hence there exist k such that $(s^\frown i, x \restriction k) \in G$ so $(s,x) \notin G$ by clause (4).

Case 2. $|s|>1$.

If $x \in G_s$, then $x \notin G_{s^\frown i}$ for all i, and hence by induction $(s^\frown i, x) \notin G$ for all i. Let $D = \{p : (s,x) \in p$ or there exist i such that $(s^\frown i, x) \in p\}$. D is dense hence $(s,x) \in G$.

If $x \notin G_s$, then $(s^\frown i, x) \in G$ for some i (by induction). Hence $(s,x) \notin G$ by clause (4).

Corollary 6. $G_\emptyset \cap X = Y$ $(\alpha \geq 2)$.

Proof. If $x \in Y$, then for every n, $((n),x) \notin G$ (by clause 5). Hence by Lemma 5 for every n, $x \notin G_{(n)}$ and so $x \in G_\emptyset$. If $x \notin Y$, then $\{p$: there exists n such that $((n),x) \in p\}$ is dense hence there exists n such that $x \in G_{(n)}$ (by Lemma 5) so $x \notin G_\emptyset$.

Remarks: (1) $\mathbb{P}_0(Y,X)$ is trivial (the empty set).

(2) $\mathbb{P}_1(Y,X)$ has nothing to do with X and Y and is isomorphic as a partial order to the usual Cohen partial order for adding a map from ω to ω.

(3) $\mathbb{P}_2(Y,X)$ is another way of viewing Solovay's "almost-disjoint-sets forcing" (see [6]).

Lemma 7. *$\mathbb{P}_\alpha(Y,X)$ has the countable chain condition.*

Proof. Suppose by way of contradiction that there exist F included in $\mathbb{P}_\alpha(Y,X)$ of cardinality $\aleph_1$ of pairwise incompatible conditions. Since there are only countably many finite subsets of T, we may assume there exist $H \subseteq T-\{\emptyset\}$ finite so that every $p \in F$ is included in $H \times (X \cup \omega^{<\omega})$. We may also assume that for every $p \in F$ and $q \in F$ and $s \in H$ with $|s|=0$ and $t \in \omega^{<\omega}$ that $[(s,t) \in p$ iff $(s,t) \in q]$. Now let

$(x_\beta : \beta < \aleph_1)$ be all the elements of X occurring in members of F. For each p in F let $p^* : G_p \to P(H)$ be defined by $G_p = \{\beta :$ there exists s, $(s, x_\beta) \in p\}$ and for $\beta \in G_p$ $p^*(\beta) = \{s : (s, x_\beta) \in p\}$. $\{p^* : p \in F\}$ is a family of $\aleph_1$ incompatible conditions in the partial order $\mathbb{Q}$, where $\mathbb{Q} = \{p :$ domain of p is a finite subset of $\aleph_1$ and range of p is $P(H)\}$, ordered by inclusion. Since it is well-known that $\mathbb{Q}$ has the countable chain condition we have a contradiction.

Remarks: (1) If $\mathbb{P} = \mathbb{P}_\alpha(Y, X)$ for any α, X, and Y, then $\mathbb{P}$ is absolutely c.c.c. That is to say if $\mathbb{P} \in M \models$ "ZFC", then $M \models$ "$\mathbb{P}$ has c.c.c.". It follows that the direct sum of any combination of the $\mathbb{P}_\alpha$'s has the c.c.c.

(2) We assume the fact that iterated c.c.c. forcing is c.c.c. (Solovay–Tennenbaum [26]) and occasionally use notation and facts from [26].

I would like to prove next an heuristic proposition. Roughly, if we add a generic $\mathbf{\Pi}^0_2$ set, then it will not be $\mathbf{\Sigma}^0_2$. This is a special case of more difficult arguments later with generic $\mathbf{\Pi}^0_\alpha$ sets.

Define $\mathbb{P}$ a partial order: $p \in \mathbb{P}$ iff p is a finite consistent set of sentences of the form "$[s] \subseteq G_n$", "$x \notin G_n$", or "$x \in \bigcap_{n\in\omega} G_n$" (where $s \in \omega^{<\omega}$ and $x \in \omega^\omega$). Order $\mathbb{P}$ by inclusion. Any G $\mathbb{P}$-generic determines a $\mathbf{\Pi}^1_2$ set $\bigcap_{n\in\omega} G_n$.

Proposition. *If G is $\mathbb{P}$-generic over M (transitive countable model of ZFC), then*

$$M[G] \models \text{“}\forall F \in F_\sigma\Big(F \cap M \neq \bigcap_{n\in\omega} G_n \cap M\Big)\text{”}.$$

Proof. Suppose not and let $p \in G$ and C_n be names such that $p \Vdash$ "C_n is closed" and such that

$$p \Vdash \text{“}\bigcup_{n\in\omega} C_n \cap M = \bigcap_{n\in\omega} G_n \cap M\text{”}.$$

It is easily seen that $\mathbb{P}$ has c.c.c. (see the proof of Lemma 7). Thus working in M we can find $Q \subseteq \mathbb{P}$ countable such that for any $\hat{G}$ $\mathbb{P}$-generic, $n \in \omega$, and $s \in \omega^{<\omega}$, if $M[\hat{G}] \models$ "$[s] \cap C_n = \emptyset$", then $\exists q \in Q \cap \hat{G}$ such that $q \Vdash$ "$[s] \cap C_n = \emptyset$". Since Q is countable, we can find $z \in \omega^\omega \cap M$ not mentioned in p or any condition in Q. Since

$$p \cup \Big\{z \in \bigcap_{n\in\omega} G_n\Big\} \Vdash \text{“}z \in \bigcup_{n\in\omega} C_n\text{”}$$

we can find $\bar{n}\in\omega$ and $\hat{p}\geqslant p$ and not mentioning z so that

$$\hat{p}\cup\left\{z\in\bigcap_{n\in\omega}G_n\right\}\Vdash\text{“}z\in C_{\bar{n}}\text{”},$$

because the only other way to mention z is “$z\notin G_n$”. By taking $\bar{m}$ large enough $\hat{p}\cup\{z\notin G_{\bar{m}}\}$ will be consistent, and since it extends p it forces “$z\notin C_{\bar{n}}$”. Let G be $\mathbb{P}$-generic with $\hat{p}\cup\{z\notin G_{\bar{m}}\}$ in G. Let $k\in\omega$ and $q\in G\cap Q$ be so that $q\Vdash$“$[z\restriction k]\cap C_{\bar{n}}=\emptyset$”. But $\hat{p}\cup q\cup\{z\in\bigcap_{n\in\omega}G_n\}$ is consistent because $q\in Q$ and so doesn't mention z. This is a contradiction since $q\Vdash$“$z\notin C_n$” and

$$\hat{p}\cup\left\{\text{“}z\in\bigcup_{n\in\omega}G_n\text{”}\right\}\Vdash\text{“}z\in C_{\bar{n}}\text{”}.$$

Define for $F\subseteq\omega^\omega$ and $p\in\mathbb{P}=\mathbb{P}_\alpha(Y,X)$,

$$|p|(F)=\max(\{|s|:\text{there is } x\notin F \text{ with } (s,x)\in p\}).$$

This is called the rank of p over F.

Lemma 8. *For all $\beta\geqslant 1$ and $p\in\mathbb{P}$ there is $\hat{p}\in\mathbb{P}$ compatible with p and $|\hat{p}|(F)<\beta+1$ so that for any $q\in\mathbb{P}$ with $|q|(F)<\beta$, if $\hat{p}$ and q are compatible, then p and q are compatible.*

Proof. First find an extension $p_0\geqslant p$ so that for all $(s,x)\in p$ and $i<\omega$ if $|s|=\lambda$ is a limit ordinal and $|s^\frown i|\leqslant\beta+1<\lambda$ (there are only finitely many such $s^\frown i$), then there is a $j<\omega$ such that $(s^\frown i^\frown j,x)\in p_0$. Now let $\hat{p}=\{(s,x)\in p_0:|s|<\beta+1$ or $x\in F\}$. We check that $\hat{p}$ has the requisite property. Suppose p and q are incompatible, $\hat{p}$ and q are compatible,and $|q|(F)<\beta$. Since $\beta\geqslant 1$ for all $(s,x)\in p$ if $|s|\leqslant 1$, then $(s,x)\in\hat{p}$, hence since $\hat{p}$ and q are compatible there are $s,t\in\omega^{<\omega}$, $i<\omega$, and $x\in\omega^\omega$ such that $(s,x)\in p$, $(t,x)\in q$, and $s=t^\frown i$ or $t=s^\frown i$.

Case 1. If $x\in F$ or $|s|<\beta+1$, then $(s,x)\in\hat{p}$ and so $\hat{p}$ and q are incompatible.

Case 2. If $x\notin F$ and $|s|\geqslant\beta+1$, then by definition of $|q|(F)<\beta$, $|t|<\beta$. So $t=s^\frown i$. If $|s|=\gamma+1$ for some γ, then $|t|=\gamma\geqslant\beta$, contradiction. If $|s|=\lambda$ is an infinite limit ordinal, then by the construction of p_0 there is $j<\omega$ with $(t^\frown j,x)\in p_0$ and hence $(t^\frown j,x)\in\hat{p}$ and so q and $\hat{p}$ are incompatible.

2. Boolean algebras

For $\mathbb{B}$ a complete boolean algebra, C included in $\mathbb{B}$, and $\alpha \geqslant 1$ define $\Sigma_\alpha(C)$, $\Pi_\alpha(C)$:

$$\Sigma_1(C) = \left\{\sum S : S \subseteq C\right\},$$

$$\Sigma_\alpha(C) = \left\{\sum S : S \subseteq \bigcup_{\beta<\alpha} \Pi_\beta(C)\right\} \quad \text{for} \quad \alpha > 1,$$

and

$$\Pi_\alpha(C) = \{-a : a \in \Sigma_\alpha(C)\}$$

Define $K(\mathbb{B})$ to be the least ordinal α such that there exists a countable C included in $\mathbb{B}$ with $\Sigma_\alpha(C) = \mathbb{B}$.

Theorem 9. *For each $\alpha \leqslant \omega_1$ there exists a complete boolean algebra $\mathbb{B}$ with countable chain condition and $K(\mathbb{B}) = \alpha$.*

Proof. For $\alpha = 0$ take $\mathbb{B}$ to be any finite boolean algebra. For $\alpha = 1$ take $\mathbb{B}$ to be $(P(\omega), \cap, \cup)$ (or more appropriately the regular open subsets of ω^ω since this corresponds to Cohen real forcing).

For $\alpha, 2 \leqslant \alpha < \omega_1$, $\mathbb{B}$ will be the complete boolean algebra associated with $\mathbf{\Pi}^0_\alpha$-forcing. Let $\mathbb{P} = \mathbb{P}_\alpha(\emptyset, X)$. Given a partial order $\mathbb{P}$ there is a canonical way of constructing a complete boolean algebra $\mathbb{B}$ in which $\mathbb{P}$ is densely embedded (see [5]). Let $[p]$ denote the image of $p \in \mathbb{P}$ under this embedding. If $p \geqslant q$, then $[p] \leqslant [q]$. For every $a \in \mathbb{B}$ if $a \neq 0$, then there is a $p \in \mathbb{P}$ such that $[p] \leqslant a$.

Lemma 10. *Suppose $F \subseteq X$ and $C = \{[p] : p \in \mathbb{P}$ and $|p|(F) = 0\}$. For any $\beta \geqslant 1$, $p \in \mathbb{P}$, and a in $\Sigma_\beta(C)$, if $[p] \leqslant a$, then there is $q \in \mathbb{P}$ such that $|q|(F) < \beta$, q and p are compatible, and $[q] \leqslant a$.*

Proof. The proof is by induction on β.

Case 1. $\beta = 1$. Suppose $a = \sum\{[q] : q \in \Gamma\}$ for some $\Gamma \subseteq C$. If $[p] \leqslant a$, then for some $q \in \Gamma$, p and q are compatible.

Case 2. β a limit ordinal. Suppose $a = \sum\{b : b \in \Gamma\}$ for some $\Gamma \subseteq \bigcup\{\Sigma_\alpha(C) : \alpha < \beta\}$. Then there is $\hat{p} \geqslant p$ and $b \in \Gamma \cap \Sigma_\alpha(C)$ for some $\alpha < \beta$ so that $[\hat{p}] \leqslant b$. Now apply the inductive hypothesis to $\hat{p}$.

Case 3. $\beta + 1$. Suppose $[p] \leqslant \sum\{b : b \in \Gamma\}$ for some $\Gamma \subseteq \Pi_\beta(C)$. Choose $\hat{p} \leqslant p$ so that for some $b \in \Gamma$, $[\hat{p}] \leqslant b$. By Lemma 8 of Section 1, there exists q compatible with $\hat{p}$ with $|q|(F) < \beta + 1$ and for any r with $|r|(F) < \beta$, if r and q are compatible, then r and $\hat{p}$ are compatible. This q works since if $[q] \nleqslant b$, then there exists $q_0 \geqslant q$ with $[q_0] \leqslant -b$. Since $-b \in \Sigma_\beta(C)$ by induction there is q_1 compatible with q_0 with

$|q_1|(F)<\beta$ and $[q_1]\leq -b$. But then q_1 would be compatible with $\hat{p}$, contradicting $[\hat{p}]\leq b$.

Now if $X=\omega^\omega$, for example, the lemma shows that $\mathbb{B}$ cannot be generated by a set of size less than the continuum in fewer than α steps. For suppose $D\subseteq\mathbb{B}$ has cardinality less than $|\omega^\omega|$, then there exists $F\supseteq\omega^\omega$ with $X-F\neq\emptyset$ and $D\subseteq \boldsymbol{\Sigma}_1\{[p]:|p|(F)=0\}$. Let $\beta<\alpha$, $z\in X-F$, and $s\in T-\{\emptyset\}$ with $|s|_T=\beta$ (where T is the normal α-tree used in the definition of $\mathbb{P}_\alpha(\emptyset, X)$). $[\{(s,z)\}]$ is not in $\Sigma_\beta(D)$. Because if it were it would be in $\boldsymbol{\Sigma}_\beta(C)$ and so by the lemma there exists q with $|q|(F)<\beta$ and $[q]\subseteq[\{(s,z)\}]$. But since $|s|_T=\beta$ and $z\notin F$ we know $(s,z)\notin q$. Thus there are n (and m) such that $q\cup\{(s^\frown n, z)\}$ ($q\cup\{(s^\frown n, z\restriction m)$ in case $|s|_T=1$) is in $\mathbb{P}$, but this is a contradiction.

Next we show $\mathbb{B}$ is countably generated in α steps. Let $\hat{C}=\{[p]:|p|(\emptyset)=0\}$.

Claim. *For all $x\in X$ and $s\in T-\{\emptyset\}$ if $|s|_T=\beta\geq 1$, then $[\{(s,x)\}]$ is in $\Pi_\beta(\hat{C})$.*

Proof. If $|s|_T=1$, then

$$[\{(s,x)\}]=\prod\{-[\{(s^\frown n, x\restriction m)\}]:n,m\in\omega\}.$$

If $|s|>1$, then

$$[\{(s,x)\}]=\prod\{-[\{(s^\frown n, x)\}]:n\in\omega\}.$$

For $A\in\mathbb{B}$, $-A=\{p\in\mathbb{P}:[p]\cap A=\emptyset\}$. If $(s,x)\in p$, then $[p]\cap[\{(s^\frown n,x)\}]=\emptyset$ all n. On the other hand if $[p]\cap[\{(sn,x)\}]=\emptyset$ for all n, then easily $(s,x)\in p$.

Now for any $p\in\mathbb{P}$ $[p]=\prod\{[\{(s,x)\}]:(s,x)\in p\}$, so $[p]\in\Sigma_\alpha(\hat{C})$. For any $A\in B$ $A=\sum\{[p]:p\in A\}$ so $A\in\Sigma_\alpha(\hat{C})$. Thus $K(\mathbb{B})\leq\alpha$.

We are now ready to consider the case of $\alpha=\omega_1$. Let $\mathbb{P}=\sum_{\alpha<\omega_1}\mathbb{P}_\alpha(\emptyset,\omega^\omega)$. Now the complete boolean algebra associated with $\mathbb{P}$ does take ω_1 steps to close (for suitable generators), however, $\mathbb{P}$ is not countably generated. So we do as follows: Let $(x_\alpha:\alpha<\omega_1)$ be any set of ω_1 distinct elements of ω^ω. Let $*:\omega^{<\omega}\times\omega^{<\omega}\to\omega$ be a 1–1 map. Let T_α be the normal tree of rank α used in the construction of $\mathbb{P}_\alpha=\mathbb{P}_\alpha(\emptyset,{}^\omega\omega)$. Any G which is $\mathbb{P}_\alpha$-generic is determined by $G\cap\{(s,t)\in\mathbb{P}_\alpha:|s|_{T_\alpha}=0 \text{ and } t\in\omega^{<\omega}\}$. That is a map y from $T_\alpha^*=\{s\in T_\alpha:|s|_{T_\alpha}=0\}$ to $\omega^{<\omega}$. Now imagine G $\mathbb{P}$-generic and let $y_\alpha:T_\alpha^*\to\omega^{<\omega}$ be the maps determined by G. Let $Y=\{(*(s,t))^\frown x_\alpha:y_\alpha(s)=t \text{ and } \alpha<\omega_1\}$. Form in the generic extension $\mathbb{P}_2(\omega^\omega-Y,\omega^\omega)=Q$ (in both cases we mean ω^ω formed in the ground model). The partial order we are interested in is $R=\mathbb{P}*Q$. $\mathbb{P}*Q=\{(p,q):p\in\mathbb{P}$ and $p\Vdash$"$q\in Q$"$\}$. $(\hat{p},\hat{q})\geq(p,q)$ iff ($\hat{p}\geq p$ and $\hat{q}\geq q$). $p\Vdash$"$q\in Q$" just in case whenever $((n),(*(s,t)^\frown x_\alpha)$ is in q, then $(s,t)\in p(\alpha)$. Now let $\mathbb{B}$ be the complete boolean algebra associated with R. Since R has the countable chain condition so does $\mathbb{B}$.

Claim. *$\mathbb{B}$ is countably generated.*

Proof. The idea is that once you know what the real is gotten by Q you know all the reals gotten by $\mathbb{P}$ — and hence everything. Let $C=\{[\langle\emptyset, q\rangle]: |q|(\emptyset)=0\}$. Then C is countable and generates $\mathbb{B}$.

For $C \subseteq \omega^\omega$ and $(p, q) \in R$ define

$$|(p, q)|(C)=\max\{|s|_{T\alpha}: \text{there exists } x \notin C,\ (s, x) \in p(\alpha) \text{ and } \alpha<\omega_1\}$$

Lemma 11. *Given $F \subseteq \omega^\omega$ $\forall p \in R$ $\forall \beta \geq 1$ $\exists \hat{p} \in R$ compatible with p, $|\hat{p}|(F)<\beta+1$ and $\forall q$ $|q|(F)<\beta$ (if $\hat{p}, q$ compatible, then p, q are compatible).*

Proof. This is proved similarly to Lemma 8. Given $p=\langle p_0, p_1\rangle$ extend each $p_0(\alpha) \leq p_0^1(\alpha)$ as in Lemma 8, then take $\hat{p}=\langle\hat{p}_0, \hat{p}_1\rangle$, $\hat{p}_1=p_1$, $\hat{p}_0(\alpha)=\{(s, x) \in p_0^1(\alpha):|s|<\beta+1 \text{ or } x \in C\}$. Note that $\hat{p}_0 \Vdash$"$\hat{p}_1 \in Q$" because requirements in Q are decided by rank zero condition in $\mathbb{P}$.

From this lemma it is easily shown as before that $K(\mathbb{B}) \geq \omega_1$. Since $\mathbb{B}$ is countably generated and has the countable chain condition we have $K(\mathbb{B}) \leq \omega_1$, hence $K(\mathbb{B})=\omega_1$.

For any σ-complete boolean algebra $\mathbb{B}$ the Sikorski–Loomis theorem [25, p. 93] says that $\mathbb{B}$ is isomorphic to a σ-field of subsets of some X modulo a σ-ideal of subsets of X.

Theorem 12 (Kunen). $\forall \alpha \leq \omega_1$ $\exists X$, *R with $R \subseteq P(X)$ such that $K(R)=\alpha$.*

Proof. By the Sikorski–Loomis theorem and Theorem 9 we can find $\hat{R}$, X, and I with $\hat{R} \subseteq P(X)/I$ where I is a σ-ideal and α is the least ordinal such that $\hat{R}_\alpha=\hat{R}_{\omega_1}$. Define $R \subseteq P(X)$ by ($A \in R$ iff $A/I \in \hat{R}$). It is easily shown by induction on $\beta \leq \omega_1$ that ($A \in R_\beta$ iff $A/I \in \hat{R}_\beta$). Hence we have $K(R)=\alpha$.

Let $\mathbb{B}_M$ be the complete boolean algebra Borel(2^ω) modulo the ideal of meager sets.

Theorem 13. *For any α, $1 \leq \alpha<\omega_1$, there is a countable $C \subseteq \mathbb{B}_M$ which is closed under finite conjunction and complementation so that α is the least ordinal such that $\Sigma_\alpha(C)=\mathbb{B}_M$.*

Proof. Let $x \in \omega^\omega$ be arbitrary and $\mathbb{B}$ be the complete boolean algebra associated with $\mathbb{P}_\alpha(\emptyset,\{x\})$. Note that if $|p|(\emptyset)=0$, then $-[p]=\sum\{[q]:|q|(\emptyset)=0$ and q is incompatible with $p\}$. Let C be the closure of $\{[p]:|p|(\emptyset)=0\}=\hat{C}$ under finite boolean combinations. Note that since $\hat{C}$ is closed under finite intersections and

$-[p]$ is in $\Sigma_1(\hat{C})$ for any p in $\hat{C}$, we have that $\Sigma_\beta(C)=\Sigma_\beta(\hat{C})$ for all $\beta\geq 1$. By Lemma 10 α is the least such that $\Sigma_\alpha(\hat{C})=\mathbb{B}$. Since $\mathbb{P}_\alpha(\emptyset,\{x\})$ is countable and separative, $\mathbb{B}$ is separable and nonatomic and hence isomorphic to $\mathbb{B}_M$.

Remark. The theorem above is false for $\alpha=\omega_1$ since for any countable C which generates $\mathbb{B}_M$, at some countable stage every clopen set is generated and after one more step all of $\mathbb{B}_M$.

3. Countably generated Borel hierarchies

A set $X\subseteq 2^\omega$ is called a Luzin set iff X is uncountable and for every meager M, $M\cap X$ is countable. The analagous definition with measure zero in place of meager is of a Sierpinski set [30]. For I a σ-ideal in Borel(2^ω) say X is I-Luzin iff $[\forall A\in \text{Borel}(2^\omega)\ (|A\cap X|<2^{\aleph_0}$ iff $A\in I)]$. The following theorem was first proved by Luzin [12] assuming I is the ideal of meager sets and CH.

Theorem 14. (MA). *If I is an ω_1 saturated σ-ideal in Borel(2^ω) containing singletons, then there exists an I-Luzin set.*

Proof. Let $\kappa=|2^\omega|$, $\{A_\alpha:\alpha<\kappa\}=I$, and $\{B_\alpha:\alpha<\kappa\}=\text{Borel}(2^\omega)-I$ each set repeated κ-many times. Choose x_α for $\alpha<\kappa$, so that for every α x_α is in $B_\alpha-(\bigcup\{A_\beta:\beta<\alpha\}\cup\{x_\beta:\beta<\alpha\})$. Clearly if this can be done, then $X=\{x_\alpha:\alpha<\kappa\}$ is I-Luzin. If $\kappa=\omega_1$, then it is trivial, and if MA, then this follows from [14, Lemma 1, p. 158].

The next theorem was proved by Poprougenko [19] and Sierpinski (see [29]).

Theorem 15. *If $X\subseteq 2^\omega$ is a Luzin set, then* $\text{ord}(X)=3$.

Proof. Since every Borel set B has the property of Baire, $B=G\Delta M$ where G is open and M is meager. But $M\cap X=F$ is countable hence F_σ, so $B\cap X=(G\Delta F)\cap X$ showing $\text{ord}(X)\leq 3$. Now choose $s\in 2^{<\omega}$ so that $[s]\cap X$ is uncountable and dense in $[s]$. If $D\subseteq[s]\cap X$ is countable and dense in $[s]$, then $D\neq G\cap X$ for all $G\in G_\delta$, so $\text{ord}(X)\geq 3$.

A modern example of a Luzin set arises when one adds an uncountable (in M) number of product generic Cohen reals X to M a countable transitive model of ZFC. $M[X]\vdash$"X is a Luzin set". See also Kunen [10] for more on Luzin sets and MA.

In contrast to the boolean algebras Szpilrajn [29] showed:

Theorem 16. *If $X\subseteq 2^\omega$ is a Sierpinski set, then* $\text{ord}(X)=2$.

Proof. The proof is similar except note that any measurable set is the union of an F_σ set and a set of measure zero.

The following theorem generalizes these classical results using a lemma of Silver (see [14, p. 162]) that assuming MA every $X \subseteq 2^\omega$ with $|X| < |2^\omega|$ is a Q set, i.e. every subset of X is an F_σ in X.

Theorem 17. (MA). *There are uncountable* $X, Y \subseteq 2^\omega$ *such that* $\operatorname{ord}(X) = 3$ *and* $\operatorname{ord}(Y) = 2$.

Proof. Let X be I-Luzin where I is the ideal of meager Borel sets. For any meager set M choose F a meager F_σ with $M \subseteq F$. By Silver's Lemma there exists F_0 an F_σ set such that $F_0 \cap F \cap X = M \cap F \cap X = M \cap X$. Thus every meager set intersected with X is an F_σ set intersected with X and this shows as before $\operatorname{ord}(X) = 3$. For I the ideal of measure zero sets analagous arguments work.

After I had shown that it is consistent with ZFC that $\forall \alpha \leq \omega_1 \, \exists X \subseteq \omega^\omega$ $\operatorname{ord}(X) = \alpha$, Kunen showed that in fact CH implies $\forall \alpha \leq \omega_1 \, \exists X \subseteq \omega^\omega$ $\operatorname{ord}(X) = \alpha$. The following theorem sharpens his result slightly.

Theorem 18. *If there exists a Luzin set, then for any* α *such that* $2 < \alpha \leq \omega_1$ *there is an* $X \subseteq 2^\omega$ *such that* $\operatorname{ord}(X) = \alpha$.

Proof. Let Y be a Luzin set with the property that for every Borel set $A \subseteq 2^\omega$ ($A \cap Y$ is countable iff A is meager). Such a set always exists if a Luzin set does. By Theorem 13 there is a $C \subseteq \mathbb{B}_M$ countable such that C generates $\mathbb{B}_M$ in exactly α steps and C is closed under finite Boolean combinations. Let $C = \{[C_n] : n \in \omega\}$ where the C_n are Borel subsets of 2^ω and $[C_n]$ is the equivalence class modulo meager of C_n. For $x, y \in 2^\omega$ define $x \sim y$ iff for all $n < \omega$ ($x \in C_n$ iff $y \in C_n$). We claim that for each $x \in 2^\omega$ the $\sim$ equivalence class of x is meager. Note that any element of the σ-algebra generated by $\{C_n : n < \omega\}$ is a union of $\sim$ equivalence classes. If some $\sim$ equivalence class E is not meager, then there are K_0 and K_1 disjoint nonmeager Borel sets such that $E = K_0 \cup K_1$. Since $\{[C_n] : n < \omega\}$ generates $\mathbb{B}_M$ there are L_0 and L_1 in the σ-algebra generated by $\{C_n : n < \omega\}$ such that $[L_0] = [K_0]$ and $[L_1] = [K_1]$. For some i, L_i is disjoint from E, but then L_i is meager, contradiction. By shrinking Y if necessary we may assume that for all $x, y \in Y$ ($x = y$ iff $x \sim y$). Let $R = \{C_n \cap Y : n < \omega\}$, then R_2 contains every countable subset of Y. It is easily seen that $K(R) = \alpha$, so by Proposition 2, we are done.

Theorem 19. (MA). *For any* $\alpha < \omega_1$ *there is an* $X \subseteq \omega^\omega$ *such that* $\alpha \leq \operatorname{ord}(X) \leq \alpha + 2$.

Proof. For $\alpha < \omega_1$ let $\mathbb{P}_\alpha$ be the partial order $\mathbb{P}_\alpha(\emptyset, \omega^\omega)$. Let T_α be the normal tree of rank α used in the definition of $\mathbb{P}_\alpha$. $T^*_\alpha=\{s\in T_\alpha:|s|_{T_\alpha}=0\}$. If G is $\mathbb{P}_\alpha$-generic, then G is completely determined by the real $y_G:T^*_\alpha\to\omega^{<\omega}$ defined by $y_G(s)=t$ iff $\{(s,t)\}\in G$. Each condition $p\in\mathbb{P}_\alpha$ can be thought of as a statement about y_G. Let $C_p=\{y\in\omega^\omega: y$ codes a map $\hat{y}:T^*_\alpha\to\omega^{<\omega}$ and $p(\hat{y})$ is true$\}$. It is easily seen that for any $p\in\mathbb{P}_\alpha$ there is $\beta<\alpha$ such that C_p is $\mathbf{\Pi}^0_\beta$.

Lemma 20. *If $\mathbb{B}_\alpha$ is the complete boolean algebra associated with $\mathbb{P}_\alpha$ and X_α is ω^ω with the topology generated by basic open sets $\{C_p:p\in\mathbb{P}_\alpha\}$, then $\mathbb{B}_\alpha$ is isomorphic to the boolean algebra of regular open subsets of X_α.*

Proof. Given $A\subseteq X_\alpha$ a regular open set let $D_A=\{p\in\mathbb{P}_\alpha:C_p\subseteq A\}$. The map $A\to D_A$ is an isomorphism.

Define I_α to the σ-ideal generated by $\mathbf{\Pi}^0_\alpha$ sets of the form $\omega^\omega-\bigcup\{C_p:p\in D\}$ where D is a maximal antichain in $\mathbb{P}_\alpha$.

Lemma 21. *α is the least ordinal such that for every Borel A there is a $\mathbf{\Sigma}^0_\alpha$ B such that $A\Delta B\in I_\alpha$.*

Proof. Note first that I_α is the ideal of meager subsets of X_α. If D is a maximal antichain in $\mathbb{P}_\alpha$, then $\bigcup\{C_p:p\in D\}$ is open dense in X_α, so every element of I_α is meager in X_α. If C is closed nowhere dense in X_α, then let $Q=\{p\in\mathbb{P}:C_p\cap C=\emptyset\}$. Since Q is open dense in $\mathbb{P}_\alpha$, we can pick $D\subseteq Q$ a maximal antichain. Thus $C\subseteq\omega^\omega-\bigcup\{C_p:p\in D\}$ and every meager subset of X_α is in I_α.

Since A is Borel in X_α there is a regular open set B in X_α such that $(A\Delta B)\in I_\alpha$. Let $Q=\{p\in\mathbb{P}_\alpha:C_p\subseteq B\}$. Pick $D\subseteq Q$ an antichain which is maximal with respect to being contained in Q. Since B is regular open, $B=\bigcup\{C_p:p\in D\}$, so B is $\mathbf{\Sigma}^0_\alpha$ in ω^ω. To see that α is minimal note that for $s\in T_\alpha$ with $|s|_{T_\alpha}=\beta$ there is no B $\mathbf{\Sigma}^0_\beta$ in ω^ω with $(C_{(s,x)}\Delta B)\in I_\alpha$.

Now let $X\subseteq\omega^\omega$ be I_α-Luzin. Then $\operatorname{ord}(X)\geq\alpha$ since for any A and B Borel in ω^ω ($(A\Delta B)\in I_\alpha$ iff $|(A\Delta B)\cap X|<|X|$). But $\operatorname{ord}(X)\leq\alpha+2$ follows from the fact that for all B in I_α there exists C in $I_\alpha\cap\mathbf{\Sigma}^0_{\alpha+1}$ with $B\subseteq C$, just as in the proof of Theorem 17. This concludes the proof of Theorem 19.

Remarks. (1) If $V=L$, then using the $\mathbf{\Delta}^1_2$ well-ordering of $L\cap 2^\omega$ we can get $X\subseteq 2^\omega$ a $\mathbf{\Delta}^1_2$ set with $\operatorname{ord}(X)=\alpha$ for any $\alpha\leq\omega_1$. If X is $\mathbf{\Pi}^1_1$ (or $\mathbf{\Sigma}^1_1$), then $X=A\Delta M$ where A is $\mathbf{\Pi}^0_\alpha$ and $M\in I_\alpha$, so X cannot be I_α-Luzin.

(2) A finer index can be given to a set $X\subseteq\omega^\omega$ by considering the classical Hausdorff difference hierarchies. A set $C\subseteq\omega^\omega$ is a $\beta-\mathbf{\Pi}^0_\alpha$ set iff there exists $D_\gamma\in\mathbf{\Pi}^0_\alpha$ for $\gamma<\beta$ such that the D_γ's are decreasing and $D_\lambda=\bigcup_{\gamma<\lambda}D_\gamma$ for λ limit and $C=\bigcup\{D_\gamma-D_{\gamma+1}:\gamma<\beta$ and γ even$\}$. It is a theorem of Hausdorff that $\mathbf{\Delta}^0_{\alpha+1}=\bigcup\{\beta-\mathbf{\Pi}^0_\alpha:\beta<\omega_1\}$ (see [11, pp. 417, 448]). It is also not hard to show,

using a universal set argument, that there exists a properly $\beta-\mathbf{\Pi}^0_\alpha$ set for all $\alpha, \beta<\omega_1$. Accordingly define $H(X)$ to be the lexicographical least pair $(\alpha,\beta)\in\omega_1^2$ such that for any Borel set A there exists B a $\beta-\mathbf{\Pi}^0_\alpha$ set such that $A\cap X=B\cap X$. If X is a Luzin set (Sierpinski set), then $H(X)=(2,2)$ $(H(X)=(2,1))$. It is easily shown that in Theorem 22 $N\models$"$H(X_{\alpha+1})=(\alpha+1,1)$". It is not hard to see that for C a countable closed set $H(C)=(1,\alpha)$ where α is the Cantor–Bendixson rank of C.

Theorem 22. *It is relatively consistent with* ZFC *that for any uncountable* $X\subseteq 2^\omega$ ord $(X)=\omega_1$. *This can be generalized to show that for any successor ordinal* β_0 *such that* $2\leq\beta_0<\omega_1$, *it is consistent that*

$$\{\beta:\exists X\subseteq 2^\omega \textit{ uncountable } \operatorname{ord}(X)=\beta\}=\{\beta:\beta_0\leq\beta\leq\omega_1\}.$$

Remark. It is true in the model obtained that for any uncountable separable metric space X the Borel hierarchy on X has length ω_1. This is true, since if $|X|=\omega_1$, then since $|2^\omega|\geq\omega_2$ and X can be embedded into $\mathbb{R}^\omega$, X must be zero dimensional. But any zero dimensional space can be embedded into 2^ω.

To prove Theorem 22 let M be a countable transitive model of ZFC+GCH. Choose $(\alpha_\lambda:\lambda<\omega_2)$ in M so that for all $\beta<\omega_1$ $\{\lambda:\alpha_\lambda=\beta\}$ is unbounded in ω_2. Define $\mathbb{P}^\gamma$ for $\gamma\leq\omega_2$ by induction $\mathbb{P}^0=\mathbb{P}_{\alpha_0}(\phi, 2^\omega\cap M)$, $\mathbb{P}^{\gamma+1}=\mathbb{P}^\gamma * Q^\gamma$ where Q^γ is a term in the forcing language of $\mathbb{P}^\gamma$ denoting $\mathbb{P}_{\alpha_\gamma}(\emptyset, M[G_\gamma]\cap 2^\omega)$ for any G_λ $\mathbb{P}^\gamma$-generic over M and at limits take the direct limit.

Call $p\in\mathbb{P}^\beta$ nice if it has the following properties for all $\gamma<\beta$.

(1) $p(\gamma)$ is a canonical name for $p^*\cup\{(s,\tau):s\in F\}$ where p^* is a function from some finite subset of $\{s\in T_{\alpha_\gamma}:|s|=0\}$, F is some finite subset of $\{s\in T_{\alpha_\gamma}:|s|>0\}$, and each τ is forced with value one to be an element of 2^ω.

(2) For each $(s,\tau)\in p(\gamma)\,\exists t_\tau\in 2^{<\omega}$ such that $p\restriction\gamma\Vdash$"$\check{t}_\tau\subseteq\tau$" and if (s,τ), $(s^\frown n,\tau')$ are in $p(\gamma)$ (or $(s^\frown n, t)\in p^*$), then t_τ and $t_\tau'(t)$ are incompatible.

It is not hard to see by induction on β that the nice p are dense. For the rest of the proof we assume all p are nice.

For $Q\subseteq\mathbb{P}$ and θ a sentence we say that Q decides θ iff $\{p\in\mathbb{P}$: there is a $q\in Q$ such that $p\geq q$ and $(q\Vdash$"θ" or $q\Vdash$"$\neg\theta$")$\}$ is dense in $\mathbb{P}$. For any $H\subseteq 2^\omega$ define $|p|(H)$ and $|\tau|(H,p)$ for $p\in\mathbb{P}^\gamma$ and τ a $\mathbb{P}^\gamma$ term for an element of 2^ω by induction on γ.

(1) For $p\in\mathbb{P}^0=\mathbb{P}_{\alpha_0}(\emptyset, 2^\omega\cap M)$ define

$$|p|(H)=\max\{|s|_{T_{\alpha_0}}:\exists x\in 2^\omega-H\ (s,x)\in p\}.$$

(2) For $p\in P^{\gamma+1}$ define

$$|p|(H)=\max\{|p\restriction\gamma|(H), |\tau|(H,p\restriction\gamma):(s,\tau)\in p(\gamma)\}.$$

(3) For $p \in P^\lambda$ define

$$|p|(H) = \sup\{|p \restriction \gamma| : \gamma < \lambda\}.$$

(4) Define $|\tau|(H, p)$ is the least β such that for any $n \in \omega$ $\{q \in \mathbb{P}^\gamma : q$ incompatible with p or $|q|(H) \leq \beta\}$ decides "$\tau(n) = 0$"

$\mathbb{P}^{\omega_2} = \mathbb{P}$ is not a lattice, however, it does have one similar property:

Lemma 23. *Suppose G is $\mathbb{P}^\alpha$-generic over M and for $i < n < \omega$ $q_i \in G$ and $|q_i|(H) < \beta$, then there is a $q \in G$ with $|q|(H) < \beta$ and $q \geq q_i$ for all $i < n$.*

Proof. The proof is by induction on α. For $\alpha = 0$ or a α a limit it is easy. So suppose $\alpha = \beta + 1$ and $G_\beta \times G^\beta$ where G_β is $\mathbb{P}^\beta$-generic over M. Find $\Gamma \subseteq G_\beta$ finite so that for any $q \in \Gamma$ with $|q|(H) < \beta$ and for any i and j less than n if $(s, \tau) \in q_i(\beta)$ and $(s ^\frown k, \hat{\tau}) \in q_j(\beta)$ (or $(s ^\frown k, t) \in q_j(\beta)$ where $t \in 2^{<\omega}$), then there is $r \in \Gamma$ such that $r \Vdash$ "$\tau \neq \hat{\tau} (t \not\subseteq \tau)$". By induction there is q in G_β such that $|q|(H) < \beta$, for all $\hat{q} \in \Gamma$ $q \geq \hat{q}$, and for all $i < n$ $q \geq q_i \restriction \beta$. Define $q(\beta)$ to be equal to $\bigcup \{q_i(\beta) : i < n\}$.

Lemma 24. *Given P_0 a countable subset of $\mathbb{P}^\alpha$ and Q_0 a countable set of $\mathbb{P}^\alpha$ terms for elements of 2^ω, there exists H countable such that for every $p \in P_0$ and $\tau \in Q_0$ $|p|(H) = |\tau|(H, \emptyset) = 0$.*

Proof. This is easy using c.c.c. of $\mathbb{P}^\alpha$.

Let $|p| = p(H)$ and $|\tau|(p) = |\tau|(H, p)$. for some fixed H.

Lemma 25. *For each $p \in \mathbb{P}^\alpha$ and β there exists $\hat{p} \in \mathbb{P}^\alpha$ compatible with p, $|\hat{p}| < \beta + 1$, and for every $q \in \mathbb{P}^\alpha$ with $|q| < \beta$, if $\hat{p}$ and q are compatible, then p and q are compatible.*

Proof. The proof is by induction on α. For $\alpha = 0$ this is just Lemma 8 of Section 1. For α limit it is easy. From now on assume the lemma is true for α.

Define for $x, y \in 2^\omega$, x is lexicographically less than y iff

$$\exists n\, \forall m < n\, (x(m) = y(m) \quad \text{and} \quad x(n) < y(n)).$$

This is the lexicographical order. For $C \subseteq 2^\omega$ a nonempty closed set let x_C be the lexicographically least element of C.

Claim 1. *Let $\dot{C}$ be a term in $\mathbb{P}^\alpha$ and $p_0 \in \mathbb{P}^\alpha$ with $|p_0| < \beta + 1$ such that $p_0 \Vdash$ "$\dot{\mathbf{C}}$ is a nonempty closed subset of 2^ω". Suppose for every G $\mathbb{P}^\alpha$-generic with $p_0 \in G$, and*

$s\in 2^{<\omega}(M[G]\models$ "$[s]\cap \dot{C}=\emptyset$" iff $\exists q\in G$, $|q|<\beta$, and $q\Vdash$ "$[s]\cap \dot{C}=\emptyset$"). Then $|x_C|(p_0)<\beta+1$.

Proof. First we show that given any $p\in\mathbb{P}^\alpha$ with $p\geqslant p_0$, if $s\in 2^{<\omega}$, $p\Vdash$ "$[s]\cap\dot{C}\neq\emptyset$", then there exist $\hat{p}\in\mathbb{P}^\alpha$ compatible with $p, |\hat{p}|<\beta+1$, and $\hat{p}\Vdash$ "$[s]\cap\dot{C}\neq\emptyset$". Let p' be as from Lemma 25 for p. By using Lemma 23 obtain $\hat{p}$ compatible with $p, \hat{p}\geqslant p'$, $\hat{p}\geqslant p_0$, and $|\hat{p}|<\beta+1$. *I* claim $\hat{p}\Vdash$ "$[s]\cap\dot{C}\neq\emptyset$". Suppose not then there exists G $\mathbb{P}^\alpha$-generic, $\hat{p}\in G$, and $M[G]\models$ "$[s]\cap\dot{C}=\emptyset$". So there exists $q\in G, |q|<\beta$, and $q\Vdash$ "$[s]\cap\dot{C}=\emptyset$". But then since q is compatible with $\hat{p}$ it is compatible with p' and hence with p, contradiction. In order to show $|x_C|(p_0)<\beta+1$ it suffices to show for every $p\geqslant p_0$ and $n\in\omega$ there exist $\hat{p}\in\mathbb{P}^\alpha$ compatible with p, $|\hat{p}|<\beta+1$, and there exists $s\in 2^n$ such that $\hat{p}\Vdash$ "$x_C\upharpoonright n=s$". So given p and n find $r\geqslant p$ and $s\in 2^n$ such that $r\Vdash$ "$x_C\upharpoonright n=s$". We have just shown there exists $\hat{r}$ compatible with r with $|\hat{r}|<\beta+1$ and $\hat{r}\Vdash$ "$[s]\cap C\neq\emptyset$". Let G be $\mathbb{P}^\alpha$-generic containing r and $\hat{r}$. For each $t\in 2^{m+1}$ with $m+1\leqslant n$ and for all $k<m$ $(t(k)=s(k))$ and $t(m)<s(m)$, choose $q_t\in G$ with $|q_t|<\beta$ and $q_t\Vdash$ "$[t]\cap C=\emptyset$". (There are only finitely many such t). Choose $q\in G$ with $|q|<\beta+1$, $q\geqslant\hat{r}$, and $q\geqslant q_t$ for each such t (q exists by Lemma 23). Then $q\Vdash$ "$x_C\upharpoonright n=s$".

For p and q compatible define $p\cup q\Vdash$ "θ" to mean that for every r, if $r\geqslant p$ and $r\geqslant q$, then $r\Vdash$ "θ". For τ a $\mathbb{P}^\alpha$ term for an element of 2^ω and $p\in\mathbb{P}^\alpha$, define $C(\tau,p)$ a $\mathbb{P}^\alpha$ term so that for any G which is $\mathbb{P}^\alpha$-generic (it need not contain p) $C^G(\tau,p)=\bigcap\{D_{\hat{\tau}}$: there exist $q\in G$, $|q|<\beta$, $|\hat{\tau}|(q)<\beta$, $q\Vdash$ "$\hat{\tau}\in 2^\omega$", p and q are compatible, and $p\cup q\Vdash$ "$\tau\in D_{\hat{\tau}}$"$\}$. D is a universal $\mathbf{\Pi}^0_1$ subset of $2^\omega\times 2^\omega$ $(\forall K\in\mathbf{\Pi}^0_1\,\exists x\in 2^\omega\; K=D_x=\{y:(x,y)\in D\})$.

Claim 2. *Let $\hat{p}$ be given by Lemma 25 for $p\in\mathbb{P}^\alpha$ (i.e. for all $q\in\mathbb{P}^\alpha$ if $|q|<\beta$, then if q and $\hat{p}$ are compatible, then q and p are compatible). Then $\hat{p}$ and $C(\tau,p)$ satisfy the hypothesis of Claim 1 for p_0 and $\dot{C}$.*

Proof. Suppose $M[G]\models$ "$[s]\cap C(\tau,p)=\emptyset$". By compactness there exists $n<\omega$, $q_i\in G$, τ_i for $i<n$ with $|q_i|<\beta$ and $|\tau_i|(q_i)<\beta$ so that $p\cup q_i\Vdash$ "$\tau\in D_{\tau_i}$" and $M[G]\models$ "$\bigcap\{D_{\tau_i}:i<n\}\cap[s]=\emptyset$". Let $\hat{\tau}$ be a term for an element of 2^ω so that $D_{\hat{\tau}}=\bigcap\{D_{\tau_i}:i<n\}$ and $q\in G$ with $q\geqslant q_i$ for $i<n$ and $|q|<\beta$. ($\hat{\tau}$ can be chosen so that $|\hat{\tau}|(q)<\beta$ assuming some nice properties of D). Since q and $\hat{p}$ are compatible, q and p are compatible and $q\cup p\Vdash$ "$\tau\in D_{\hat{\tau}}$". Since $M[G]\models$ "$D_{\hat{\tau}}\cap[s]=\emptyset$" by compactness there exists $m\in\omega$ so that if $t=\hat{\tau}^G\upharpoonright m$ then for every $x\supseteq t$, $x\in 2^\omega$ $D_x\cap[s]=\emptyset$. Since $|\hat{\tau}|(q)<\beta$ there exists $\hat{q}\geqslant q$ an element of $G, |\hat{q}|<\beta$, and $\hat{q}\Vdash$ "$\hat{\tau}\upharpoonright m=t$"; hence $\hat{q}\Vdash$ "$[s]\cap C(\tau,p)=\emptyset$". The fact that $\hat{p}\Vdash$ "$C(\tau,p)\neq\emptyset$" follows from this since if not there exists q compatible with $\hat{p}$, $|q|<\beta$, and $q\Vdash$ "$[\emptyset]\cap C(\tau,p)=\emptyset$". But then q is compatible with p contradiction.

We now return to the proof of the $\alpha+1$ step of Lemma 25.

Assume $p\in\mathbb{P}^{\alpha+1}$ is nice. Let (s_i,τ_i) for $i<n$ be all $(s,\tau)\in p(\alpha)$ with $|s|\geqslant 1$ and

let $\bar{\tau}=(\tau_0, \tau_1, \ldots, \tau_{n-1})$ (where $(\ ,\ ,\ldots,\ ):(2^\omega)^n \to 2^\omega$ is some recursive coding). Let $\hat{p}\restriction_\alpha$ be as given from Lemma 25 for $p\restriction_\alpha$. Let $\bar{\tau}^l$ be the lexicographical least element of $C(\bar{\tau}, p\restriction_\alpha)$. By Claim 1 and 2 $|\bar{\tau}^l|(\hat{p}\restriction_\alpha)<\beta+1$. Now let

$$\hat{p}(\alpha)=\{(s,t)\in p(\alpha):|s|=0\}\cup\{(s_i,\tau_i^l):i<n\}$$

$(\bar{\tau}^l=(\tau_0^l,\ldots,\tau_{n-1}^l))$. Since $\emptyset \Vdash$ "$C(\bar{\tau}, p_\alpha)$ is included in $\prod_{i<n}[s_{\tau_i}]$", $\hat{p}$ is a condition, $\hat{p}$ and p are compatible, also $|\hat{p}|<\beta+1$. Now suppose $q\in\mathbb{P}^{\alpha+1}$ compatible with $\hat{p}$, $|q|<\beta$, and q and p are not compatible. Let G be $\mathbb{P}^\alpha$-generic with $\hat{p}\restriction_\alpha$ and $q\restriction_\alpha$ elements of G and $M[G]\models$"$\hat{p}(\alpha)$ and $q(\alpha)$ are compatible". If we think of $p(\alpha)$ as a statement about $\bar{\tau}$ i.e. $p(\alpha)(\bar{\tau})$, then $\hat{p}(\alpha)=p(\alpha)(\bar{\tau}^l)$. Since p and q are incompatible but p_α and q_α are compatible $(p\restriction_\alpha\cup q\restriction_\alpha)\models$"$p(\alpha)$ and $q(\alpha)$ are incompatible". $D(\bar{\tau})\equiv$"$p(\alpha)(\bar{\tau})$ and $q(\alpha)$ are incompatible" is a $\mathbf{\Pi}_1^0$ statement with parameters from $q(\alpha)$ about $\bar{\tau}$. Thus we conclude that $M[G]\models$"$p(\alpha)(\bar{\tau}^l)$ and $q(\alpha)$ are incompatible", contradiction. This concludes the proof of Lemma 25.

From now on let $\mathbb{P}=\mathbb{P}^{\omega_2}$.

Lemma 26. *Suppose $|\tau|=0$, $B(v)$ is a $\mathbf{\Sigma}_\beta^0$ predicate, $\beta\geq 1$, with parameters from M, and $p\in\mathbb{P}$ is such that $p\Vdash$"$B(\tau)$"; then there exists $q\in\mathbb{P}$ compatible with p, $|q|(H)<\beta$ and $q\Vdash$"$B(\tau)$".*

Proof. The proof is by induction on β.

Case 1. $\beta=1$.

Suppose $p\Vdash$"$\exists n\, R(x\restriction n, \tau\restriction n)$" for R recursive and $x\in M$. Let G be $\mathbb{P}$-generic with $p\in G$. Choose $n\in\omega$ and $s\in 2^n$ so that $M[G]\models$"$R(\restriction n, \tau\restriction n)$ and $\tau\restriction n=s$". Choose $q\in G$ with $|q|=0$ and $q\Vdash \tau\restriction n=s$".

Case 2. β is a limit ordinal.

If $p\Vdash$"$\exists n\, B(n,\tau)$", then $\exists\hat{p}\geq p$ $\hat{p}\Vdash$"$B(n_0,\tau)$" and $B(n_0,v)$ $\mathbf{\Sigma}_\gamma^0$ for $\gamma<\beta$, so apply induction hypothesis to $\hat{p}$.

Case 3. $\beta+1$.

Suppose $p\Vdash$"$\exists n\, B(n,\tau)$" where $B(n,v)$ is $\mathbf{\Pi}_\beta^0$ with parameters from M. Choose $r\geq p$ and $n_0\in\omega$ so that $r\Vdash$"$B(n_0,\tau)$". By Lemma 25 there is q compatible with r, $|q|<\beta+1$, and for every s, $|s|<\beta$, if q and s are compatible, then r and s are compatible. $q\Vdash$"$B(n_0,\tau)$" because if not, then there is $q'\geq q$ such that $q'\Vdash$"$B(n_0,\tau)$", and so by induction there is s with $|s|<\beta$ compatible with q' and $s\Vdash$"$B(n_0,\tau)$"; but then s is compatible with r, contradiction.

Now let us prove the first part of Theorem 22. Let G be $\mathbb{P}$-generic over M. We claim $M[G]\models$"for every $X\subseteq 2^\omega$ and $\alpha<\omega_1$ if $|X|=\omega_1$, then $\operatorname{ord}(X)\geq\alpha+1$". But since any such X is in some $M[G_\beta]$ for $\beta<\omega_2$, we may as well assume $X\in M$, $\alpha_0=\alpha+1$, and we must show $M[G]\models$"$\operatorname{ord}(X)\geq\alpha+1$". Let $G_{(0)}$ be the $\mathbf{\Pi}_\alpha^0$ set created by $G\cap\mathbb{P}_{\alpha_0}(\emptyset, 2^\omega\cap M)$. Suppose that $M[G]\models$"there is K a $\mathbf{\Sigma}_\beta^0$ set such that

$K \cap X = G_{(0)} \cap X$". Let τ be a term for the parameter of K. Choose $p \in G$ such that $p \vDash$ "$\forall z \in X$ ($x \in K$ iff $z \in G_{(0)}$)". By Lemma 24 there exists H in M countable so that $|\tau|(H, \emptyset) = |p|(H) = 0$. Let $z \in X - H$. Define $\hat{p} \in \mathbb{P}$ by $\hat{p}(0) = p(0) \cup \{((0), z)\}$ and $\hat{p}(\alpha) = p(\alpha)$ for $\alpha > 0$. Since $\hat{p}$ says $z \in G_{(0)}$, $\hat{p} \Vdash$ "$z \in K$". By Lemma 26 there exists q compatible with $\hat{p}$, $|q|(H) < \beta$, and $q \Vdash$ "$z \in K$". By Lemma 23 there exists $\hat{q}$ with $|\hat{q}(H) < \beta$, $\hat{q} \geq q$, and $\hat{q} \geq p$. Since $|(0)|_{T_{\alpha_0}} = \alpha$, $((0), z) \notin \hat{q}(0)$, there exists $m \in \omega$ such that r defined by $r(0) = q(0) \cup \{((0, m), z)\}$ and $r(\alpha) = \hat{q}(\alpha)$ for $\alpha > 0$ is a condition. But this is a contradiction since $r \Vdash$ "($z \in G_{(0)}$ iff $z \in K$) and $z \in K$ and $z \notin G_{(0)}$".

Now we prove the second sentence of Theorem 22. Let $X = \bigcup \{X_\alpha : \beta_0 \leq \alpha < \omega_1$ and α a successor$\}$ where each X_α is a set of ω_1 product generic Cohen reals. Let $M_0 = M[X]$. Define in M_0 the partial order $\mathbb{P}^\gamma$ for $\gamma \leq \omega_2$ so that $\mathbb{P}^{\gamma+1} = \mathbb{P}^\gamma * Q_\gamma$ where Q_γ is a term denoting:

Case 1. $\mathbb{P}_{\beta_0}(\emptyset, M_0[G_\gamma] \cap 2^\omega)$ or

Case 2. $\mathbb{P}_\beta(Y_\gamma, X_\beta \cup F)$ where Y_γ is a Borel subset of X_β in $M_0[G_\gamma]$ and $F = \{x \in 2^\omega : x$ eventually zero$\}$.

Case 1 is done cofinally in ω_2 and Case 2 is done in such a way as to insure: $M_0[G_{\omega_2}] \vDash$ "For every successor ordinal β with $\beta_0 \leq \beta < \omega_1$ and Y Borel in X_β there is a γ such that $Y = Y_\gamma$". First we show that essentially the same arguments as before show that $M_0[G_{\omega_2}] \vDash$ "For every $X \subseteq 2^\omega$ uncountable ord $(X) \geq \beta_0$". This will not use that the X_α are made up of Cohen reals, hence any of the intermediate models would serve as the ground model. So suppose Case 1 occurs on the first step, $Y \in M_0$ is uncountable, $\beta_0 = \gamma + 1$, and $M_0[G_{\omega_2}] \vDash$ "$Y \cap G_{(0)} = Y \cap J$ for some $J \in \Sigma^0_\gamma$". Given $L \subseteq \omega_2$ define $\mathbb{P}^\alpha_L$ as follows.

For $\alpha \in L$:

Case 1. $\mathbb{P}^{\alpha+1}_L = \mathbb{P}^\alpha_L * \mathbb{P}_{\beta_0}(\emptyset, M[G^L_\alpha] \cap 2^\omega)$ where G^L_α is $\mathbb{P}^\alpha_L$-generic over M_0.

Case 2. $\mathbb{P}^{\alpha+1}_L = \mathbb{P}^\alpha_L * \mathbb{P}_\beta(Y_\alpha - F, X_\beta \cup F)$ (where we assume L has the property that when Case 2 happens for $\alpha \in L$ then Y_α is a Borel subset of X_β coded by some term τ_α in $\mathbb{P}^\alpha_L$).

For $\alpha \notin L$:

$$\mathbb{P}^{\alpha+1}_L = \mathbb{P}^\alpha_L * \quad \text{(singleton partial order)}.$$

Note that by using c.c.c. of $\mathbb{P}^{\omega_2}$ we can find $L \subseteq \omega_2$ countable, so that the Borel code for the above J is a $\mathbb{P}^{\omega_2}_L$ term and L has the property mentioned under Case 2. For α a limit $\mathbb{P}^\alpha_L$ is the direct limit of $(\mathbb{P}^\beta_L : \beta < \alpha)$.

Lemma 27[1]. *If $N \supseteq M$ is a model of ZFC and G is $\mathbb{P}_\beta(\emptyset, N \cap 2^\omega)$ generic over N, then $G \cap \mathbb{P}_\beta(\emptyset, M \cap 2^\omega)$ is $\mathbb{P}_\beta(\emptyset, M \cap 2^\omega)$ generic over M.*

[1] I would like to thank the referee for suggesting this proof of Lemma 27 and thus eliminating the need for Lemma 28. A similar argument is utilized by J. Truss, "Sets having calibre $\aleph_1$", in: *Logic Colloquium 76*, Studies in Logic, Vol. 87 (North-Holland, Amsterdam, 1977).

Proof. It is sufficient to show that if $A \in M$ and A is a maximal antichain in $\mathbb{P}_\beta(0, M \cap 2^\omega)$ (where $\beta < \omega^M$), then A is also a maximal antichain in $\mathbb{P}_\beta(0, N \cap 2^\omega)$ for any $N \supseteq M$ which is a transitive model of ZFC. But by c.c.c. (in M), A is countable in M, so this result is immediate by absoluteness of $\mathbf{\Pi}^1_1$ predicates.

Given any G $\mathbb{P}^{\omega_2}$-generic let G_L be the subset of $\mathbb{P}_L$ generated by the rank zero conditions in G. The preceding lemma enables us to prove:

Lemma 29. *For any α if G_α is $\mathbb{P}^\alpha$-generic over M_0, then G^L_α is $\mathbb{P}^\alpha_L$-generic over M_0.*

Proof. This is proved by induction on α. For $\alpha+1 \notin L$ it is immediate. For $\alpha+1 \in L$ Case 1 is handled by Lemma 27 and the product lemma. Case 2 is easy as $\mathbb{P}_\beta(Y_\alpha - F, X_\beta \cup F)$ is the same partial order in either case. For α limit ordinal let $\Delta \subseteq \mathbb{P}^\alpha_L$ be dense, we show $\{q \in \mathbb{P}^\alpha : \exists p \in \Delta,\ p \leq q\}$ is dense in $\mathbb{P}^\alpha$. If $q \in \mathbb{P}^\alpha$, then $q \in \mathbb{P}^\beta$ for some $\beta < \alpha$. Let $\Delta_\beta = \{p \restriction \beta : p \in \Delta\}$, then Δ_β is dense in $\mathbb{P}^\beta_L$. Hence if G_α is $\mathbb{P}^\alpha$-generic with $q \in G_\alpha$, then since G^L_β is $\mathbb{P}^\beta_L$-generic it meets Δ_β — say at $p \restriction \beta$. But then q and p are compatible.

Define for $H \subseteq 2^\omega$ $|p|(H)$, $|\tau|(H, p)$ for $p \in \mathbb{P}^\alpha_L$ and τ a $\mathbb{P}^\alpha_L$-term for a subset of ω by induction on α.

Case 1. $\mathbb{P}^{\alpha+1} = \mathbb{P}^\alpha * \mathbb{P}_{\beta_0}(\emptyset, M[G^\alpha_L] \cap 2^\omega)$.

$$|p|(H) = \max\{|p \restriction \gamma|(H), |p(\gamma)|(H, p \restriction \gamma)\} \quad \text{(same as before)}.$$

Case 2. $\mathbb{P}^{\alpha+1} = \mathbb{P}^\alpha * \mathbb{P}_\beta(Y_\alpha - F, X_\alpha \cup F)$.

$$|p|(H) = \max\{|p \restriction \alpha|(H), |s|_{T_\beta} : x \notin H\ \langle s, x\rangle \in p(\alpha)\}.$$

$|\tau|(H, p)$ is defined as it was just before Lemma 23. Lemma 23 is easily proven since in Case 2 we have a lattice. Lemma 24 is also easily proven if in addition H is taken with the property that $\forall x \in H\ \forall \alpha \in L\ \{p : |p|(H) = 0\}$ decides "$x \in Y_\alpha$" whenever Case 2 happens at stage α. Lemma 25 can be proven for $\beta < \beta_0$ by the same argument in Case 1 and by the argument of Theorem 34 in Case 2. Lemma 26 follows and so does the claim that $M_0[G_{\omega_2}] \models$ "$K \subseteq \{\beta : \beta_0 \leq \beta < \omega_1\}$".

Next we show $M_0[G_{\omega_2}] \models$ "$\operatorname{ord}(X_\beta) = \beta$ for each β successor $\beta_0 \leq \beta < \omega_1$". If not, then again we can reduce to some $L \subseteq \aleph_2$ countable; and since each X_α is present in M_0, we can relabel L so that for some $\hat\beta < \omega_1$ and β_1 with $\beta_0 \leq \beta_1 < \omega_1$, $M_0[G_{\hat\beta}] \models$ "$\operatorname{ord}(X_{\beta_1}) < \beta_1$" for $G_{\hat\beta}$ $\mathbb{P}^\beta$-generic over M_0, and on some step before $\hat\beta$ we force with $\mathbb{P}_{\beta_1}(\emptyset, X_{\beta_1} \cup F)$. Suppose $X = \{x_\alpha : \alpha < \omega_1\}$ and $M_0 = M[\{\langle \alpha, x_\alpha\rangle : \alpha < \omega_1\}]$. Given $H \subseteq \omega_1$, $H \in M$ let $\hat H = \{\langle \alpha, x_\alpha\rangle : \alpha \in H\}$. Define $\mathbb{P}^\alpha_H \in M[\hat H]$ for each $\alpha < \hat\beta$.

Case 1. $\mathbb{P}^{\alpha+1}_H = \mathbb{P}^\alpha_H * \mathbb{P}_{\beta_0}(\emptyset, M[G^H_\alpha] \cap 2^\omega)$.

Case 2. $\mathbb{P}^{\alpha+1}_H = \mathbb{P}^\alpha_H * \mathbb{P}_\beta((Y_\beta - F) \cap \hat H, (X_\beta \cap \hat H) \cup F)$ (assuming Y_α is a Borel subset of X_β given by the term τ_α in forcing language of $\mathbb{P}^\alpha_H$).

Lemma 30. *For any $\alpha \leq \hat{\beta}$ if G^{α} is $\mathbb{P}^{\alpha}$-generic over M_0, then G_H^{α} is $\mathbb{P}_H^{\alpha}$-generic over $M[\hat{H}]$.*

Proof. The proof is like Lemma 29 except on $\alpha+1$ under Case 2. $\mathbb{P}_1 = \mathbb{P}_\beta(Y_\alpha - F, X_\beta \cup F)$ in $M[X][G^\alpha] = M_1$, $\mathbb{P}_2 = \mathbb{P}_\beta((Y_\alpha - F) \cap \hat{H}, (X_\beta \cap \hat{H}) \cup F)$ in $M[\hat{H}][G_H^\alpha] = M_2$. Again suppose $\Delta \in M_2$ is dense in $\mathbb{P}_2$, we show $\{p \in \mathbb{P}_1 : \exists q \in \Delta, q \leq p\}$ is dense in $\mathbb{P}_1$. Given $p \in \mathbb{P}_1$ let $p = r \cup \{\langle s_n, x_n\rangle : n < N\}$ where $x_n \in X_\alpha - \hat{H}$, $N < \omega$, and $r \in \mathbb{P}_2$. Let Q_N be the partial order for adding N Cohen reals. By the product lemma $\{x_n : n < N\}$ is Q_N-generic over M_2, and also $p \in M_2[\{x_n : n < N\}]$. Hence if $\forall q \in \Delta\, p$ and q are incompatible in

$$\mathbb{P}_3 = \mathbb{P}_\beta((Y_\alpha - F) \cap (H \cup \{x_n : n < N\}), (X_\beta \cap (H \cup \{x_n : n < N\})) \cup F),$$

then $\exists \hat{p} \in Q_N\ \hat{p} \Vdash$ "$\forall q \in \Delta\, p$ and q are incompatible in $\mathbb{P}_3$". Choose $y_n \in F$ for $n < N$ so that $p_0 = r \cup \{\langle s_n, y_n\rangle : n < N\} \in \mathbb{P}_2$ and $\forall m < \omega\ \exists \hat{p}' \geq \hat{p}\ \forall n < N\ \hat{p}' \Vdash$ "$y_n \restriction_m = x_n \restriction_m$". Since $\exists q \in \Delta\ p_0$ and q are compatible, then as before p and q can be forced compatible by an extension of $\hat{p}$. So p and q are compatible in $\mathbb{P}_3$ and hence in $\mathbb{P}_1$.

Lemma 31. *Given $\hat{\tau}$ a term in forcing language of $\mathbb{P}_H^{\hat{\beta}}$ if $p \in \mathbb{P}^{\hat{\beta}}$ $p \Vdash_{\mathbb{P}^{\hat{\beta}}}$ "$B(\tau)$" where $B(v)$ is a Σ_1^1 predicate with parameters in $M[\hat{H}]$, then $\exists q \in \mathbb{P}_H^{\beta}$ compatible with p such that $q \Vdash_{\mathbb{P}^{\hat{\beta}}}$ "$B(\tau)$".*

Proof. Let G be $\mathbb{P}^{\hat{\beta}}$-generic over M_0 with $p \in G$. Then by Lemma 9 G_H^{β} is $\mathbb{P}_H^{\hat{\beta}}$-generic over $M[\hat{H}]$. Since Σ_1^1 sentences are absolute and $M_0[G] \models$ "$B(\tau)$" we have $M[\hat{H}][G_H] \models$ "$B(\tau)$". So $\exists q \in G_H\ q \Vdash_{\mathbb{P}_H^{\hat{\beta}}}$ "$B(\tau)$". But for any G $\mathbb{P}^{\hat{\beta}}$-generic containing q, $M[H][G_H] \models$ "$B(\tau)$" whence by absoluteness $M_0[G] \models$ "$B(\tau)$". We conclude $q \Vdash_{\mathbb{P}^{\hat{\beta}}}$ "$B(\tau)$".

Lemma 32. *Given $H = X - \{z\}$ where $z \in X_{\alpha+1}$, $\gamma \leq \hat{\beta}$, $1 \leq \beta < \alpha$, $p \in \mathbb{P}^\gamma$, then $\exists \hat{p} \in \mathbb{P}^\gamma$, $|\hat{p}|(M[\hat{H}] \cap 2^\omega) < \beta + 1$, $\hat{p}$ compatible with p, and $\forall q \in \mathbb{P}^\gamma$ if $|q|(M[\hat{H}] \cap 2^\omega) < \beta$, then ($\hat{p}, q$ compatible $\Rightarrow p, q$ compatible).*

Proof. This is proved by induction on γ. For γ limit it is easy, also for $\gamma+1$ in which Case 1 occurs the proof is the same as Lemma 25. So we only have to do $\gamma+1$ in Case 2.

$p \in \mathbb{P}^\gamma * \mathbb{P}_{\beta_1}(Y_\gamma - F, X_{\beta_1} \cup F)$. Extend $p(\gamma)$ if necessary so that $\forall \langle s, x\rangle \in p(\gamma)\ \forall i < \omega$ if $|s| = \lambda$ infinite limit $|s^\frown i| \leq \beta + 1 < \lambda$, then $\exists j < \omega\ \langle s^\frown i^\frown j, x\rangle \in p(\gamma)$. Let $\hat{p}(\gamma) = \{\langle s, x\rangle \in p(\gamma) : |s| < \beta + 1 \text{ or } x \neq z\}$. If $\hat{p} = \langle \hat{p} \restriction \gamma, \hat{p}(\gamma)\rangle$ were a condition, then just as in Lemma 8, $\hat{p}$ would have the required properties. To be a condition we need to know that whenever $\langle\langle n\rangle, x\rangle \in \hat{p}(\gamma)$ $\hat{p} \restriction \gamma \Vdash$ "$x \notin (Y_\gamma - F)$".

Note that none of these x's are equal to z because $z \in X_{\alpha+1}$ so $\langle\langle n\rangle, z\rangle \in p(\gamma) \rightarrow |\langle n\rangle| = \alpha \geq \beta + 1$ so $\langle\langle n\rangle, z\rangle \notin \hat{p}(\gamma)$. Let G be $\mathbb{P}^\gamma$-generic containing $p \restriction \gamma$, and $\hat{p} \restriction \gamma$. By Lemma 31 $\exists q \in \mathbb{P}_H^\gamma \cap G$ (so $|q|(M[H] \cap 2^\omega) = 0$) such that $\forall x\, \forall n$ if

$\langle\langle n\rangle, x\rangle \in \hat{p}(\gamma)$, then $q \Vdash$ "$x \notin Y_\gamma - F$". By Lemma 23, $\exists p_0 \geq q$, $\hat{p}\restriction\gamma$ so that $|p_0|(M[H]\cap 2^\omega) < \beta + 1$. So $\langle p_0, \hat{p}(\gamma)\rangle$ works.

Immediate from Lemma 32 we get that: If J is any $\Sigma^0_{\alpha+1}$ predicate with parameters ($H = X - \{z\}$, $z \in X_{\alpha+1}$, and τ is in the forcing language of $\mathbb{P}_H$), then $\forall p \in \mathbb{P}$ if $p \Vdash$ "$z \in J$", then $\exists q \in \mathbb{P}$ $|q|(M[H]\cap 2^\omega) < \beta$, q and p are compatible, and $q \Vdash$ "$z \in J$". So we get our result $\operatorname{ord}(X_{\alpha+1}) = \alpha + 1$ in $M_0[G_{\omega_2}]$.

Remark. Assuming large amounts of the axiom of determinacy and therefore getting more absoluteness in inner models (see [7]) it is easy to produce an inner model N such that $N \models$ "For every $\alpha < \omega_1$ there exist $X \subseteq 2^\omega$ such that $\operatorname{ord}(X) = \alpha$ and for every $n < \omega$ and A Π^1_n, $A \cap X$ is Borel in X". Similar improvements for Theorem 43 are possible.

4. The σ-algebra generated by the abstract rectangles

For any cardinal λ let $R^\lambda = \{A \times B : A, B \subseteq \lambda\}$. If $R^\lambda_{\omega_1}$ (the σ-algebra generated by R^λ) is the set of all subsets of $\lambda \times \lambda$, then $\lambda \leq |2^\omega|$ (see [9, 21]).

Theorem 33. *If $\alpha_0 < \omega_1$ and there is an $X \subseteq \omega^\omega$ such that $|X| = \kappa \geq \omega$ and every subset of X of cardinality less than κ is $\Pi^0_{\alpha_0}$ in X, then $R^\kappa_{\alpha_0} = P(\kappa\times\kappa)$. The same is true if every subset of X of cardinality less than κ is $\Sigma^0_{\alpha_0}$ in X.*

Proof. Consider $A \subseteq \kappa\times\kappa$ and suppose $(\alpha,\beta)\in A$ implies $\alpha \leq \beta$. It is enough to show such sets are in $R^\kappa_{\alpha_0}$ since every subset of $\kappa\times\kappa$ can be written as the union of a set above the diagonal and a set below the diagonal. Let T be a normal α_0 tree and $T^* = \{s \in T : |s|_T = 0\}$. For any $y : T^* \to \omega^{<\omega}$ define G^s_y as follows. If $s \in T^*$, then $G^s_y = [y(s)]$, otherwise $G^s_y = \bigcap\{\omega^\omega - G^{s\frown n}_y : n < \omega\}$. Let $X = \{x_\alpha : \alpha < \kappa\}$ and for each $\beta < \kappa$ choose β so that for all α ($(\alpha,\beta)\in A$ iff $x_\alpha \in G^\phi_{y_\beta}$). For $s \in T$ define $B_s \subseteq \kappa\times\kappa$ as follows. If $s \in T^*$, then $B_s = \bigcup\{\{\alpha : t \subseteq x_\alpha\}\times\{\beta : y_\beta(s) = t\} : t \in \omega^{<\omega}\}$, otherwise $B_s = \bigcap\{(\kappa\times\kappa) - B_{s\frown n} : n < \omega\}$. Clearly $B_\emptyset = A$ and $B_\emptyset$ is "$\Pi^0_{\alpha_0}$" in R^κ, and so every subset of $\kappa\times\kappa$ is "$\Pi^0_{\alpha_0}$" in R^κ. Note that $(\kappa\times\kappa)-(A\times B) = ((\kappa - A)\times\kappa)\cup(\kappa\times(\kappa - B))$ and thus if α_0 is even (odd), then $R^\kappa_{\alpha_0}$ is the class of sets "$\Pi^0_{\alpha_0}$" ("$\Sigma^0_{\alpha_0}$") in R^κ. By passing to complements if necessary we have that $R^\kappa_{\alpha_0} = P(\kappa\times\kappa)$. The second sentence of the theorem is proved similarly.

Corollary (Kunen [9]; Rao [21]). *If there is an $X \subseteq 2^\omega$ such that $|X| = \omega_1$, then $R^{\omega_1}_2 = P(\omega_1\times\omega_1)$.*

The converse of this corollary is also true. Suppose $R \subseteq P(\omega_1)$ is a countable field of sets and $\{(\alpha,\beta):\alpha<\beta<\omega_1\}\in\{A\times B:A,B\in R\}_{\omega_1}$. Since this set is antisymetric we conclude that the map given in Proposition 2 is a 1–1 embedding of ω_1 into 2^ω.

Corollary (Kunen [9]; Silver). (MA). *If* $\kappa=|2^\omega|$, *then* $R_2^\kappa=P(\kappa\times\kappa)$.

Proof. If X is I-Luzin where I is the ideal of meager sets, then every subset of X of smaller cardinality is Σ_2^0 in X (see proof of Theorem 17).

For any $\alpha\leq\omega_1$ $X\subseteq\omega^\omega$ is a Q_α set iff $\text{ord}(X)=\alpha$ and every subset of X is Borel in X.

Theorem 34. *If M is countable transitive model of ZFC,* $1\leq\alpha_0<\omega_1^M$, *and* $X=M\cap\omega^\omega$, *then there is a Cohen extension $M[G]$ such that* $M[G]\models$"*X is a* Q_{α_0+1} *set*".

Remark. This shows that the Baire order of the constructible reals can be any countable successor ordinal greater than one. In fact the argument shows that in $M[G]$ for any uncountable $Y\subseteq 2^\omega$ with $Y\in M$, Y is a Q_{α_0+1} set. Thus, for example, if M models $V=L$, then in $M[G]$ there are Π_1^1 Q_{α_0+1} sets. In Theorem 55 we show that it is consistent with ZFC that for every $\alpha<\omega_1$ there is a Q_α set (in that model the continuum is $\aleph_{\omega_1+1}$).

The proof of Theorem 34. $M[G]$ is gotten by iterated $\Pi^0_{\alpha_0+1}$-forcing. Let $\kappa=|2^{2^\omega}|$. Suppose we are given $\mathbb{P}^a$ for some $a<\kappa$ and Y_α a term in the forcing language of $\mathbb{P}^\alpha$ for a subset of X ($\emptyset\Vdash$"$Y_\alpha\subseteq X$"), then let $\mathbb{P}^{\alpha+1}=\mathbb{P}^\alpha*\mathbb{P}_{\alpha_0+1}(Y_\alpha,X)$. At limit ordinals take direct limits. $\mathbb{P}^\kappa$ may be viewed as a sub-lower lattice of $\sum_k\mathbb{P}_{\alpha_0+1}(\emptyset,X)$. We may assume that for every set $B\subseteq X$ in $M[G]$ (G $\mathbb{P}^\kappa$-generic over M) there exists α such that $Y_\alpha=B$. This is because $\mathbb{P}^\kappa$ has c.c.c. It follows from Corollary 6 that $M[G]\models$"$\text{ord}(X)\leq\alpha_0+1$ and every subset of X is Borel in X".

We assume $\mathbb{P}^0=\mathbb{P}_{\alpha_0+1}(\emptyset,X)$. Let $G_{(0)}$ be one of the $\Pi^0_{\alpha_0}$ set determined by $G\cap\mathbb{P}^0$. We want to show that $M[G]\models$"For every K in $\Sigma^0_{\alpha_0}$, $K\cap X\neq G_{(0)}\cap X$". To this end we make the following definition: For $H\subseteq\omega^\omega$, $|p|(H)=\max\{|s|:\text{there exists } x\notin H\ (s,x)\in p(\alpha)\text{ for some }\alpha<\kappa\}$. Let $\text{supp}(p)=\{\alpha<\kappa:p(\alpha)\neq\emptyset\}$. Given τ a term in the forcing language of $\mathbb{P}^\kappa$ denoting a subset of ω, we can find H included in ω^ω and K included in κ with the following properties:

(a) H and K are countable;

(b) for each $n\in\omega$ $\{p\in\mathbb{P}^\kappa:\text{supp}(p)\subseteq K, |p|(H)=0\}$, decides "$n\in\tau$";

(c) $\forall x\in H\,\forall\alpha\in K\ \{p\in\mathbb{P}^\kappa:\text{supp}(p)\subseteq K, |p|(H)=0\}$ decides "$x\in Y_\alpha$".

H and K can be found by repeatedly using the c.c.c. of $\mathbb{P}^\kappa$.

Lemma 35. *If H and K have property* (c), *then for any $p \in \mathbb{P}^\kappa$ and β with $1 \leq \beta < \alpha_0$, there exists $\hat{p} \in \mathbb{P}^\kappa$ compatible with p, $|\hat{p}|(H) < \beta + 1$,* $\operatorname{supp}(\hat{p}) \subseteq K$, *and for any $q \in \mathbb{P}^\kappa$ if $|q|(H) < \beta$ and* $\operatorname{supp}(q) \subseteq K$, *then [if $\hat{p}$ and q are compatible, then p and q are compatible].*

Proof. The proof of this is like Lemma 8. Let G be P^κ-generic over M with $p \in G$. Choose $\Gamma \subseteq G$ finite with the properties:

(1) $\forall q \in \Gamma\,(|q|(H) = 0$ and $\operatorname{supp}(q) \subseteq K)$.

(2) If $((n), x) \in p(\alpha)$ for some $n < \omega$, $\alpha \in K$, and $x \in H$ (so $p \restriction \alpha \Vdash$ "$x \notin Y_\alpha$"), then there is $q \in \Gamma \cap \mathbb{P}^\alpha$ such that $q \Vdash$ "$x \notin Y_\alpha$".

(3) If $(s, x) \in p(\alpha)$, $\alpha \in K$, and $|s| = \lambda$ is an infinite limit ordinal, and $|s^\frown i| \leq \beta + 1 < \lambda$, then there is a $j \in \omega$ such that $\{(s^\frown i^\frown j, x)\} \in p$.

Now let $\hat{p} \in \mathbb{P}^\kappa$ be defined by

$$\hat{p}(\alpha) = \bigcup \{r(\alpha) : r \in \Gamma\} \cup \{(s, x) \in p(\alpha) : |s| < \beta + 1 \text{ or } x \in H\}$$

when $\alpha \in K$ and $\hat{p}(\alpha) = \emptyset$ for $\alpha \notin K$. Note if $((n), x) \in \hat{p}(\alpha)$, then $x \in H$ since $|(n)| = \alpha_0 \geq \beta + 1$. By choice of Γ $\hat{p}$ is a condition and also $|\hat{p}|(H) < \beta + 1$ and is compatible with p since $\hat{p}$, $p \in G$. It is easily checked as in Lemma 8 that $\hat{p}$ has the required property.

Lemma 36. *Let H and K have properties* (b) *and* (c) *for τ. Let $B(v)$ be a Σ^0_β ($1 \leq \beta \leq \alpha_0$) predicate with parameters from M and $p \in \mathbb{P}^\alpha$ such that $p \Vdash$ "$B(\tau)$". Then there exists $q \in \mathbb{P}^\alpha$ compatible with p, $|q|(H) < \beta$, $q \Vdash$ "$B(\tau)$", and* $\operatorname{supp}(q) \subseteq K$.

Proof. The proof is by induction on β.

$\beta = 1$: $p \Vdash$ "$\exists n\, R(n, \tau \restriction n, x \restriction n)$", $x \in M$, and R primitive recursive. Let G be $\mathbb{P}$-generic over m with $p \in G$. There exist $n \in \omega$ and $s \in 2^n$ such that $M[G] \vDash$ "$R(n, \tau \restriction n, x \restriction n)$ and $\tau \restriction n = s$". By property (b) there exists $q \in G$ such that $q \Vdash$ "$\tau \restriction n = s$", $\operatorname{supp}(q) \subseteq K$, and $|q|(H) = 0$. q does it.

β *limit*: $p \Vdash$ "$\exists n\, B_n(\tau)$", $B_n \in \Sigma^0_{\beta_n}$, $\beta_n < \beta$. Choose $r \geq p$ such that $r \Vdash$ "$B_n(\tau)$" for some n. By induction there exist q such that $q \Vdash$ "$B_n(\tau)$", q is compatible with r (and hence with p), and $|q|(H) < \beta$, $\operatorname{supp}(q) \subseteq K$. q does it.

$\beta + 1$: If $p \Vdash$ "$\exists n\, B_n(\tau)$" we could extend p to force $B_n(\tau)$ for some particular n. So we may as well assume $p \Vdash$ "$B(\tau)$" where $B(v)$ is Π^0_β with parameter in M. Since $1 \leq \beta < \alpha_0$ by Lemma 35 there is $\hat{p}$ compatible with p, $|\hat{p}|(H) < \beta + 1$, etc. Then $\hat{p} \Vdash$ "$B(\tau)$" because otherwise there is $p_0 \geq \hat{p}$ such that $p_0 \Vdash$ "$\neg B(\tau)$", and so by induction there is q compatible with p_0 (hence with $\hat{p}$) $|q|(H) < \beta$, $\operatorname{supp}(q) \subseteq K$, and $q \Vdash$ "$\neg B(\tau)$". By our assumption on $\hat{p}$, since $\hat{p}$ and q are compatible, p and q are compatible, but $p \Vdash$ "$B(\tau)$".

We now use Lemma 36 to show that for any G $\mathbb{P}^\kappa$-generic over M, $M[G]\Vdash$"For every L a $\Sigma^0_{\alpha_0}$ set $(L\cap X\neq G_{(0)}\cap X)$" where $G_{(0)}$ is one of the $\Pi^0_{\alpha_0}$ sets determined by $G\cap\mathbb{P}_{\alpha_0+1}(\emptyset, X)$. Suppose not; then let τ be a term in forcing language of $\mathbb{P}^\kappa$, L a $\Sigma^0_{\alpha_0}$ set with parameter τ, and $p\in G$ such that $p\Vdash$"for every $x\in X$, $x\in L$ iff $x\in G_{(0)}$". Choose H and K with properties (a), (b), and (c) with respect to τ and also so that $\operatorname{supp}(p)\subseteq K$ and $|p|(H)=0$. Since H is countable there exists $x\in X-H$. Let $r=p\cup\{(0,((0),x))\}$ (so $r\Vdash x\in G_{(0)}$). Since $r\Vdash$"$x\in L$", by Lemma 36 there exists q compatible with r, $|q|(H)<\alpha_0$, and $q\Vdash$"$x\in L$". Since $|q|(H)<\alpha_0$, $((0),x)\notin q(0)$. Let $\hat{q}$ be defined by:

$$\hat{q}(\alpha)=\begin{cases}p(\alpha)\cup q(\alpha) & \text{if } \alpha>0,\\ p(0)\cup q(0)\cup\{((0,m),x)\} & \text{otherwise } (m \text{ sufficiently large so that } \hat{q}(0) \text{ is condition}).\end{cases}$$

$\hat{q}\Vdash$"$x\in L$ and $x\notin G_{(0)}$ and ($x\in L$ iff $x\in G_{(0)}$))". This a contradiction and concludes the proof of Theorem 34.

Theorem 37. *For any α_0 a successor ordinal such that $2\leqslant\alpha_0<\omega_1$, it is relatively consistent with ZFC that $|2^\omega|=\omega_2$ and α_0 is the least ordinal such that $R^{\omega_2}_{\alpha_0}=P(\omega_2\times\omega_2)$.*

Remark. In Theorem 52 we remove the restriction that α_0 is a successor (but the continuum in that model is $\aleph_{\omega+1}$). In [1] it is shown that α_0 cannot be ω_1.

Proof. Let M be a countable transitive model of "ZFC+$|2^\omega|=|2^{\omega_1}|=\omega_2$". Let $X=\omega^\omega\cap M$ and define $\mathbb{P}^\alpha$ for $\alpha\leqslant\omega_2$ so that $\mathbb{P}^{\alpha+1}=\mathbb{P}^\alpha*\mathbb{P}_{\alpha_0}(Y_\alpha, X)$ where Y_α is a $\mathbb{P}^\alpha$ term for a subset of X, and at limits take the direct limit. Dovetail so that in $M[G_{\omega_2}]$ for every $Y\subseteq X$ such that $|Y|\leqslant\omega_1$ there are ω_2 many $\alpha<\omega_2$ such that $Y_\alpha=Y$. By Theorem 33 $R^{\omega_2}_{\alpha_0}=P(\omega_2\times\omega_2)$.

Now comes the difficulty: we must show some subset of $\omega_2\times\omega_2$ is not in $R^{\omega_2}_{\alpha_0-1}$. For the remainder of the proof let $(A_s:s\in\omega^{<\omega})$ and $(B_s:s\in\omega^{<\omega})$ be fixed terms in the forcing language of $\mathbb{P}^{\omega_2}$ such that for every $s\in\omega^{<\omega}$ $\emptyset\Vdash$"$A_s\subseteq X$ and $B_s\subseteq\omega_2$". For $p\in\mathbb{P}^{\omega_2}$ define $\operatorname{supp}(p)=\{\alpha<\omega_2:p(\alpha)\neq\emptyset\}$ and $\operatorname{trace}(p)=\{x\in X:\exists\alpha\,\exists t\,(t,x)\in p(\alpha)\}$. By using the c.c.c. of $\mathbb{P}^{\omega_2}$ choose for each $x\in X$ countable sets $I_x\subseteq X$ and $J_x\subseteq\omega_2$ so that:

(1) for each $s\in\omega^{<\omega}$ $\{p\in\mathbb{P}^{\omega_2}:\operatorname{trace}(p)\subseteq I_x$ and $\operatorname{supp}(p)\subseteq J_x\}$ decides "$x\in A_s$", and

(2) for each $y\in I_x$ and $\alpha\in J_x$ $\{p\in\mathbb{P}^{\omega_2}:\operatorname{trace}(p)\subseteq I_x$ and $\operatorname{supp}(p)\subseteq J_x\}$ decides "$y\in Y_\alpha$".

Similarly for $\alpha<\omega_2$ we can pick countable sets $I_\alpha\subseteq X$ and $J_\alpha\subseteq\omega_2$ having properties (1) and (2) with $\alpha, B_s, I_\alpha, I_\alpha$ in place of x, A_s, I_x, I_x.

For $x\in X$ and $\alpha<\omega_2$ let $L(x,\alpha)=(I_x\times J_x)\cup(I_\alpha\times J_\alpha)$ and define for $p\in\mathbb{P}^{\omega_2}$,

$$|p|(x,\alpha)=\max\{|s|_{T_{\alpha_0}}:(s,u)\in p(\gamma) \text{ and } (u,\gamma)\notin L(x,\alpha)\}.$$

Lemma 38. *Fix $x \in X$ and $\alpha < \omega_2$ and let $|p| = |p|(x, \alpha)$. For any $\beta \geq 1$ and $p \in \mathbb{P}^{\omega_2}$ there is a $\hat{p} \in \mathbb{P}^{\omega_2}$ with $|\hat{p}| < \beta + 1$, $\hat{p}$ compatible with p, and for any $q \in \mathbb{P}^{\omega_2}$ if $|q| < \beta$ and $\hat{p}$ and q are compatible, then p and q are compatible.*

Proof. The proof of this is like that of Lemma 35. Let $p_0 \geq p$ so that if $(s, x) \in p(\gamma)$ with $|s| = \lambda$ a limit ordinal greater than β and $|s^\frown i| \leq \beta + 1$, then there is $j < \omega$ so that $(s^\frown i^\frown j, x) \in p_0(\gamma)$. Let G be $\mathbb{P}^{\omega_2}$-generic with $p_0 \in G$. Choose $\Gamma \subseteq G$ finite so that if $((n), u) \in p_0(\gamma)$ (so $p_0 \upharpoonright \gamma \Vdash$ "$u \notin Y_\gamma$") and $(u, \gamma) \in L(x, \alpha)$, then there is a $q \in \Gamma$ such that $q \Vdash$ "$u \notin Y_\gamma$". Define $\hat{p}$ by

$$\hat{p}(\gamma) = \bigcup \{q(\gamma) : q \in \Gamma\} \cup \{(s, u) \in p_0(\gamma) : |s| < \beta + 1 \text{ or } (u, \gamma) \in L(x, \alpha)\}.$$

For any well-founded tree $\hat{T}$ define $C_s(\hat{T})$ for $s \in \hat{T}$ as follows. If $|s|_{\hat{T}} = 0$, then $C_s(\hat{T}) = A_s \times B_s$, otherwise

$$C_s(\hat{T}) = \bigcup \{(X \times \omega_2) - C_{s^\frown i}(\hat{T}) : i < \omega\}.$$

Lemma 39. *If $x \in X$, $\alpha \in \omega_2$, $\hat{T} \in M$ is a well-founded tree, $s \in \hat{T}$ with $|s|_{\hat{T}} = \beta$ where $1 \leq \beta \leq \alpha_0 - 1$, and $p \in \mathbb{P}^{\omega_2}$ such that $p \Vdash$ "$(x, \alpha) \notin C_s(T)$", then there exist q compatible with p, $|q|(x, \alpha) < \beta$, and $q \Vdash$ "$(x, \alpha) \notin C_s(T)$".*

Proof. The proof is by induction on β.

Case 1. $\beta = 1$: Suppose

$$p \Vdash \text{“}(x, \alpha) \in \bigcup_{i \in \omega} (A_{s^\frown i} \times B_{s^\frown i})\text{”}.$$

So there exists $i_0 \in \omega$ and $\hat{p}$ and $\hat{q}$ elements of $\mathbb{P}^{\omega_2}$ so that $(p \cup \hat{p} \cup \hat{q}) \in \mathbb{P}^{\omega_2}$ and using (1) above,

$$(t, u) \in \hat{p}(\gamma) \to (u, \gamma) \in I_x \times J_x$$

and

$$(t, u) \in \hat{q}(\gamma) \to (u, \gamma) \in I_\alpha \times J_\alpha$$

and

$$\hat{p} \Vdash \text{“}x \in A_{s^\frown i_0}\text{”}, \qquad \hat{q} \Vdash \text{“}y \in B_{s^\frown i_0}\text{”}.$$

So $\hat{p} \cup \hat{q} = q$ does the job.

Case 2. *β a limit ordinal:* Suppose

$$p \Vdash \text{“}(x, \alpha) \in \bigcup_{i \in \omega} C_{s^\frown i}(\hat{T})$$

where $|s|_{\hat{T}} = \beta$. Find $q \geq p$ and $i_0 \in \omega$ such that $q \Vdash$ "$(x, y) \in C_{s\frown i_0}(\hat{T})$. Let

$$T_0 = \{t \in \hat{T} : s^\frown i_0 \subseteq t \text{ or } t \subseteq s^\frown i_0\}.$$

Then

$$|s|_{T_0} = |s^\frown i|_{\hat{T}} + 1 < \beta, \quad \text{and} \quad C_s(T_0) = (X \times \omega_2) - C_{s\frown i_0}(T),$$

hence $q \Vdash$ "$(x, \alpha) \notin C_s(T_0)$" where $|s|_{T_0} < \beta$; so by induction hypothesis there exists r compatible with q (and hence with p), $|r|\,(x, \alpha) < \beta$, and $r \Vdash$ "$(x, \alpha) \in C_{s\frown i_0}(T)$". r does the trick.

Case 3. $\beta + 1$: Since $\beta + 1 < \alpha_0$, let q be as from Lemma 38.

Define $D \subseteq X \times \omega_2$ by $D = \{(x, \alpha) : x \in G^\alpha_{(0)}$ where $G^\alpha_{(0)}$ is one of the $\mathbf{\Pi}^0_{\alpha_0 - 1}$ sets created on the αth step. D is $\mathbf{\Pi}^0_{\alpha_0-1}$ in the rectangles on $X \times \omega_2$. We want to show it is not $\mathbf{\Sigma}^0_{\alpha_0 - 1}$ in the rectangles on $X \times \omega_2$ in $M[G_{\omega_2}]$.

Define: (x, α) is free (with respect to $(A_s : s \in \omega^{<\omega})$, $(B_s : s \in \omega^{<\omega})$) iff $[x \notin I_\alpha$ and $\alpha \notin J_x]$.

Lemma 40. *If $T \subseteq \omega^{<\omega}$ is well-founded and $T \in M$, $s \in T$ with $|s|_T \leq \alpha_0 - 1$, (x, α) is free, and $Y_\alpha = \emptyset$; then for every $p \in \mathbb{P}^{\omega_2}$ such that $|p|\,(x, \alpha) = 0$ it is not the case that $p \Vdash$ "$(x, \alpha) \in D$ iff $(x, \alpha) \notin C_s(T)$".*

Proof. Let $\hat{p} \geq p$ by defining $\hat{p}(\gamma) = p(\gamma)$ for $\gamma \neq \alpha$ and $\hat{p}(\alpha) = p(\alpha) \cup \{((0), x)\}$. Then $\hat{p} \Vdash$ "$(x, \alpha) \in D$" so by Lemma 39 there exists q compatible with $\hat{p}$, $|q|\,(x, \alpha) < \alpha_0$, and $q \Vdash$ "$(x, \alpha) \notin C_s(T)$". But (x, α) free implies that $(x, \alpha) \notin L(x, \alpha)$ so q does not say "$x \in G^{\gamma_\alpha}_{(0)}$". Thus for a sufficiently large $m < \omega$ r defined by $r(\gamma) = p(\gamma) \cup q(\gamma)$ for $\gamma \neq \alpha$ and $r(\alpha) = p(\alpha) \cup q(\alpha) \cup \{((0, m), x)\}$ is a member of $\mathbb{P}^{\omega_2}$. But $r \Vdash$ "$(x, \alpha) \notin D$ and $(x, \alpha) \notin C_s(T)$", a contradiction since r extends p.

Since the terms $(A_s : s \in \omega^{<\omega})$ and $(B_s : s \in \omega^{<\omega})$ were arbitrary to start with it will complete the proof of the theorem to find lots of (x, α) free.

The next lemma generalized Kunen [9, p. 74].

Lemma 41. *Given $|I_\alpha| < \kappa$ for $\alpha < \kappa^+$, there exists $G \subseteq \kappa^+$ with $|G| = \kappa^+$ and there is S with $|S| \leq \kappa$ so that for any $\alpha, \beta \in G$ if $\alpha \neq \beta$, then $I_\alpha \cap I_\beta \subseteq S$.*

Proof. We can assume $I_\alpha \subseteq \kappa^+$.

Define $\mu_\alpha, z_\alpha < \kappa^+$ for $\alpha < \kappa^+$ nondecreasing so that:

(1) $\mu_\lambda = \sup\{\mu_\alpha : \alpha < \lambda\}$ for λ limit;

(2) z_α's are strictly increasing;

(3) for α a successor and for distinct $\beta, \gamma < \alpha$ $I_{z_\beta} \cap I_{z_\gamma} \subseteq \mu_\alpha$;

(4) if $\mu_{\alpha+1} > \mu_\alpha$, then for any $z > z_\alpha$ $\mu_\alpha \not\supseteq I_z \cap \bigcup\{I_{z_\beta} : \beta \leq \alpha\}$ and $\bigcup\{I_{z_\beta} : \beta \leq \alpha\} \subseteq \mu_{\alpha+1}$.

Let $G=\{z_\alpha:\alpha<\kappa^+\}$ and $S=\sup\{\mu_\alpha:\alpha<\kappa^+\}$. To see that $S<\kappa^+$ note that for any $\alpha<\kappa^+$ $|\{\beta:\mu_{\beta+1}>\mu_\beta$ and $\beta<\alpha\}|<\kappa$. This is because $I_{z_\alpha}\cap(\mu_{\beta+1}-\mu_\beta)\neq\emptyset$ for all $\beta<\alpha$ such that $\mu_{\beta+1}>\mu_\beta$.

Lemma 42. *There exists $\Sigma_0\subseteq X$ and $\Sigma_1\subseteq\omega_2$ with $|\Sigma_0|=|\Sigma_1|=\omega_2$, for every $\alpha\in\Sigma_1$ $Y_\alpha=\emptyset$, and for every $(x,\alpha)\in\Sigma_0\times\Sigma_1$ (x,α) is free.*

Proof. By Lemma 41 there exists $\hat{\Sigma}_0\subseteq X$ and $S\subseteq\omega_2$ with $|\hat{\Sigma}_0|=\omega_2$ and $|S|<\omega_2$ so that for every distinct $x, y\in\hat{\Sigma}_0$ $J_x\cap J_y\subseteq S$. Since $\{J_x-S:x\in\hat{\Sigma}_0\}$ is a disjoint family, we can cut down $\hat{\Sigma}_0$ (maintaining $|\hat{\Sigma}_0|=\omega_2$) and find $\hat{\Sigma}_1\subseteq\omega_2$ so that $|\hat{\Sigma}_1|=\omega_2$, for every $\alpha\in\hat{\Sigma}_1$ $Y_\alpha=\emptyset$, and for every $x\in\hat{\Sigma}_0$ $J_x\cap\hat{\Sigma}_1=\emptyset$. Applying Lemma 41 again find $\Sigma_1\subseteq\hat{\Sigma}_1$ with $|\Sigma_1|=\omega_2$ and $T\subseteq X$ with $|T|<\omega_2$ so that for every distinct α, $\beta\in\Sigma_1$ $I_\alpha\cap I_\beta\subseteq T$. Since $\{I_\alpha-T:\alpha\in\Sigma_1\}$ are disjoint by cutting down Σ_1 (maintaining $|\Sigma_1|=\omega_2$) we can assume Σ_0 defined to be equal to $\hat{\Sigma}_0-(T\cup\bigcup\{I_\alpha:\alpha\in\Sigma_1\})$ has cardinality ω_2. Σ_0 and Σ_1 do the job.

Lemma 42 finishes the proof of Theorem 37.

Remark. There is nothing special about ω_2 in the above theorem; we could have replaced it by any larger cardinal κ with $\kappa^{<\kappa}=\kappa$.

Now we turn to a slightly different problem. For X a topological space a set $A\subseteq X^n$ is projective iff it is in the smallest class containing the Borel sets (in the product topology on X^m for any $m\in\omega$) and closed under complementation and projection ($B\subseteq X^m$ is the projection of $C\subseteq X^{m+1}$ iff ($\bar{y}\in B$ iff $\exists x\in X\, x\bar{y}\in C$)).

Theorem 43. *If M is a countable transitive model of ZFC, then there exists N a c.c.c. Cohen extension of M such that if $M\cap\omega^\omega=X$, then $N\models$"Every projective set in X is Borel and the Borel hierarchy of X has ω_1 distinct levels* (ord$(X)=\omega_1$)".

This shows the relative consistency of an affirmative answer to a question of Ulam [31, p. 10]. Note that since $X\times X$ is homeomorphic to X (take any recursive coding function), if for every $B\subseteq X\times X$ Borel $\{x:\exists y(x,y)\in B\}$ is Borel in X, then every projective set in X is Borel in X.

Proof. The proof is slightly simpler if we assume that CH holds in M. We give the proof in that case and then later indicate the necessary modifications. In any case $|2^\omega|^M=|2^\omega|^N$.

Construct a sequence $M=M_0\subseteq M_1\subseteq\cdots\subseteq M_{\omega_1}=N$, by iterated forcing so that $M_{\alpha+1}$ is obtained from M_α by $\mathbf{\Pi}^0_{\alpha+1}$-forcing. On the αth stage we are presented with a term τ_α in the forcing language of $\mathbb{P}^\alpha$ denoting a real. Then letting Y_α be the projective set (over X) determined by τ_α we let $\mathbb{P}^{\alpha+1}=\mathbb{P}^\alpha*\mathbb{P}_{\alpha+1}(Y_\alpha,X)$. What is being done is that at stage α we make Y_α a $\mathbf{\Pi}^0_{\alpha+1}$ set intersected with X. The reason this will work is that after the αth stage our forcing will not interfere with the Borel hierarchy on X up to the αth level. Since this is c.c.c. forcing we can imagine that each X-projective set in N is eventually caught by some τ_α for $\alpha<\omega_1$. So it is clear that $N\Vdash$"Every X-projective set is Borel in X", for any $N=M[G]$, where G is $\mathbb{P}^{\omega_1}$-generic over M. Define for $H\subseteq X$ and $p\in\mathbb{P}$, $|p|(H)=\max\{|s|_{T_{\alpha+1}}$: there exist $\alpha<\omega_1$ and $x\notin H$, $(s,x)\in p(\alpha)\}$. Given τ a term in the forcing language of $\mathbb{P}^\gamma$ denoting a subset of ω $(\gamma<\omega_1)$, there exists $H\subseteq X$ such that:

(a) H is countable;
(b) $\forall n\in\omega$, $\{p\in\mathbb{P}^\gamma : |p|(H)=0\}$ decides "$n\in\tau$";
(c) $\forall\beta<\gamma$ and $x\in H$, $\{p\in\mathbb{P}^\gamma : |p|(H)=0\}$ decides "$x\in Y_\beta$".

Lemma 44. (Write $|p|=|p|(H)$). "*Exactly statement of Lemma* 38" *for* $\mathbb{P}^\gamma$.

Proof. Extend $p\le p_0$ as before. Let G be $\mathbb{P}^\gamma$-generic with $p_0\in G$. Choose $\Gamma\subseteq G$ finite so that:

(1) $q\in\Gamma\to|q|(H)=0$;
(2) if $\langle\langle n\rangle, x\rangle\in p_0(\alpha)$ (so $p\restriction_\alpha\Vdash$"$x\notin Y_\alpha$"), then $\exists q\in\Gamma\cap\mathbb{P}^\alpha$ such that $q\Vdash$"$x\notin Y_\alpha$".
Define

$$\hat{p}(\alpha)=\bigcup\{r(\alpha): r\in\Gamma\}\cup\{\langle s,x\rangle\in p_0(\alpha): |s|_{T_\alpha}<\beta+1 \text{ or } x\in H\}.$$

$\hat{p}$ is a condition because if $\langle\langle n\rangle, x\rangle\in p(\alpha)$ and $|\langle n\rangle|_{T_{\alpha+1}}<\beta+1$, then $\hat{p}\restriction_\alpha\ge p\restriction_\alpha$ (so $\hat{p}\restriction_\alpha\Vdash$"$x\notin Y_\alpha$" as required).

The $r\in\Gamma$ take care of such requirements about $x\in H$. The rest of the proof is the same.

Lemma 45. *If τ, H, γ are as above, $B(v)$ is a $\mathbf{\Sigma}^0_\beta$ predicate for some $\beta\ge 1$ with parameter from M, and $p\in\mathbb{P}^\gamma$ such that $p\Vdash$"$B(\tau)$", then there is a $q\in\mathbb{P}^\gamma$ compatible with p, $|q|(H)<\beta$ and $q\Vdash$"$B(\tau)$".*

Proof. The proof is the same as before.

We can assume that for unboundedly many $\alpha<\omega_1$ $Y_\alpha=\emptyset$. Let $G_\alpha(G^\alpha_{\langle 0\rangle})$ be one of the $\mathbf{\Pi}^0_\alpha$ sets determined by $G\cap P_{\alpha+1}(\emptyset, X)$ where $Y_\alpha=\emptyset$.

Claim. $M[G]\vDash$"*for any* $L\in\mathbf{\Sigma}^0_\alpha$ $(L\cap X\neq G_\alpha\cap X)$".

Proof. Otherwise let τ be a term for a real in the forcing language $\mathbb{P}^\gamma$ for some $\gamma<\omega_1$ such that for some L a $\mathbf{\Sigma}^0_\alpha$ set with parameter τ and some $p\in P^\gamma$ $p\Vdash$"$L\cap X=G_\alpha\cap X$". Choose H with properties (a), (b), and (c) with respect to τ, and also $|p|(H)=0$. Let $x\in X-H$. Define $r(\alpha)=p(\alpha)\cup\{(0),x)\}$ and for $\beta\neq\alpha$ $r(\beta)=p(\beta)$. Note that $r\Vdash$"$x\in G_\alpha$" hence $r\Vdash$"$x\in L$". By Lemma 45 there exists $q\in\mathbb{P}^\gamma$ compatible with r, $|q|(H)<\beta$, and $q\Vdash$"$x\in L$". Since $x\notin H$ we know

$((0), x) \notin q(\alpha)$. Define $\hat{q} \in \mathbb{P}^{\omega_1}$ by $\hat{q}(\beta) = p(\beta) \cup q(\beta)$ for $\beta \neq \alpha$ and $\hat{q}(\alpha) = p(\alpha) \cup q(\alpha) \cup \{((0, n), x)\}$ where n is picked sufficiently large so $\hat{q}(\alpha)$ is a condition. But then $\hat{q} \Vdash$ "$x \in L$ and $x \notin G_\alpha$ and ($x \in L$ iff $x \in G_\alpha$)" and this is a contradiction. This concludes the proof of Theorem 43.

When the continuum hypothesis does not hold in M the construction of N still has ω_1 steps but at each step we must take care of all reals in the ground model. That is $\mathbb{P}^{\alpha+1} = \mathbb{P}^\alpha * Q_\alpha$ where Q_α is a term denoting $\sum \{\mathbb{P}_{\alpha+1}(H_x, X) : x \in \omega^\omega \cap M[G_\alpha]\}$ for G $\mathbb{P}^\alpha$-generic over M. This works since all reals in $N = M[G]$ for G $\mathbb{P}^{\omega_1}$-generic over M are caught at some countable stage.

Remark. It is easy to see that if $V = L$ there is an $X \subseteq \omega^\omega$ uncountable Π^1_1 set such that $X \in L$ and $X \times X$ is homeomorphic to X. Also by absoluteness it is possible to make sure that for every A $\mathbf{\Sigma}^1_2$ in ω^ω, $A \cap X$ is Borel in X. This family of sets includes those obtained by the Souslin operation from Borel sets in X.

Theorem 46. (MA). $\exists X \subseteq 2^\omega$ ord$(X) = \omega_1$ *and* $\forall A \in \Sigma^1_1$ *in* 2^ω $\exists B$ Borel(2^ω) $A \cap X = B \cap X$.

Proof. Let $\mathbb{B}$ be the c.c.c. countably generated boolean algebra of Theorem 9 with $K(\mathbb{B}) = \omega_1$. $\mathbb{B} \simeq$ Borel$(2^\omega)/J$ for some J an ω_1-saturated σ-ideal in the Borel sets.

Lemma 47. *If I is an ω_1-saturated σ-ideal in* Borel(2^ω), *then* $B_I = \{A \subseteq 2^\omega : \exists B$ Borel $\exists C \in I$ $(A \Delta B) \subseteq C\}$ *is closed under the Souslin operation.*

For a proof the reader is referred to [11, p. 95].

By Theorem 14 MA implies there is $X \subseteq 2^\omega$ a J-Luzin set. For any $\alpha < \omega_1$ there is A $\mathbf{\Pi}^0_\alpha$ so that for every B $\mathbf{\Sigma}^0_\alpha$, $(A \Delta B) \notin J$, hence $|(A \Delta B) \cap X| = |2^\omega|$, so $A \cap X \neq B \cap X$, and thus ord$(X) = \omega_1$. If A is $\mathbf{\Sigma}^1_1$, then by Lemma 47 there is B Borel and C in J with $A \Delta B \subseteq C$. Since $|C \cap X| < |2^\omega|$ by MA $\exists D \in$ Borel(2^ω) $(A \Delta B) \cap X = D \cap X$. So $A \cap X = (B \Delta D) \cap X$.

This suggests the following question:

Can you have $X \subseteq 2^\omega$ such that every subset of X is Borel in X and the Borel hierarchy on X has ω_1 distinct levels? The answer is no.

Theorem 48. *If $X \subseteq 2^\omega$ and every subset of X is Borel in X, then* ord$(X) < \omega_1$.

Proof. Let $X = \{x_\alpha : \alpha < \kappa\}$ and $X_\alpha = \{x_\beta : \beta < \alpha\}$.

Lemma 49. *If $|X| \leq \kappa$, every subset of X is Borel in X, and $R^\kappa_{\omega_1} = P(\kappa \times \kappa)$, then* ord$(X) < \omega_1$.

Proof. Since every rectangle in $X\times X$ is Borel in $X\times X$ and $R^{\kappa}_{\omega_1}=P(\kappa\times\kappa)$, every subset of $X\times X$ is Borel in $X\times X$. Suppose for contradiction $\forall\alpha<\omega_1\,\exists H_\alpha\subseteq X$ not Π^0_α in X. Let $H=\bigcup_{\alpha<\omega_1}\{x_\alpha\}\times H_\alpha$. For some $\alpha<\omega_1$, H is Π^0_α in $X\times X$. But then every cross section of H is Π^0_α in X contradiction.

The proof of the theorem is by induction on $|X|=\kappa$.

For $\kappa=\omega_1$ it follows from Lemma 49 since $R^{\omega_1}_2=P(\omega_1\times\omega_1)$.

For $\mathrm{cof}(\kappa)=\omega$ it is trivial.

For $\mathrm{cof}(\kappa)>\omega_1$: $\forall\alpha<\kappa$ choose β_α minimal $<\omega_1$ so that every subset of X_α is $\Pi^0_{\beta_\alpha}$ in X (we can do this since X_α is Π^0_β in X some $\beta<\omega_1$). Since $\mathrm{cof}(\kappa)>\omega_1$ there exists $\alpha_0<\omega_1$ such that for a final segment of ordinal less than κ, $\beta_\alpha=\alpha_0$. By Theorem 33 $R^{\kappa}_{\omega_1}=P(\kappa\times\kappa)$ so by Lemma 49 $\mathrm{ord}(X)<\omega_1$.

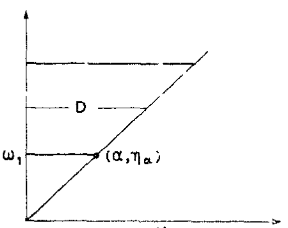
For $\mathrm{cof}(\kappa)=\omega_1$: Let $\eta_\alpha\uparrow\kappa$ for $\alpha<\omega_1$ be an increasing continuous cofinal sequence.

Lemma 50. *$\exists\beta_0<\omega_1\,\forall\alpha<\omega_1 X_{\eta_\alpha}$ is $\Pi^0_{\beta_0}$ in X.*

Proof. If $G\subseteq\kappa\times\kappa$ is the graph of a partial function, then $G\in R^{\kappa}_2$ (Rao [21]). This is because if $f:D\to\kappa$ where $D\subseteq\kappa$, then viewing $x\subseteq$ irrational real numbers we have: $(f(\alpha)=\beta)$ iff ($\alpha\in D$ and $\forall r\in Q(r<x_{f(\alpha)}$ iff $r<x_\beta$)) where Q is the set of rational numbers.

Then $D=\{(\alpha,\beta):\alpha<\omega_1\wedge\beta<\eta_\alpha\}$ is the complement in $\omega_1\times\kappa$ of a countable union of graphs of functions from κ into ω_1. Hence the set $\bigcup_{\alpha<\omega_1}\{x_\alpha\}\times X_{\eta_\alpha}$ is Borel in $X\times X$. Say it is $\Pi^0_{\beta_0}$. It follows that each X_{η_α} is $\Pi^0_{\beta_0}$.

For all $\lambda<\omega_1$ let $\beta(\lambda)$ be minimal so that every subset of X_{η_λ} is $\Pi^0_{\beta(\lambda)}$ in X. If the hypothesis of Theorem 33 fails, then $\exists f:\omega_1\to\omega_1$ increasing so that for all $\lambda<\omega_1$ $\beta(f(\lambda))<\beta(f(\lambda+1))$. So for all $\lambda<\omega_1$ there is some $H_\lambda\subseteq X_{\eta_{f(\lambda+1)}}$ which is not $\Pi^0_{\beta(f(\lambda))}$ in X. Since every subset of $X_{\eta f(\beta)}$ is $\Pi^0_{\beta(f(\beta))}$ in X we can assume $H_\lambda\subseteq(X_{\eta_{f(\lambda+1)}}-X_{\eta_{f(\lambda)}})$. Let $H=\bigcup_{\lambda<\omega_1}H_\lambda$. Then H is $\Pi^0_{\alpha_0}$ in X for some $\alpha_0<\omega_1$. But for each λ, $H_\lambda=H\cap(X_{\eta_{f(\lambda+1)}}-X_{\eta_{f(\lambda)}})$. so each H_λ is $\Pi^0_{\max(\alpha_0,\beta_0+1)}$ in X, contradiction. This ends the proof of Theorem 48.



Remark. Kunen has noted that Theorem 48 may be generalized to nonseparable metric spaces. Let $\mathbb{B}$ be a σ-discrete basis for X and assume that every subset of X is Borel in X. By using σ-discreteness it is easily seen that $\exists \mathcal{H} \subseteq \mathbb{B}\, \exists \beta < \omega_1$ so that $\mathbb{B} - \mathcal{H}$ is countable and $\forall U \in \mathcal{H}\, \mathrm{ord}(U) \leq \beta$. But $Y = \{x \in X : \forall U \in \mathbb{B}\, (x \in U \to U \notin \mathcal{H})\}$ is separable and hence by the theorem $\mathrm{ord}(Y) < \omega_1$, and so $\mathrm{ord}(X) < \omega_1$.

As a partial converse of Theorem 33 we have:

Theorem 51. *If $\kappa = |2^\omega|$, $\kappa^{<\kappa} = \kappa$, and $R^\kappa_{\alpha_0} = P(\kappa \times \kappa)$, then there is $X \subseteq 2^\omega$ with $|X| = \kappa$ and every subset of X of cardinality less than κ is $\mathbf{\Pi}^0_{\alpha_0}$ in X.*

Proof. Let Z_α for $\alpha < \kappa$ be all the subsets of κ of cardinality less than κ. Put $Z = \bigcup_{\alpha<\kappa} \{\alpha\} \times Z_\alpha$ and $W = \{(\alpha, \beta) : \alpha < \beta < \kappa\}$. Let $\{A_n : n < \omega\}$ be closed under finite boolean combinations and $Z, W \in \{A_n \times A_m : n, m < \omega\}_{\alpha_0}$. The map $F : \kappa \to 2^\omega$ defined by $(F(\alpha)(n) = 1$ iff $\alpha \in A_n)$ is 1–1 and the set $X = F''\kappa$ has the required property.

For any cardinal κ let $R(\kappa)$ be the least $\beta < \omega_1$ such that $R^\kappa_\beta = P(\kappa \times \kappa)$ or ω_1 if no such β exists.

Theorem 52. *It is relatively consistent with ZFC that $|2^\omega| = \omega_{\omega+1}$, for every $n \leq \omega$ $R(\omega_n) = 1 + n$, and $R(\omega_{\omega+1}) = \omega$. This can be generalized to show that for any $\lambda < \omega_1$ a limit ordinal it is consistent with ZFC that $R(|2^\omega|) = \lambda$.*

Proof. Let $M \models$ "ZFC+MA+$|2^\omega| = \omega_{\omega+1}$" be countable and transitive. Let $\kappa = \omega_{\omega+1}$ and define $\mathbb{P}^\alpha$ for $\alpha \leq \kappa$ so that $\mathbb{P}^{\alpha+1} = \mathbb{P}^\alpha * \mathbb{P}_{2+\beta+1}(X_\alpha, Y_\alpha)$ where $Y_\alpha \subseteq 2^\omega$, $Y_\alpha \in M$, $|Y_\alpha| = \omega_{\beta+1}$, and $\emptyset \Vdash$ "$X_\alpha \subseteq Y_\alpha$". At limits take the direct limit. By dovetailing arrange that for any G $\mathbb{P}^\kappa$-generic over M, $M[G] \models$ "If $Y \subseteq 2^\omega$, $Y \in M$, and $|Y| = \omega_{\beta+1}$ for some $\beta < \omega$, then every subset of Y is $\mathbf{\Pi}^0_{2+\beta+1}$ in Y".

As in the proof of Theorem 34 given any τ a term for a subset of ω, find in M, $H \subseteq 2^\omega$, $K \subseteq \kappa$ so that: Let $Q = \{p \in \mathbb{P}^\kappa : \mathrm{supp}(p) \subseteq K,\ |p|(H) = 0\}$:

(1) $|H| \leq \omega_{\beta_0}$, $|K| \leq \omega_{\beta_0}$.

(2) $\forall n \in \omega$ Q decides "$n \in \tau$".

(3) $\forall \beta \in K\ \forall x \in H$ Q decides "$x \in X_\beta$".

(4) If $\alpha \in K$ and $|Y_\alpha| \leq \omega_{\beta_0}$, then $Y_\alpha \subseteq H$.

Lemma 53. *If H, K have property* (3), (4) *above, then for any $p \in \mathbb{P}^\kappa$ and β with $1 \leq \beta < 2 + \beta_0$ there is $\hat{p}$ compatible with p, $|\hat{p}|(H) < \beta + 1$, $\mathrm{supp}(\hat{p}) \subseteq K$, and for any q if $|q|(H) < \beta$, $\mathrm{supp}(q) \subseteq H$, and $\hat{p}$ and q are compatible, then p and q are compatible.*

Proof. The proof of this is just like the proof of Lemma 35. To check that the $\hat{p}$

gotten there is an element of $\mathbb{P}^\kappa$, note that if $((n), x) \in \hat{p}(\alpha)$, then $x \in H$. Because if $x \notin H$ and $\alpha \in K$, then $|Y_\alpha| \geq \omega_{\beta_0+1}$ because of (4). Say $|Y_\alpha| = \omega_{\gamma+1}$, so $\mathbb{P}^{\alpha+1} = \mathbb{P}^\alpha * \mathbb{P}_{2+\gamma+1}(X_\alpha, Y_\alpha)$ and $|(n)|_{T_{2+\gamma+1}} = 2+\gamma \geq 2+\beta_0 \geq \beta+1$, but then it was thrown out, contradiction.

Lemma 54. *Suppose H and K have properties* (2), (3), *and* (4) *for* $\tau \subseteq \omega$. *Suppose* $1 \leq \beta \leq 2+\beta_0$ *and* $B(v)$ *is a* Σ^0_β *predicate with parameters from* M, $p \in \mathbb{P}^\kappa$ *and* $p \Vdash$ "$B(\tau)$". *Then* $\exists q \in \mathbb{P}^\kappa$ *compatible with* p, $|q|(H) < \beta$, $\text{supp}(q) \subseteq K$ *and* $q \Vdash$ "$B(\tau)$".

Proof. This follows from Lemma 53 just as in Theorem 34.

From Lemma 54 we have that:

(A) For any $Y \subseteq 2^\omega$ with $Y \in M$ and n with $1 \leq n \leq \omega$ ($|Y| = \omega_n$ iff Y is a Q_{2+n}-set). We claim that:

(B) For any $n < \omega$ there are $X, Y \subseteq 2^\omega$ with $|X| = |Y| = \omega_{n+2}$ so that if U is the usual Π^0_{n+2} set universal for Π^0_{n+2} sets, then $U \cap (X \times Y)$ is not Σ^0_{n+2} in the abstract rectangles on $X \times Y$.

To prove (B) just generalize the argument of Theorem 37, for $n = 0$ the argument is the same. Let $X \subseteq 2^\omega$ be in M with $|X| = \omega_{n+2}$. Choose $K \subseteq \kappa$, $|K| = \omega_{n+2}$, and $K \in M$, so that for any $\alpha \in K$ $Y_\alpha = X$ and $\emptyset \Vdash$ "$X_\alpha = \emptyset$". Let $Y = \{y_\alpha : \alpha \in K\}$ where y_α is the Π^0_{n+2} code (with respect to U) for $G^\alpha_{(0)}$. To generalize the argument allow $I_x, J_x, I_\alpha, J_\alpha$ to have cardinality $\leq \omega_n$ and also whenever $\gamma \in J_x (\gamma \in J_\alpha)$ and $|Y_\gamma| \leq \omega_n$, then $Y_\gamma \subseteq I_x (Y_\gamma \subseteq I_\alpha)$.

In $M[G]$ for any $n < \omega$ $R(\omega_n) = 1+n$. To see this, let $Y \subseteq 2^\omega$ with $Y \in M$ and $|Y| = \omega_{n+1}$. If $X \subseteq Y$ and $|X| \leq \omega_n$, then there is $Z \in M$ with $|Z| \leq \omega_n$ and $X \subseteq Z$. Because $M \vDash$ "MA" Z is Π^0_2 in Y and since X is Π^0_{2+n} in Z by (A), we have X is Π^0_{2+n} in Y. By Theorem 33 $R^{\omega_{n+1}}_{n+2} = P(\omega_{n+1} \times \omega_{n+1})$. By (B) $n+2$ is the least which will do.

Thus $R(\omega_\omega) = \omega$. To see that $R(\kappa) = \omega$ let $Y \subseteq 2^\omega$ with $Y \in M$ $|Y| = \kappa$, and every subset $Z \subseteq Y$ such that $|Z| < \kappa$ and $Z \in M$ is Σ^0_2 in Y (see Theorem 17). In $M[G]$ every $Z \subseteq Y$ with $|Z| < \kappa$ is Σ^0_ω in Y, so by Theorem 33 $R^\kappa_\omega = P(\kappa \times \kappa)$.

Remark. It is easy to generalize Theorem 52 to show that for any $\lambda < \omega_1$ a limit ordinal and $\kappa > \omega$ of cofinality ω, it is consistent that $|2^\omega| = \kappa^+$ and $R(\kappa^+) = \lambda$.

Theorem 55. *It is relatively consistent with ZFC that*

(a) $|2^\omega| = \omega_{\omega_1+1}$,

(b) *for any* $\alpha < \omega_1$ *there is a* Q_α *set*.

(c) $R(\omega_n) = n+1$ for $n < \omega$,

(d) $R(\omega_\lambda) = \lambda$ *for* $\lambda < \omega_1$ *a limit ordinal*,

(e) $R(\omega_{\lambda+n+1}) = \lambda + n$ *for* $\lambda < \omega_1$ *a limit ordinal and* $n < \omega$.

The proof of this is an easy generalization of Theorem 52 and is left to the reader.

A set $U \subseteq 2^\omega \times 2^\omega$ is universal for the Borel sets iff for every $B \subseteq 2^\omega$ there exists $x \in 2^\omega$ such that $B = U_x = \{y : (y, x) \in U\}$.

Theorem 56. *It is relatively consistent with ZFC that no set universal for the Borel sets is in the σ-algebra generated by the abstract rectangles in $2^\omega \times 2^\omega$.*

Proof. Let $M \models$"ZFC$+\neg$CH" and let

$$Q = \sum_{\beta<\omega_2} \left(\sum\{\mathbb{P}_\alpha(\emptyset, 2^\omega \cap M) : \alpha < \omega_1\}\right).$$

Let G be Q-generic over M, then in $M[G]$ there is no set U universal for the Borel sets in the σ-algebra generated by the rectangles. Suppose G is given by $(y^\alpha_\beta : T^*_{\alpha+1} \to 2^{<\omega} : \alpha < \omega_1$ and $\beta < \omega_2)$ where $T_{\alpha+1}$ is the normal $\alpha+1$ tree used in the definition of $\mathbb{P}_{\alpha+1}$ and $G^{\langle 0\rangle}_{y^\alpha_\beta}$ are the $\mathbf{\Pi}^0_\alpha$ sets determined by y^α_β. Then as before we can easily get for each $\alpha < \omega_1$ that $V^\alpha = \{(x, \beta : x \in G^{\langle 0\rangle}_{y^\alpha_\beta}\}$ is not $\mathbf{\Sigma}^0_\alpha$ in the abstract rectangles on $(2^\omega \times \omega_2)$. Now suppose such a U existed and were $\mathbf{\Sigma}^0_\alpha$ in the abstract rectangles on $2^\omega \times 2^\omega$. Choose $F : \omega_2 \to 2^\omega$ (necessarily 1–1) so that $\forall \beta < \omega_2 \forall x \in 2^\omega$ $((x, \beta) \in V^\alpha \leftrightarrow (x, f(\beta)) \in U)$. If U is $\mathbf{\Sigma}^0_\alpha$ in $\{A_n \times B_n : n < \omega\}$, then V^α is $\mathbf{\Sigma}^0_\alpha$ in $\{A_n \times f^{-1}(\beta_n) : n < \omega\}$, contradiction.

Remarks. (1) In [9] Kunen shows that if one adds ω_2 Cohen reals to a model of GCH, then no well-ordering of ω_2 is in $R^{\omega_2}_{\omega_1}$.

(2) In [1] it is shown that if G is a countable field of sets with Borel$(2^\omega) \subseteq G_{\omega_1}$, the order of G is ω_1.

In the model of Theorem 56 for any countable G and $\alpha < \omega_1$ Borel(2^ω) is not included in G_α. This can be seen as follows. Let $G = \{A_n : n < \omega\}$ and let $\{s_n : n < \omega\} = T^*$ where T is a normal α tree. Define for any $y \in \omega^\omega$ and $s \in T$ the set G^s_y as follows. For $s = s_n$ let $G^s_y = A_{y(n)}$, otherwise $G^s_y = \bigcap \{\omega^\omega - G^{s_n}_y : n < \omega\}$. If $U = \{(x, y) : x \in G^0_y\}$, then U is "$\mathbf{\Pi}^0_\alpha$" in the abstract rectangles and universal for all Borel sets, contradicting Theorem 56.

5. Problems

Show:

(1) If $|X| = \omega_1$, then X is not a Q_ω set.

(2) If $R^{\omega_2}_\omega = P(\omega_2 \times \omega_2)$, then there is $n < \omega$ with $R^{\omega_2}_n = P(\omega_2 \times \omega_2)$.

(3) If there exists a Q_ω set, then there exists a Q_n set for some $n < \omega$.

(4) If $R^{\omega_2}_{\omega_1} = P(\omega_2 \times \omega_2)$ and $|2^\omega| = \omega_2$, then $|2^{\omega_1}| = \omega_2$.

(5)[2] If there is a Q_2 set of size ω_1, then every subset of 2^ω of size ω_1 is a Q_2 set.

[2] Answered by William Fleissner in the negative; cf. "On Q-sets" by Fleissner and Miller, Proc. AMS, to appear.

(6) If X is a Q_α set and Y is a Q_β set, then $2 \leq \alpha < \beta$ implies $|X| < |Y|$.

Show consistency of:

(7) $\{\alpha : X \subseteq 2^\omega \text{ ord}(X) = \alpha\} = \{1\} \cup \{\alpha \leq \omega_1 : \alpha \text{ is even}\}$.

(8) $|2^\omega| = \omega_3$ and for any $X \subseteq 2^\omega$ if $|X| = \omega_1$, then X is a Q_7 set, if $|X| = \omega_2$, then X is a $Q_{\omega+3}$ set, and if $|X| = \omega_3$, then $\text{ord}(X) = \omega_1$.

(9) For any $\alpha \leq \omega_1$ there is a $\mathbf{\Pi}^1_1$ X with $\text{ord}(X) = \alpha$.

(10) For any $X \subseteq 2^\omega$ if $|X| \geq \omega_1$ then there is an X-projective set not Borel in X.

(11) There is no G countable with $\mathbf{\Sigma}^1_1 \subseteq G_{\omega_1}$. (This is a problem of Ulam, see Fund. Math. 30 (1938) 365.)

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