Universal sets for pointsets properly on the n^{th} level of the projective hierarchy

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Abstract

The Axiom of Projective Determinacy implies the existence of a universal $\underset{\sim}{\Pi}^1_n \setminus \underset{\sim}{\Delta}^1_n$ set for every $n \geq 1$. Assuming MA$(\aleph_1) + \aleph_1 = \aleph_1^{\mathbb{L}}$ there exists a universal $\underset{\sim}{\Pi}^1_1 \setminus \underset{\sim}{\Delta}^1_1$ set. In ZFC there is a universal $\underset{\sim}{\Pi}^0_\alpha \setminus \underset{\sim}{\Delta}^0_\alpha$ set for every α.

1 Introduction

It is a classical result of descriptive set theory that universal sets exist for various natural pointclasses[2] such as $\underset{\sim}{\Pi}^0_\alpha$, $\underset{\sim}{\Sigma}^0_\alpha$, $\underset{\sim}{\Pi}^1_n$, and $\underset{\sim}{\Sigma}^1_n$.

Definition 1.1. *For Γ a pointclass and X a Polish space, a subset $\mathcal{U} \subseteq 2^\omega \times X$ is a universal set for Γ iff*
(i) $\mathcal{U} \in \Gamma$ and
(ii) for all $B \subseteq X$
$B \in \Gamma$ iff there exists $x \in 2^\omega$ such that $B = \mathcal{U}_x =_{df} \{y \ : \ (x, y) \in \mathcal{U}\}$.

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[2]A pointclass is a collection of pointsets and a pointset is a subset of a Polish space.

The existence of universal sets for pointclasses of the form $\underset{\sim}{\Pi}^1_n \setminus \underset{\sim}{\Delta}^1_n$ has not been extensively investigated. Hjorth [4] shows that the existence of a set universal for $\underset{\sim}{\Pi}^1_1 \setminus \underset{\sim}{\Delta}^1_1$ is independent of ZFC, answering a question of Mauldin (recalled in Miller [13]). In particular, Hjorth [4] shows that the existence of such a set follows from $\underset{\sim}{\Pi}^1_1$-Determinacy, but is inconsistent with $\mathbb{V} = \mathbb{L}$. See §33 of Jech [5] for an elementary discussion of games and descriptive set theory.

In Section 2 we extend this result by showing that the Axiom of Projective Determinacy implies that for each n there is a universal $\underset{\sim}{\Pi}^1_n \setminus \underset{\sim}{\Delta}^1_n$ set. The proof we use is quite unlike that of the original result, in that it requires only the closure properties of the $\underset{\sim}{\Pi}^1_n$ pointclasses and the determinacy hypothesis is only utilized via Wadge's Lemma. Using the same argument we show that there are universal $\underset{\sim}{\Pi}^0_\alpha \setminus \underset{\sim}{\Delta}^0_\alpha$ for each countable ordinal $\alpha \geq 3$. For the Borel classes we need Borel Determinacy but that is a Theorem of ZFC (Martin [12]).

In Section 3 we show that the existence of a Universal $\underset{\sim}{\Pi}^1_1 \setminus \underset{\sim}{\Delta}^1_1$ set follows from ZFC + MA($\aleph_1$) + $\aleph_1 = \aleph_1^L$. This theory is equiconsistent with ZFC. For contrast $\underset{\sim}{\Pi}^1_1$-Determinacy is a large cardinal axiom and Projective Determinacy is a much stronger large cardinal axiom. See Kanamori [6] Chapter 6 for a discussion of determinacy and large cardinals.

For simplicity's sake, we state these results for the space 2^ω but analogous results can be proven for any other uncountable Polish space.

We will follow the notation that lowercase Greek letters denote ordinals, lowercase Roman letters denote reals, uppercase Roman letters stand for sets of reals, and capital Greek letters stand for pointclasses (sets of sets of reals).

General background on descriptive set theory can be found in Kechris [8], Moschovakis [14], and Sacks [16].

2 Determinacy

Recall that the Axiom of Projective Determinacy (PD) states that for every Gale-Stewart game with payoff set in the projective hierarchy, one of players has a winning strategy. We will use only the consequence of this axiom that appears in Lemma 2.2.

Theorem 2.1. *(a) For every $\alpha \geq 3$ there is a universal $\underset{\sim}{\Pi}^0_\alpha \setminus \underset{\sim}{\Delta}^0_\alpha$ set.*

(b) Assuming PD, for every $n \geq 1$ there is a universal $\underset{\sim}{\Pi}^1_n \setminus \underset{\sim}{\Delta}^1_n$ set.

proof

Let Γ be one of the classes $\underset{\sim}{\Pi}^1_n$ for $n \geq 1$ or $\underset{\sim}{\Pi}^0_\alpha$ for $\alpha \geq 3$. Assuming Projective Determinacy in the former case, or using Borel Determinacy in the later, we employ the following lemma due to Harrington (see Steel [17]).

Lemma 2.2 (Harrington)**.** *Suppose $A, B \subseteq 2^\omega$. If $A \in \Gamma$ and $B \in \Gamma \setminus \Delta$, then $A \leq_1 B$. i.e. A is one-to-one Wadge reducible to B. This means that there exists a one-to-one continuous map $f : 2^\omega \to 2^\omega$ such that $f^{-1}(B) = A$.*

Let $T_0 \subseteq 2^{<\omega}$ be a perfect subtree such that the corresponding closed set $[T_0]$ is nowhere dense. Define

$$T_0^* = \{s \in 2^{<\omega} \setminus T_0 \ : \ s{\restriction}n \in T_0 \text{ where } \ell h(s) = n + 1\}.$$

These are the nodes which are just outside of T_0.

Take $C_0, C_1 \in \Gamma \setminus \Delta$ such that $C_0 \subseteq [T_0]$ and note that the following set C is in Γ.

$$C = C_0 \cup \bigcup_{s \in T_0^*} \{s^\frown x : x \in C_1\}.$$

Claim 2.3. *Let $P_0 = [T_0]$. For every $A \in \Gamma \setminus \Delta$ there exists continuous maps $f : P_0 \to 2^\omega$ and $g : 2^\omega \to 2^\omega$ and a closed set $P_1 \subseteq 2^\omega$ such that the following five conditions are satisfied.*

1. $g^{-1}(C) = A$
2. $g(P_1) \subseteq P_0$
3. $f(P_0) \subseteq P_1$
4. $f(g(y)) = y$ *for all* $y \in P_1$*, and*
5. $g(f(x)) = x$ *for all* $x \in P_0$.

Before proving the claim, note that the existence of such an f, g, and P_1 implies that $A \in \Gamma \setminus \Delta$. The set A is in Γ since g witnesses that $A \leq_W C$. Conditions (2)-(5) guarantee that $f : P_0 \to P_1$ is a homeomorphism with $f^{-1} = g \restriction P_1$. Condition (1) is equivalent to:

$$\forall x \in 2^\omega (x \in A \iff g(x) \in C)$$

which implies

$$\forall x \in P_1 (x \in A \iff g(x) \in C_0).$$

But, since f is the inverse of $g \restriction P_1$, this implies

$$\forall y \in P_0 (y \in C_0 \iff f(y) \in P_1 \cap A).$$

Since $C_0 \notin \Delta$ it follows that $A \notin \Delta$.

Proof of Claim:

We will define our continuous functions by means of Wadge strategies. A Wadge strategy is a function $\sigma : 2^{<\omega} \to 2^{<\omega}$ which satisfies:

1. for all $s, t \in 2^{<\omega}$ if $s \subseteq t$ then $\sigma(s) \subseteq \sigma(t)$ and
2. for all $n \in \omega$ there exists $m \in \omega$ such that $\ell h(\sigma(s)) \geq n$ whenever $\ell h(s) \geq m$.

Let $f_\sigma : 2^\omega \to 2^\omega$ denote the continuous function corresponding to the strategy σ, i.e., $f(x) = \bigcup_{n \in \omega} \sigma(x \restriction n)$.

Define the set D to consist of the triples (σ, τ, T_1) satisfying the following conditions:

1. $\sigma : T_0 \to 2^{<\omega}$ is a Wadge strategy,
2. $\tau : 2^{<\omega} \to 2^{<\omega}$ is a Wadge strategy,
3. $T_1 \subseteq 2^{<\omega}$ is a nonempty subtree without terminal nodes,
4. $\sigma(T_0) \subseteq T_1$,
5. $\tau(T_1) \subseteq T_0$,
6. $t \subseteq \sigma(\tau(t))$ or $\sigma(\tau(t)) \subseteq t$ for all $t \in T_1$, and
7. $s \subseteq \tau(\sigma(s))$ or $\tau(\sigma(s)) \subseteq s$ for all $s \in T_0$.

Note that D is $\underset{\sim}{\Pi}^0_2$. We show that if $A \in \Gamma \setminus \Delta$, then there exists $(\sigma, \tau, T_1) \in D$ such that $f_\tau^{-1}(C) = A$.

By Lemma 2.2 there exists a one to one continuous map $f : P_0 \to 2^\omega$ such that $C_0 = f^{-1}(A)$. Let $P_1 = [T_1]$ be the range of f and let σ be such that $f_\sigma = f$. By compactness, $f : P_0 \to P_1$ is a homeomorphism. Let $\tau : T_1 \to T_0$ be a Wadge strategy corresponding to f^{-1}. We extend τ to $2^{<\omega}$

as follows: Suppose $t \in T_1^*$, where $\ell h(t) = n+1$. If $\tau(t \upharpoonright n) = s$, then take any $s^* \in T_0^*$ extending s. This is possible since $[T_0]$ is nowhere dense. But we know from Wadge's Lemma that $A^t \leq_W C_1$ where

$$A^t = \{x \in 2^\omega : t^\frown x \in A\}$$

Hence we can find a Wadge strategy τ_{t,s^*} which takes t to s^* and reduces $A \cap [t]$ to $C \cap [s^*]$. We use τ_{t,s^*} to define τ for all extensions of t. This proves the Claim.

□

We now define the universal $\Gamma \setminus \Delta$ set W by sections:

$$W_{(\sigma,\tau,T_1)} = \begin{cases} \{x : f_\tau(x) \in C\} & \text{if } (\sigma,\tau,T_1) \in D \\ C & \text{otherwise.} \end{cases}$$

This proves the Theorem.

□

This leaves the case $\underset{\sim}{\Pi}^0_\alpha$ for $\alpha = 1, 2$.

Proposition 2.4. *There does not exist a closed $U \subseteq 2^\omega \times 2^\omega$ universal for non-clopen closed sets but there does exist a Π^0_1 set $V \subseteq \omega^\omega \times 2^\omega$ universal for non-clopen closed subsets of 2^ω. There exists a Π^0_2 set $W \subseteq \omega^\omega \times 2^\omega$ universal for $\underset{\sim}{\Pi}^0_2 \setminus \underset{\sim}{\Delta}^0_2$ subsets of 2^ω.*

proof

U cannot exist because for each n there would be an x_n with

$$U_{x_n} = \{y \in 2^\omega : \forall m > n \;\; y(m) = 0\}.$$

But some subsequence of the x_n must converge to (say) $x \in 2^\omega$, but then $U_x = 2^\omega$.

V can be defined as follows: Put $(T, f) \in Q$ if and only if $T \subseteq 2^{<\omega}$ is a nonempty subtree without terminal nodes and $f : \omega \to 2^{<\omega}$ has the property that for every n if $f(n) = s$ then $|s| > n$, $s{\upharpoonright}n \in T$, but $s \notin T$. Note that Q is homeomorphic to ω^ω. To see this note that

$$P(2^{<\omega}) \times (2^{<\omega})^\omega \approx 2^\omega \times \omega^\omega \approx \omega^\omega$$

and that Q is a Π^0_2 subspace of the first space and therefor a zero dimensional Polish space. Note that given any $(T, f) \in Q$ and m it is easy to construct a sequence $(T_n, f_n) \in Q$ with $T \cap 2^m = T_n \cap 2^m$, $f_n \restriction m = f \restriction m$, and $|f_n(m)| > n$ for every n. But this sequence has no convergent subsequence. It follows that compact sets have empty interior. Hence by the Alexandrov-Urysohn Theorem, Q is homeomorphic to ω^ω (see Kechris [8] 7.7 p.37.) Put $V_{(T,f)} = [T]$.

Define W analogously to the proof of Theorem 2.1. The set D above is homeomorphic to ω^ω by a similar argument to the one for Q. (Construct a sequence of Wadge strategies to witness non-compactness.) Take P_0 a perfect closed nowhere dense set and let $C_0 \subseteq P_0$ be such that $P_0 \setminus C_0$ is countable and dense in P_0. By Hurewicz's Theorem (See [8]) for any analytic $A \subseteq 2^\omega$ which is not F_σ there is a perfect P_1 and a homeomorphism $f : P_0 \to P_1$ which takes C_0 to $A \cap P_1$. For C_1 take any universal G_δ set.

□

The existence of a universal $\mathbf{\Pi}^0_2 \setminus \mathbf{\Delta}^0_2$ set for the case of $2^\omega \times 2^\omega$ is unknown to us.

3 Martin's Axiom

Theorem 3.1. *Assume MA($\aleph_1$) and $\aleph_1^{\mathbb{L}} = \aleph_1$. Then there exists $\mathcal{W} \subseteq \omega^\omega \times 2^\omega$ with the properties:*

1. $\mathcal{W} \in \Pi^1_1$,
2. $\forall x \in \omega^\omega \;\; \mathcal{W}_x \in \mathbf{\Pi}^1_1 \setminus \mathbf{\Delta}^1_1$, *and*
3. $\forall A \in \mathbf{\Pi}^1_1 \setminus \mathbf{\Delta}^1_1 \; \exists x \in \omega^\omega \;\; A = \mathcal{W}_x$.

Hence $\mathcal{W}$ is a universal set for $\mathbf{\Pi}^1_1 \setminus \mathbf{\Delta}^1_1$.

We use the following standard results, details of which can be found in chapter 4 of Moschovakis [14]. Recall that LO is the set of binary predicates on ω which are linear orderings and WO $\subseteq$ LO are the well-orderings. For each countable ordinal α the set WO$_{<\alpha}$ are the elements of WO of order type less than α. Then WO is a Π^1_1 set and the sets WO$_{<\alpha}$ are Borel.

Theorem 3.2 (The Boundedness Lemma)**.** *Let $A \subseteq WO$ be $\mathbf{\Sigma}^1_1$. Then there exists $\alpha < \aleph_1$ such that $A \subseteq WO_{<\alpha}$.*

Theorem 3.3. *For any Π^1_1 set $A \subseteq \omega^\omega$ there is a continuous function $f : \omega^\omega \to$ LO such that $f^{-1}(WO) = A$.*

We identify each such f with the norm on A defined by $\phi(x) =$ order type of $f(x)$.

We also need the following Theorem.

Theorem 3.4 (Kleene, Kripke-Platek). *(a) The Π^1_1 predicates are closed under the quantifier $\exists x \in \Delta^1_1(y)$, e.g., $\quad Q(y,z) \equiv \exists x \in \Delta^1_1(y)\ P(x,y,z)$.*

(b) The $\Delta^1_1(z)$ sets (or hyperarithmetic in z sets) can be described from the constructible hierarchy as follows:

$$\Delta^1_1(z) = \mathbb{L}_{\omega_1^z}[z] \cap 2^\omega$$

where ω_1^z is the first ordinal which is not the order type of a well-ordered relation recursive in z. The countable ordinal ω_1^z is also known as Church-Kleene ω_1 of z and the first admissible in z.

Part (a) can be found in Moschovakis [14] 4D.3 p.220. For part (b) Sacks [16] exercise 9.12 gives the unrelativized version as an exercise and Barwise, Gandy, Moschovakis [1] 3.1(b) prove it using admissible sets language. For nonrelativized versions of (a) and (b) see Mansfield and Weitkamp [10] 4.19 and 5.19.

Intuitively, the proof of Theorem 3.1 proceeds by taking a special universal Π^1_1 set $\mathcal{U}$ and identifying each non-$\underset{\sim}{\Delta}^1_1$ section $\mathcal{U}_z$ by means of a function f designed to witness the unboundedness of the norm on $\mathcal{U}_z$. The function f will be coded by a real a which we create using almost disjoint forcing and MA($\aleph_1$). If the function f_a fails to perform as required, we fall back to a default position, and code into the cross section some canonical $\underset{\sim}{\Pi}^1_1 \setminus \underset{\sim}{\Delta}^1_1$ set.

Next we discuss almost disjoint sets forcing.

Definition 3.5. *Fix a recursive enumeration $(s_n)_{n\in\omega}$ of $2^{<\omega}$. For $b \in 2^\omega$ let $b^\star = \{n : s_n \subset b\}$.*

Lemma 3.6. *Assume MA($\aleph_1$). Suppose $X \subseteq 2^\omega$ for has size $\aleph_1$ and $B \subseteq X$ is arbitrary. Then there exists $a \subseteq \omega$ such that for every $b \in X$*

$$b^* \cap a \text{ is infinite iff } b \in B.$$

This lemma is standard and can be found in the textbooks Kunen [9] p.57, Jech [5] p.276, Fremlin [2] 21C, or the handbook article Rudin [15]. Almost disjoint sets forcing was originally invented for its use in definability by Jensen and Solovay around 1968.

We use it in a way similar to Martin-Solovay [11] who showed that assuming MA($\aleph_1$) + $\aleph_1 = \aleph_1^{\mathbb{L}}$ every set of reals of cardinality $\aleph_1$ is $\underset{\sim}{\Pi}^1_1$.

Given $a = (a_n \subseteq \omega : n < \omega)$ define the function $f_a : 2^\omega \to 2^\omega$ as follows:

$$f_a(b) = c \text{ iff } \forall n\ (c(n) = 1 \Leftrightarrow b \cap a_n \text{ is infinite}).$$

Note that f_a is a Borel function, in fact $\Delta^0_3(a)$ uniformly in a. Consider a set $X \subseteq 2^\omega$ of size $\aleph_1$ and an arbitrary function $f : X \to 2^\omega$. For each $n < \omega$ let

$$B_n = \{b \in X \ : \ f(b)(n) = 1\}.$$

By the Lemma there exists a_n such that:

$$\forall b \in X\ (b^* \cap a_n \text{ is infinite iff } b \in B_n).$$

It follows that if we set $a = (a_n \subseteq \omega : n < \omega)$ that $f \restriction X = f_a \restriction X$.

To summarize:

Lemma 3.7. *Assume MA($\aleph_1$). There exists a Δ^0_3 function*

$$f : (\mathcal{P}(\omega))^\omega \times 2^\omega \to 2^\omega$$

such that for any $g : X \to 2^\omega$ an arbitrary function with domain $X \subseteq 2^\omega$ of size $\aleph_1$, there exist $a \in (\mathcal{P}(\omega))^\omega$ with $f = f_a \restriction X$.

Lemma 3.8. *There exists a Π^1_1 set $\mathcal{U} \subseteq \omega^\omega \times 2^\omega$ and a Π^0_1 set $P \subseteq \omega^\omega \times 2^\omega$ such that*

(1) for all $x \in \omega^\omega \quad P_x \subseteq 2^\omega$ is a nonempty perfect set disjoint from $\mathcal{U}_x$ and

(2) for any $\underset{\sim}{\Pi}^1_1$ set $A \subseteq 2^\omega$ with uncountable complement there exists x with $\mathcal{U}_x = A$.

proof

Let $Q \subseteq \omega^\omega \times 2^\omega$ be a Π^0_1 set universal for perfect subsets of 2^ω, i.e., every cross section is perfect and every perfect set occurs as a cross section.

Let $V \subseteq \omega^\omega \times 2^\omega$ be a Π^1_1 set universal for $\underset{\sim}{\Pi}^1_1$ sets.

Let $\mathcal{U}_{\langle u,v\rangle} = V_u \setminus Q_v$ and let $P_{\langle u,v\rangle} = Q_v$.

Since every uncountable $\underset{\sim}{\Sigma}^1_1$ set contains a perfect set we are done.

□

Note that the results established until this point do not use the hypothesis $\aleph_1 = \aleph_1^{\mathbb{L}}$. From this point on, however, we will be assuming both the hypotheses of Theorem 3.1.

The self-constructible reals are defined by

$$\mathcal{C} = \{x \in 2^\omega \ : \ x \in L_{\omega_1^x}\}$$

where ω_1^x is the Church-Kleene ω_1 of x. The set $\mathcal{C}$ is Π^1_1 and has cardinality $\aleph_1$ but does not contain a perfect set. Since uncountable Borel sets contain perfect sets, for any norm g for $\mathcal{C}$ and countable ordinal α it must be that $g^{-1}(\mathrm{WO}_{<\alpha})$ is countable.

The self-constructible reals were studied by Guaspari, Kechris, and Sacks, see Kechris [7] §2. See also Mansfield and Weitkamp [10] 6.20.

Define the (relativized) self-constructible reals:

$$C_x = \{z \in P_x \ : \ z \in L_{\omega_1^{x,z}}[x]\}.$$

The set $C = \{(x, z) \ : \ z \in C_x\}$ is Π^1_1. Each $C_x \subseteq L[x]$ has cardinality $\aleph_1$ but contains no perfect set. Note that the perfect set P_x from Lemma 3.8 is the set of branches of tree recursive in x, so the restriction $C_x \subseteq P_x$ is harmless.

Let $\mathcal{U}$ be from Lemma 3.8. Take $h, g : \omega^\omega \times 2^\omega \to \mathrm{LO}$ to be recursive continuous maps such that $h^{-1}(\mathrm{WO}) = \mathcal{U}$ and $g^{-1}(\mathrm{WO}) = C$. (i.e., norms for the two sets). As usual we use $g_x(\cdot) = g(x, \cdot)$ and $h_x(\cdot) = g(x, \cdot)$ to denote the cross sectional functions. Take f_a from Lemma 3.7.

Recall the standard prewell-ordering predicate for Π^1_1. For L_1 and L_2 linear orders define $L_1 \preceq L_2$ iff L_1 can be order embedded into L_2, i.e., there exists $f : \omega \to \omega$ such that for all $a, b \in \omega$ if $a <_{L_1} b$ then $f(a) <_{L_2} f(b)$. The predicate $\preceq$ is Σ^1_1 as a subset of $LO \times LO$.

Definition of $\mathcal{W}$

Define $\mathcal{W} \subseteq ((\mathcal{P}(\omega))^\omega \times \omega^\omega) \times 2^\omega$ as follows:

$z \in \mathcal{W}_{\langle a,x\rangle}$ iff either
  (a) $z \in \mathcal{U}_x$
  or
  (b) $z \in C_x$ and $\exists w, b \in \Delta^1_1(x, z, a)$ with $w \in C_x$ and either
    (1) $g_x(w) \not\preceq h_x(f_a(w))$ or
    (2) $b \subseteq \omega$ is nonempty with no $h_x(f_a(w))$ least element.

We show that $\mathcal{W}$ is universal for $\underset{\sim}{\Pi}^1_1 \setminus \underset{\sim}{\Delta}^1_1$ via three Claims.

Claim 3.9. $\mathcal{W} \in \Pi^1_1$.

proof

Clause (a) is Π^1_1 and omitting all super-sub-scripts-1, the logical form of (b) is

$$\Pi \wedge \exists w, b \in \Delta(x, z, a)\ (\Pi \wedge (\neg\Sigma \vee \Delta))$$

Since the quantifier $\exists u \in \Delta^1_1(v)$ preserves the class of Π^1_1 predicates (3.4) we have that $\mathcal{W}$ is Π^1_1.

□

Suppose that $\mathcal{U}_x$ is properly $\underset{\sim}{\Pi}^1_1$. Choose a so that $f_a : C_x \to \mathcal{U}_x$ and $g_x(z) \preceq h_x(f_a(z))$ for every $z \in C_x$. This is possible because the set C_x has cardinality $\aleph_1$ and because the order types of $h_x(u)$ for $u \in \mathcal{U}_x$ are unbounded. So we may find a using Lemma 3.7.

We say that a is good for x iff

for all $z \in C_x \quad f_a(z) \in \mathcal{U}_x$ and $g_x(z) \preceq h_x(f_a(z))$.

Claim 3.10. *If a is good for x, then $\mathcal{W}_{<a,x>} = \mathcal{U}_x$ and $\mathcal{U}_x$ is properly $\underset{\sim}{\Pi}^1_1$.*

proof

No z can enter $\mathcal{W}_{<a,x>}$ because of clause (b)(2) because $w \in C_x$ implies $f_a(w) \in \mathcal{U}_x$ and so $h_x(f_a(w))$ is a well-ordering. No z can enter by clause (b)(1) since it would directly contradict our choice of f_a. So $\mathcal{W}_{<a,x>} = \mathcal{U}_x$.

Since a is good for x the norm h_x is unbounded on $\mathcal{U}_x$ and so by the Boundedness Theorem 3.2, $\mathcal{U}_x$ cannot be $\underset{\sim}{\Sigma}^1_1$.

□

Claim 3.11. *If a is bad for x, then there is a countable set Z such that*

$$\mathcal{W}_{\langle a,x\rangle} \cap P_x = C_x \setminus Z.$$

proof

Since P_x is disjoint from $\mathcal{U}_x$ and $C_x \subseteq P_x$ no $z \in P_x$ enters $\mathcal{W}_{\langle a,x\rangle}$ because of clause (a), so $\mathcal{W}_{\langle a,x\rangle} \cap P_x \subseteq C_x$. For the reverse inclusion, note that since $C_x \subseteq P_x$, it suffices to show that for all but countably many $z \in C_x$ that $z \in \mathcal{W}_{\langle a,x\rangle}$. The point is that for any countable α for all but countably many $z \in C_x$ will have $\omega_1^{x,z} > \alpha$.

Case 1. $f_a(w) \notin \mathcal{U}_x$ for some $w \in C_x$.

Since $h_x(f_a(w))$ is not a well-ordering there is a set b with no $h_x(f_a(w))$ least element. By absoluteness such a set b exists in $L[x, a, w]$. Choose a countable ordinal α with $w \in L_\alpha[x]$ and $b \in L_\alpha[x, a]$. Since $L_\alpha[x]$ is countable for all but countably many $z \in C_x$ we have that $\omega_1^{x,z} > \alpha$, hence $w, b \in \Delta^1_1(x, z, a)$ by Theorem 3.4, so z is put into $\mathcal{W}_{\langle a,x\rangle}$ by (b)(2).

Case 2. $g_x(w) \not\preceq h_x(f_a(w)$ for some $w \in C_x$.

Choose a countable ordinal α with $w \in L_\alpha[x]$. Then for all but countably many $z \in C_x$ we have that $\omega_1^{x,z} > \alpha$ and hence $w \in \Delta^1_1(x, z)$. It follows from clause (b)(1) that $z \in \mathcal{W}_{\langle a,x\rangle}$.

This proves the Claim.

$\square$

Since C_x cannot be Borel even if a countable set is extracted, $\mathcal{W}_{\langle a,x\rangle}$ is properly $\underset{\sim}{\Pi}^1_1$ for every a. This concludes the proof of Theorem 3.1.

$\square$

We do not know what the consistency strength of the existence of a universal $\underset{\sim}{\Pi}^1_2 \setminus \underset{\sim}{\Delta}^1_2$ set is. Perhaps a variant of the model of Harrington [3] could be made to work.

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