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SET THEORY

On Generating the Category Algebra and the Baire Order Problem

by

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Summary. Let **B** denote the complete boolean algebra of Borel subsets of 2^ω modulo the σ-ideal of meager sets. It is shown that for every α with $1 \leqslant \alpha < \omega_1$, **B** can be countably generated in exactly α steps. As a corollary to this we prove a theorem of Kunen that assuming the continuum hypothesis there are separable metric spaces of all Baire orders less than or equal to ω_1.

B is also isomorphic to the regular open algebra of 2^ω. Given G included in **B** and closed under finite boolean combinations define $G_0 = G$ and for $\alpha \geqslant 1$ let G_α be the collection of ΣD for D such that for every A in D, $-A$ is in G_β for some $\beta < \alpha$. It is easily checked that for all α, G_α is closed under finite intersections and for all $\alpha < \beta$, $G_\alpha \subseteq G_\beta$. Since **B** satisfies the countable chain condition all infinite sums (or products) are countable. Thus G_{ω_1} is the complete subalgebra generated by G.

THEOREM 1. *For every α, $1 \leqslant \alpha < \omega_1$, there exists a countable G included in* **B** *such that α is the least such that $G_\alpha = \mathbf{B}$.*

Proof. For clarity we first prove Theorem 1 for the case α is finite and afterwards we indicate the changes necessary to prove the general case. Note that $\alpha = \omega_1$ is impossible because the clopen sets modulo meager must be generated at some countable stage. Fix N with $1 \leqslant N < \omega$.

Let ω^n denote the set of finite sequences of elements of ω of length n. Let $S = \omega^{\leqslant N} = \bigcup \{\omega^n : n \leqslant N\}$ (including the empty sequence $\emptyset$). Given s in ω^m and $n < \omega$ let sn be the unique extension, of s in ω^{m+1} whose last element is n. Let X be $\{x \in 2^S : \forall s \in S - \omega^N\ \forall n < \omega\ (x(sn) = 1$ implies $x(s) = 0)\}$. It easily checked that X is a perfect closed subset of 2^S, whence X is homeomorphic to 2^ω, so **B** is isomorphic to Borel (X) modulo meager (X).

Let **P** be the set of maps p into 2 with domain F $(F = \mathrm{dom}\,(p))$ such that F is a finite subset S, and if s and sn are in F then $p(sn) = 1$ implies $p(s) = 0$. The topology on X is given by basic clopen neighborhoods $N(p) = \{x \in X : x$ extends $p\}$ for p in **P**. Let G be the class of clopen subsets of X which are finite unions of $N(p)'s$ for p in **P** such that dom (p) is included in ω^N. Note that G is also closed under complementation. Define $G_0 = G$ and let G_{n+1} be the set of countable unions of complements of elements of G_n. Note that fora each n, G_n is closed under finite intersections and $G_n \subseteq G_{n+1}$. Let $A \triangle B$ denote $(A-B) \cup (B-A)$.

Lemma 1. *For every Borel set A there exists B in G_{N+1} such that $A \triangle B$ is meager.*

Proof. For s in S and i equal to 0 or 1 let $N(s,i)=\{x \in X: x(s)=i\}$. These are the subbasic clopen sets and note that $N(s,i)=X-N(s,1-i)$. $N(s,0)-\bigcup\{N(sn,1):n<\omega\}$ is closed nowhere dense. This is true since given any p in $\mathbf{P}$ with $p(s)=0$ we can find $n<\omega$ so that no extension of sn is in dom (p), hence there is a q in $\mathbf{P}$ extending p with $q(sn)=1$, and thus $N(q) \subseteq N(p) \cap N(sn,1)$. On the other hand, $N(sn,1)$ is included in $N(s,0)$ by the definition of X. Hence an easy induction shows that for every s in ω^{N-n} there exists A in G_n such that $A \triangle N(s,0)$ is meager. For p in $\mathbf{P}$ $N(p)=\cap\{N(s,p(s)): s \in \text{dom}(p)\}$. G_{N+1} contains G_N and complements of elements of G_N and is closed under finite intersection and countable union, thus since every Borel set in X is equivalent modulo meager to an open set, the lemma is proved. ■

Define for s in S, $|s|=N-n$ where s is in ω^n. For p in $\mathbf{P}$ let $|p|=\max\{|s|: s \in \text{dom}(p)\}$. For $p, q \in \mathbf{P}$ we say p and q are compatible just in case $N(p) \cap N(q) \neq \emptyset$.

Lemma 2. *Suppose $1 \leqslant n \leqslant N$ and $p \in \mathbf{P}$, then there exists $\hat{p}$ compatible with p, $|\hat{p}| \leqslant n$, and for all q if $|q|<n$ then ($\hat{p}$ and q are compatible implies p and q are compatibile).*

Proof. Let $D=\{s \in \text{dom}(p): |s| \leqslant n\}$ and let $\hat{p}=p \upharpoonright D$. Suppose $|q|<n$ and p and q are not compatible. Then there are $s \in \text{dom}(p)$ and $t \in \text{dom}(q)$ which demonstrate their incompatibility.

Case 1. $s=t$ and $p(s) \neq q(t)$. Since $|q|<n$, $|t|<n$, and so $s \in \text{dom}(\hat{p})$.

Case 2. $t=sm$ for some m and $p(s)=q(t)=1$. Since $|t|<n$, $|s| \leqslant n$, and again $s \in \text{dom}(\hat{p})$.

Case 3. $s=tm$ for some m and $p(s)=q(t)=1$. But $|s|=|t|-1$ so $s \in \text{dom}(\hat{p})$.

In all three cases $s \in \text{dom}(\hat{p})$ and so $\hat{p}$ and q are incompatible. ■

Lemma 3. *Suppose $1 \leqslant n \leqslant N$, $p \in \mathbf{P}$ and $A \in G_n$. If $N(p) \cap A$ is not meager then there is $\hat{p} \in \mathbf{P}$ such that $|\hat{p}|<n$, $\hat{p}$ and p are compatible, and $N(\hat{p})-A$ is meager.*

Proof. The proof is by induction on n.

Case 1. $n=1$. Since G_0 is closed under complementation and $A \in G_1$ then $A=\cup\{N(p_i): o<\omega\}$ where $||p_i|=0$ for each i. Let $\hat{p}$ be any p_i compatible with p.

Case 2. $n+1$. Suppose $A=\cup\{X-A_m: m<\omega\}$ where each $A_m \in G_n$. Then there is m such that $N(p) \cap (X-A_m)$ is not meager. Choose q extending p so that $N(q)-(X-A_m)$ is meager. By Lemma 2 there is $\hat{p}$ compatible with q, $|\hat{p}| \leqslant n$, and for every $r \in \mathbf{P}$ if $|r|<n$ then ($\hat{p}$ and r compatible implies q and r compatible).

Claim. *$N(\hat{p})-(X-A_m)=N(\hat{p}) \cap A_m$ is meager.*

If not then by induction there is $|r|<n$ compatible with $\hat{p}$ (and hence with q) such that $N(r)-A_m$ is meager. But then $N(q \cup r) \subseteq (N(q) \cap A_m) \cup (N(r)-A_m)$ is meager, contradiction. ■

Finally to prove Theorem 1 we show that for every A in G_N, $A \triangle N(\emptyset, 1)$ is not meager ($\emptyset$ denotes the empty sequence). If it were meager then by the lemma there exists p in $\mathbf{P}$ with $|p| \leqslant N-1$ and $N(p)-A$ meager. Since $|p| \leqslant N-1$, $\emptyset \notin \operatorname{dom}(p)$, so if q is the one element extension of p defined by letting $q(\emptyset)=0$, then q is in $\mathbf{P}$. But then $N(q)-A$ is meager and $N(q) \cap N(\emptyset, 1)$ is empty, contradiction.

To prove Theorem 1 for any $\alpha<\omega_1$ proceed as follows. Construct $T_\alpha \subseteq \omega^{<\omega}$ for $\alpha<\omega_1$ by induction. For $\alpha=0$ let $T_0=\{\emptyset\}$. For $\alpha=\beta+1$ let $T_\alpha=\{ns : n<\omega$ and $s \in T_\beta\}$. For α a limit ordinal choose a strictly increasing sequence β_n for $n<\omega$ cofinal in α with $0<\beta_0$ and let $T=\{ns : n<\omega$ and $s \in T_{\beta_n}\}$. Fix $\alpha<\omega_1$ and $T=T_\alpha$. Let $X=\{x \in 2^T : \forall s \in T-\{\emptyset\}$ if $sn \in T$ then $(x(sn)=1$ implies $x(s)=0)\}$. Define by induction for any $s \in T$ $|s|=\sup\{|sn|+1 : n<\omega\}$. and $|p|=\max\{|s| : s \in \operatorname{dom}(p)\}$ for $p \in \mathbf{P}$ where $\mathbf{P}$ is defined analogously. Let $G=G_0$ be the family of finite unions of $N(p)$'s with $|p|=0$. For any $\beta<\omega_1$ define G_β to be the family of countable unions of complements of elements of $\cup\{G_\gamma : \gamma<\beta\}$. It is easy to generalize Lemma 1 to show that for every A Borel in X there is B in G such that $A \triangle B$ is meager. Lemma 2 can be generalized to show that given any β, $1 \leqslant \beta \leqslant \alpha$, and p in $\mathbf{P}$, $\exists \hat{p}$ compatible with p, $|\hat{p}| \leqslant \beta$, and for every q in $\mathbf{P}$ if $|q|<\beta$ then ($\hat{p}$ and q compatible implies p and q compatible). To see this let $F=\{sn : s \in \operatorname{dom}(p)$, $p(s)=1$, $|s|=\lambda$ a limit ordinal $>\beta$, and $|sn|<\beta\}$. By the construction of T, F is finite and $\forall t \in F$ $\forall m<\omega$ $tm \in T$. Thus we can find r extending p so that $\forall t \in F \exists m<\omega$ $tm \in \operatorname{dom}(r$ and $r(tm)=1$. Let $D=\{s \in \operatorname{dom}(r) : |s| \leqslant \beta\}$ and $\hat{p}=r \restriction D$, then $\hat{p}$ works as before except for Case 2.

Case 2. $t=sm$ and $p(s)=q(t)=1$. Since $|t|<\beta$ either $|s| \leqslant \beta$ and so $s \in \operatorname{dom}(\hat{p})$ or $|s|=\lambda$ a limit ordinal $>\beta$ in which case $t \in F$ so there exists $n<\omega$ such that $r(tn)=1$ and thus $tn \in \operatorname{dom}(\hat{p})$ and so $\hat{p}$ and q are incompatible. The proof of Lemma 3 is the same except for $A \in G_\beta$ for β a limit ordinal in which case it is easy since if $A=\cup\{X-A_n : n<\omega\}$ where each $A_n \in G_{\beta_n}$ with $\beta_n<\beta$, then for each $n<\omega$ $X-A_n \in$ $\in G_{\beta_n+1}$ where $\beta_n+1<\beta$. ∎

For X a separable metric space define ord (X) (the Baire order of X) to be the least ordinal $\alpha \leqslant \omega_1$ such that every Borel in X subset of X is $\mathbf{\Sigma}^0_\alpha$. Recall that $\mathbf{\Sigma}^0_\alpha$ is the additive Borel class of rank α; that is, $\mathbf{\Sigma}^0_1$ is the class of open sets, and for $\alpha>1$, $\mathbf{\Sigma}^0_\alpha$ is the class of countable union of complements of elements of $\cup\{\mathbf{\Sigma}^0_\beta : \beta<\alpha\}$. Note that since the Borel sets are closed under complementation, we could have equally as well define ord (X) in terms of $\mathbf{\Pi}^0_\alpha$ (the α^{th} multiplicative class) or $\mathbf{\Delta}^0_\alpha$ (the α^{th} ambiguous class). Note that X is discrete iff ord $(X)=1$. For $\mathbf{Q}$ (the rationals) ord $(\mathbf{Q})=2$ and in general for X countable ord $(X) \leqslant 2$. It is a classical theorem of Lebesque (see [2]) that for X an uncountable Polish space (or more generally X analytic, since every uncountable analytic space contains a perfect subset) the Baire order of X is ω_1. Mazurkiewicz (see [5]) asked for which α does there exist X such that ord $(X)=\alpha$.

Assuming the continuum hypothesis we prove the following theorem of Kunen's.

THEOREM 2. *For every $\alpha \leqslant \omega_1$ there exists X a separable metric space with* ord $(X)=\alpha$.

Proof.

LEMMA 3. *For all $\alpha<\omega_1$ there exists I a σ-ideal in Borel (2^ω) such that α is the least ordinal such that* $\forall A \in \text{Borel}(2^\omega)\, \exists B \in \Sigma^0_\alpha\ (A \triangle B) \in I$.

Proof. For $\alpha=1$ let I be the ideal of meager sets. Again we assume $\alpha=N+1$ and $1 \leqslant N < \omega$. Let $\omega^N=\{s_n\ \bar{n}<\omega\}$ and define $h\colon X \dashrightarrow 2^\omega$ by $h(x)(n)=x(s_n)$. For $A \subseteq 2^\omega$ let $H(A)=h^{-1}(A)$. H maps G_0 1–1, onto the clopen subsets of 2^ω and preserves unions and complements, and so for each α it maps the levels G_α 1–1, onto Σ^0_α. Let I be defined by $A \in I$ iff $H(A)$ is meager in X. Given A Borel in 2^ω let B be in G_{N+1} such that $H(A) \triangle B$ is meager, then $H^{-1}(B) \triangle A$ is in I and $H^{-1}(B)$ is in Σ^0_{N+1}. Also if B_s in G_{N+1} and there exists D in Σ^0_N such that $H^{-1}(B) \triangle D$ is in I, then $B \triangle H(D)$ is meager and $H(D)$ is in G_N. This proves the lemma for α finite and the general case is similar. ■

LEMMA 4. (Luzin [3]) *Assuming the continuum hypothesis if I is a σ-ideal in the Borel subsets of 2^ω with $2^\omega \notin I$ and $\{x\} \in I$ for all x 2^ω, then there exists Y contained in 2^ω such that for every Borel A ($A \cap Y$ is countable iff A is in I).*

Proof. Let $\{C_\alpha\ \alpha<\omega_1\}=I$ and $\{\beta_\alpha\ \alpha<\omega_1\}=\text{Borel}(2^\omega)-I$ (where each element is listed uncountably often). Choose x_α in $B_n-(\bigcup\{C_\beta\ \beta<\alpha\}\bigcup\{x_\beta . \beta<\alpha\})$, and let $Y=\{x_\alpha\colon \alpha<\omega_1\}$.

Note that the ideal given by Lemma 3 satisfies the hypothesis of Lemma 4 since $H(\{x\})$ is closed nowhere dense in X. To prove Theorem 2 let $3 \leqslant \alpha < \omega$, and let Y be contained in 2^ω be the set given by Lemma 4 for I the ideal given by Lemma 3 for α. Then in its relative topology ord $(Y)=\alpha$. For every Borel set A there is B in Σ^0_α such that $A \triangle B$ is in I, so $(A \triangle B) \cap Y=F$ is countable, $(B \triangle F) \cap Y=A \cap Y$, and since $\alpha \geqslant 3$, $B \triangle F$ is in Σ^0_α. On the other hand, if β is less than α there is a Borel set A such that for every Σ^0_β set B, $A \triangle B$ is not in I, hence $(A \triangle B) \cap X$ is uncountable and so $A \cap X \neq B \cap X$.

REMARKS

(1) A set Y as in Lemma 3 for I the ideal of meager sets is called a Luzin set. In [5] Poprougenko and Sierpiński showed that a Luzin set has Baire order 3. This is because every Borel set is equal to a G_δ (Π^0_2) set union a meager set, and no countable dense subset of a Luzin set is G_δ in the relative topology. A set Y as in Lemma 3 for I the ideal of measure zero sets is called a Sierpiński set. In [6] Szpilrajn showed that a Sierpiński set has Baire order 2. This is because every Borel set is equal to an F_σ (Σ^0_2) set union a set of measure zero.

(2) The author had previously shown that it is consistent with ZFC that for every $\alpha \leqslant \omega_1$ there exists Y contained in 2^ω with ord $(Y)=\alpha$ using a much more difficult proof than the proof of Theorem 2 given here. In [4] it is shown that it is also consistent with ZFC that for every Y an uncountable separable metric space,

ord $(Y)=\omega_1$, and hence the only Baire orders in that model of set theory are 1,2, and ω_1. This confirms a conjecture of Banach (see [6]).

(3) Kunen also points out that Kolmogorov's problem ([1]) is solved. Given $\alpha<\omega_1$ let I be the σ-ideal in the Borel sets given by Lemma 3. Let $R=\{C:C$ is Borel in 2^ω and there exists D clopen such that $C\triangle D$ is in $I\}$, then for all $\beta<\omega_1$ and $A\subseteq 2^\omega$ (A is in R_β iff there exists B in Σ^0_β with $A\triangle B$ in I), and so the hierarchy generated by R has exactly α levels.

(4) In [4] it is also shown that for every $\alpha\leqslant\omega_1$ there exists a complete boolean algebra $\mathbf{B}$ with the countable chain condition such that there is C included in $\mathbf{B}$ countable with $C_\alpha=\mathbf{B}$ and for every D included in $\mathbf{B}$ of cardinality less than the continuum and $\beta<\alpha$ $D_\beta\neq\mathbf{B}$. This answers a question of Tarski's. It can be generaized to show, for example, that if the continuum is at least $\aleph_n$ then there is a complete boolean algebra $\mathbf{B}$ such that for every m if $1\leqslant m\leqslant n<\omega$ then m is the least number such that there is C included in $\mathbf{B}$ with cardinality $\aleph_{n-m}$ and $C_m=\mathbf{B}$.

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А. В. Миллер, О порождении категорной алгебры и проблеме порядка Бэра

Содержание. Показано*i* что полная булева алгебра борелевских множеств 2^ω по модулю σ-идеала множеств *I*-ой категории может быть счетно порождена ровно за α шагов*i* где α— произвольный ординал меньший ω_1. В качестве следствия доказана теорема Кунена: предполагая континуум-гипотезй*i* можно построить сепарабельные метрические пространства, все бэровские порядки которых меньше или равны ω_1.