The γ-Borel conjecture

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Abstract

In this paper we prove that it is consistent that every γ-set is countable while not every strong measure zero set is countable. We also show that it is consistent that every strong γ-set is countable while not every γ-set is countable. On the other hand we show that every strong measure zero set is countable iff every set with the Rothberger property is countable.

A set of reals X has strong measure zero iff for any sequence $(\epsilon_n : n < \omega)$ of positive reals there exists a sequence of intervals $(I_n : n < \omega)$ covering X with each I_n of length less than ϵ_n. Laver [8] showed that it is relatively consistent with ZFC that the Borel conjecture is true, i.e., every strong measure zero set is countable.

Sets of reals called γ-sets were first considered by Gerlits and Nagy [5]. They showed that every γ set has strong measure zero and that Martin's Axiom implies every set of reals of size smaller than the continuum is a γ-set. A γ-set of size continuum is constructed in Galvin and Miller [4] using MA.

Next we define γ-set. An open cover $\mathcal{U}$ of a topological space X is an ω-cover iff for every finite $F \subseteq X$ there exists $U \in \mathcal{U}$ with $F \subseteq U$ and $X \notin \mathcal{U}$. An open cover $\mathcal{U}$ of X is a γ-cover iff $\mathcal{U}$ is infinite and each $x \in X$ is in all but finitely many $U \in \mathcal{U}$. Finally, X is a γ-set iff X is a separable metric space in which every ω-cover contains a γ-subcover.

Paul Szeptycki asked if it was possible to have a sort of weak Borel conjecture be true, i.e., every γ-set countable, while the Borel conjecture is false. We answer his question positively. We use Hechler [6] forcing, $\mathbb{H}$, for adding a dominating real, an analysis of it due to Baumgartner and Dordal [1], properties of Laver forcing $\mathbb{L}$, and a characterization of $\mathbb{H}$ due to Truss [11].

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Theorem 1 *If $\mathbb{H}$ is iterated ω_2 times with finite support, $\mathbb{H}_{\omega_2}$, over a model of CH, then in the resulting model every γ-set is countable but every set of reals of cardinality ω_1 has strong measure zero.*

Proof

For $f \in \omega^\omega$, define $\mathcal{U}_f$ to be the following family of clopen subsets of 2^ω.

$$\mathcal{U}_f = \{C_F : \exists n \;\; F \subseteq 2^{f(n)}, |F| \leq n\} \text{ where } C_F = \{x \in 2^\omega : x \restriction f(n) \in F\}\}$$

Note that for any finite $A \subseteq 2^\omega$ there exists $C \in \mathcal{U}_f$ with $A \subseteq C$. Also $2^\omega \notin U_f$ provided that $2^{f(n)} > n$ all n. Let $\mathbb{L}$ denote Laver forcing [8].

Lemma 2 *Suppose M is a model of set theory, f is $\mathbb{L}$-generic over M, and $X \subseteq 2^\omega$ is in M. Then*

$$M[f] \models \forall \mathcal{C} \in [\mathcal{U}_f]^\omega \;\; |\bigcap \mathcal{C} \cap X| \leq \omega$$

Proof

For a tree $p \subseteq \omega^{<\omega}$ and $s \in p$ we define

$$p_s = \{t \in p : t \subseteq s \text{ or } s \subseteq t\}$$

A Laver condition (or Laver tree) is a tree $p \subseteq \omega^{<\omega}$ with a special node $s \in p$, called the stem of p, which has the properties that

1. $p_s = p$ and
2. for every $t \in p$ with $|t| \geq |s|$ there exist infinitely many $n < \omega$ with $tn \in p$.

The order is $p \leq q$ iff $p \subseteq q$. As usual we define $p \leq_0 q$ iff $p \leq q$ and $\text{stem}(p) = \text{stem}(q)$. Somewhat nonstandardly let us write

$$\text{leaves}(p) = \{r \in p : \text{stem}(p) \subseteq r\}$$

and for each $s \in \text{leaves}(p)$ define

$$\text{split}(p, s) = \{n \in \omega : sn \in p\}$$

Suppose that the lemma is false. Let p^* be a Laver condition such that

$$p^* \Vdash \text{“} \bigcap \overset{\circ}{\mathcal{C}} \cap X = \overset{\circ}{Y} \text{ is uncountable and } \overset{\circ}{\mathcal{C}} \subseteq \mathcal{U}_f \text{ is infinite”}$$

By cutting $\mathcal{C}$ down (if necessary) we may suppose that $\mathcal{C} = \{C_{F_n} : n \in Q\}$ where $F_n \subseteq 2^{f(n)}$ with $|F_n| \leq n$ and $Q \in [\omega]^\omega$.

Working in M using standard arguments of Laver forcing [8] we can prove the following Claims.

Claim. Suppose that p is an arbitrary condition such that

$$p \Vdash \{\overset{\circ}{s}_i : i < k\} \subseteq 2^{f(k)}$$

where $k = |s|$ and $s = \text{stem}(p)$. Then there exists $r \leq_0 p$ and $(x_i \in 2^\omega : i < k)$ such that for any $m < \omega$ for all but finitely many $n \in \text{split}(r, s)$ for every $i < k$

$$r_{sn} \Vdash x_i \restriction m = \overset{\circ}{s}_i \restriction m$$

Proof

One of the basic properties of Laver forcing is that if p is any Laver tree and θ any sentence in the forcing language, then there exists $q \leq_0 p$, which decides θ, i.e.,

$$q \Vdash \theta \quad \text{or} \quad q \Vdash \neg\theta.$$

Note that for $sn \in p$ we have that $p_{sn} \Vdash f(k) = n$. Hence we can find $q \leq_0 p$ and $(s_i^n \in 2^n : i < k, n \in \text{split}(q, s))$ so that for each $n \in \text{split}(q, s)$ we have that

$$q_{sn} \Vdash \text{“}s_i^n = \overset{\circ}{s}_i \text{ for all } i < k\text{”}$$

It follows by compactness that there exists $x_i \in 2^\omega$ and an infinite set $E \subseteq \text{split}(q, s)$ so that for every $m < \omega$ we have that for all but finitely many $n \in E$ that

$$s_i^n \restriction m = x_i \restriction m \text{ for all } i < k.$$

Now let $r = \bigcup\{q_{sn} : n \in E\}$ so that $r \leq_0 q$. This proves the Claim.

QED

Note that if $y \in 2^\omega \setminus \{x_i : i < k\}$, then

$$r_{sn} \Vdash y \notin C_{\{s_i : i < k\}}$$

for all but finitely many $n \in \text{split}(r, s)$. By the usual fusion arguments we obtain:

Claim There exists $q \leq_0 p^*$ and $(K_s \in [2^\omega]^{\leq|s|} : s \in \text{leaves}(q))$ such that

1. for each $s \in \text{leaves}(q)$ either $q_s \Vdash |s| \in \overset{\circ}{Q}$ or $q_s \Vdash |s| \notin \overset{\circ}{Q}$, and

2. for each $s \in \text{leaves}(q)$ if $q_s \Vdash |s| \in \overset{\circ}{Q}$ then for any $x \in 2^\omega \setminus K_s$ for all but finitely many n if $sn \in q$, then

$$q_{sn} \Vdash x \notin \overset{\circ}{C}_{F_{|s|}}$$

Proof
We repeat the first Claim at each node starting at the stem and continuing with longer and longer nodes; and finally taking the fusion.
QED

Now since p^* forces that Y is uncountable we must be able to find

$$x \notin \cup\{K_s \ : \ s \in \text{leaves}(q)\}$$

and $r \leq q$ such that $r \Vdash x \in Y$. But this is a contradiction, since there must be some $s \in \text{leaves}(r)$ such that

$$r_s \Vdash \text{“}|s| \in \overset{\circ}{Q}\text{”}$$

and then for all but finitely many $n \in \text{split}(r, s)$ we have that

$$r_{sn} \Vdash x \notin \overset{\circ}{C}_{F_{|s|}}$$

But even one such n gives a contradiction. This proves the Lemma.
QED

Now we note that this property is preserved when we add a Cohen real.

Lemma 3 *Suppose N is a model of set theory, $x \in \omega^\omega$ is a Cohen real over N, $X \subseteq 2^\omega$ in N, and $\mathcal{U} \in N$ is a family of sets such that*

$$N \models \text{“}\forall \mathcal{C} \in [\mathcal{U}]^\omega \ \ |\bigcap \mathcal{C} \cap X| \leq \omega\text{”}$$

Then

$$N[x] \models \text{“}\forall \mathcal{C} \in [\mathcal{U}]^\omega \ \ |\bigcap \mathcal{C} \cap X| \leq \omega$$

Proof

Suppose not and let $\mathring{\mathcal{C}}$ be name for some $\mathcal{C} \in [\mathcal{U}]^\omega$ and p a Cohen condition so that

$$p \Vdash X \cap \bigcap \mathring{\mathcal{C}} \text{ is uncountable and } \mathring{\mathcal{C}} \in [\mathcal{U}]^\omega.$$

Working in N since the Cohen partial order is countable, there would exist $q \leq p$ so that

$$Y = \{x \in X : q \Vdash x \in \bigcap \mathring{\mathcal{C}}\}$$

is uncountable. Let $\mathcal{C}' = \{U \in \mathcal{U} : Y \subseteq U\}$. Note that $\mathcal{C} \subseteq \mathcal{C}'$ so $\mathcal{C}'$ is infinite, $\mathcal{C}' \in N$ and $\bigcap \mathcal{C}' \cap X$ contains Y which is uncountable. This contradicts the assumption that

$$N \models \text{“}|\bigcap \mathcal{C}' \cap X| \leq \omega\text{”}$$

QED

It follows from the two Lemmas that if f is Laver over M, x is Cohen over $N = M[f]$, and $X \subseteq 2^\omega$ is an uncountable set in M, then in $M[f, x]$ every infinite $\mathcal{C} \subseteq \mathcal{U}_f$ has the property that $\bigcap \mathcal{C} \cap X$ countable.

The following Lemma applies to the Laver real f since it is dominating.

Lemma 4 *(Truss [11]) Suppose f is a dominating real over M, i.e., $g \leq^* f$ for every $g \in M \cap \omega^\omega$ and $x \in \omega^\omega$ is a Cohen real over $M[f]$, then $h = f + x$ is $\mathbb{H}$-generic over M.*

Lemma 5 *Let f be $\mathbb{L}$-generic over M, $x \in \omega^\omega$ a Cohen real over $M[f]$, and $h = f + x$. Then for every uncountable $X \subseteq 2^\omega$ in M*

$$M[h] \models \text{“}\forall \mathcal{C} \in [\mathcal{U}_h]^\omega \quad \bigcap \mathcal{C} \cap X \textit{ is countable }\text{”}.$$

Proof

Since $M[h] \subseteq M[f, x]$ the lemma follows from the stronger claim that

$$M[f, x] \models \text{“}\forall \mathcal{C} \in [\mathcal{U}_h]^\omega \quad \bigcap \mathcal{C} \cap X \text{ is countable }\text{”}.$$

If this were false, then working in $M[f, x]$ we could obtain $Q \subseteq \omega$ infinite and $(C_{H_n} : n \in Q)$ with $H_n \subseteq 2^{h(n)}$ and $|H_n| \leq n$ such that $\bigcap_{n \in Q} C_{H_n} \cap X$ uncountable. Define $F_n = \{s \restriction f(n) : s \in H_n\}$. Now since $h(n) \geq f(n)$ we have that $C_{H_n} \subseteq C_{F_n}$. The set $\{C_{F_n} : n \in Q\}$ must be infinite because f is increasing and $|F_n| \leq n$ (if nothing else their measures, $\mu(C_{F_n}) \leq \frac{n}{2^{f(n)}}$, go to zero).

QED

Note that the lemma applies to every Hechler generic real and not just the sum of a Laver and a following Cohen. This is because if it is false it must be forced false by a particular Hechler condition. Then just take a Laver real in that condition and follow it with a Cohen to get a contradiction. In more detail let

$$\mathbb{H} = \{(n, f) : f \in \omega^\omega, n \in \omega\}$$

and define the Hechler neighborhoods

$$[n, f] = \{g \in \omega^\omega : g \upharpoonright n = f \upharpoonright n \text{ and } \forall i \;\; g(i) \geq f(i)\}$$

Then $(m, g) \leq (n, f)$ iff $m \geq n$ and $g \in [n, f]$. Also for G $\mathbb{H}$-generic over M the Hechler real is

$$h = \bigcup \{f \upharpoonright n : (n, f) \in G\}$$

and it has the property that

$$G = \{(n, f) \in \mathbb{H} : h \in [n, f]\}$$

The lemma must be true in every Hechler extension, If not, there would exist some condition (n, g) forcing it is false. It is easy to find a Laver real $f \in [n, g]$ and letting $x \in \omega^\omega$ be a Cohen real over $M[f]$ with $x \upharpoonright n$ constantly zero, we would get a Hechler real $h = f + x$ with $h \in [n, g]$ which gives a contradiction.

Question 6 *(Ramiro de la Vega) Given a countable transitive model of set theory M, is it true that for every Hechler real h over M there exists a Laver real f over M and a Cohen real x over $M[f]$ such that $h = f + x$?*

Define $(a_\alpha \in [\omega]^\omega : \alpha < \omega_1)$ is eventually narrow iff for every $b \in [\omega]^\omega$ there exists $\alpha < \omega_1$ so that $b \setminus a_\beta$ is infinite for all $\beta > \alpha$.

Lemma 7 *(Baumgartner and Dordal [1]) Suppose N is a model of set theory and*

$$N \models (a_\alpha \in [\omega]^\omega : \alpha < \omega_1) \text{ is eventually narrow.}$$

Then for any G_{ω_2} which is $\mathbb{H}_{\omega_2}$-generic over N, we have that

$$N[G_{\omega_2}] \models (a_\alpha \in [\omega]^\omega : \alpha < \omega_1) \text{ is eventually narrow.}$$

Now we prove that every γ-set in $M[G_{\omega_2}]$ countable. Since γ-sets are zero dimensional we need only worry about uncountable $Y \subseteq 2^\omega$. Let $X \subseteq Y$ be a subset of size ω_1. Construct $g : \omega \to \omega$ so that for every $n < \omega$ if $m = g(n)$, then

$$|\{x \upharpoonright m : x \in X\}| > n$$

By the usual ccc finite support iteration arguments we can find $\alpha < \omega_2$ so that $X, g \in M[G_\alpha]$ and letting $h = h_\alpha$ be the next Hechler real added we have that $h(n) > g(n)$ for all n. From Lemma 5 and the remark following it we see that in $N = M[G_{\alpha+1}]$ the set $\bigcap \mathcal{C} \cap X$ is countable for every infinite $\mathcal{C} \subseteq \mathcal{U}_h$. Now since $h(n) > g(n)$ there is no $U \in \mathcal{U}_h$ which covers X, however $\mathcal{U}_h$ is an ω-cover of 2^ω and hence of Y.

Now let $X = \{x_\alpha : \alpha < \omega_1\}$ and $\mathcal{U}_h = \{U_n : n < \omega\}$. In the model $N = M[G_{\alpha+1}]$ define $a_\alpha = \{n < \omega : x_\alpha \in U_n\}$. Note that

$$N \models (a_\alpha \in [\omega]^\omega : \alpha < \omega_1) \text{ is eventually narrow.}$$

Otherwise if $b \subseteq^* a_\alpha$ for uncountably many α, then for some infinite $c \subseteq b$

$$Z = \{x_\alpha : c \subseteq a_\alpha\}$$

is uncountable. But then $Z \subseteq \bigcap\{U_n : n \in c\}$ which contradicts Lemma 5.

Since the tail of a finite iteration of $\mathbb{H}$ is itself a finite support iteration of $\mathbb{H}$, the Baumgartner-Dordal Lemma applies and so,

$$N[G_{[\alpha+2,\omega_2)}] = M[G_{\omega_2}]$$

models that $(a_\alpha : \alpha < \omega_1)$ is eventually narrow. But this implies that Y is not a γ-set since if $(U_n \in \mathcal{U} : n \in b)$ is a γ-cover of $X \subseteq Y$, then for some infinite $c \subseteq b$, we would have that $X \cap \bigcap\{U_n : n \in c\}$ is uncountable, which implies that for uncountably many α that $c \subseteq a_\alpha$. Contradicting the fact the a_α are eventually narrow.

On the other hand, it is well known that forcing with $\mathbb{H}$ adds Cohen reals and adding Cohen reals makes sets of reals of small cardinality into strong measure zero sets. To see this suppose that $(\epsilon_n > 0 : n < \omega) \in M$ a model of set theory. In M let $(I_{nm} : m < \omega)$ list all intervals with rational end points and of length less than ϵ_n. If $x : \omega \to \omega$ is a Cohen real over M, then it is an easy density argument to prove that

$$M \cap \mathbb{R} \subseteq \bigcup_{n<\omega} I_{nx(n)}$$

The usual arguments show that in the iteration every set of reals of cardinality ω_1 has strong measure zero. This proves Theorem 1.
QED

Remark. It is also true in the Hechler real model that every set of reals of size ω_1 is both in $\mathsf{S}_1(\Gamma,\Gamma)$ and $\mathsf{S}_1(\Omega,\Omega)$. For definitions, see Just, Miller, Scheepers, and Szeptycki [7]. This follows from the fact that $\mathfrak{b} > \omega_1$ and $cov(\mathcal{M}) > \omega_1$, see Figure 4 [7].

Define. X is C'' iff for every sequence $(\mathcal{U}_n : n < \omega)$ of open covers of X there exist $(U_n \in \mathcal{U}_n : n < \omega)$ an open cover of X. Equivalent terminology for C'' is the Rothberger property or $\mathsf{S}_1(\mathcal{O},\mathcal{O})$.

Define. C''-BC to be the statement that every set of reals with the property C'' is countable and let SMZ-BC denote the standard Borel conjecture, every strong measure zero set is countable.

Proposition 8 *SMZ-BC is equivalent to C''-BC.*

Proof
It is only necessary to prove right to left.

If $\mathfrak{b} = \omega_1$, then there exists an uncountable set of reals Z concentrated on a countable subset of itself, i.e., there exist countable $Q \subseteq Z$ with the property that $Z \setminus U$ is countable for every open set U containing Q (Besicovitch [2], Rothberger [10]). Any such set has property C''. To see these two results, let

$$X = \{f_\alpha \in \omega^\omega \;:\; \alpha < \omega_1\}$$

be well ordered by $\leq^*$ and unbounded. Let $h : \omega^\omega \to [0,1]$ be a homeomorphism with range the irrationals in $[0,1]$. We claim that $Y = h(X)$ is concentrated on Q where Q is the set of rationals in $[0,1]$. Note that for any open $U \subseteq [0,1]$ containing Q the set $[0,1] \setminus U$ is compact and therefore $C = h^{-1}([0,1] \setminus U)$ is a compact subset of ω^ω. Since compact sets correspond to finitely branching trees, there exists $f \in \omega^\omega$ such that $g \leq f$ for every $g \in C$. Since X is unbounded we have that $C \cap X$ is countable, hence $Y \setminus U$ is countable. Hence $Z = Y \cup Q$ is concentrated on Q. To see that concentrated sets have property C'', suppose Z is concentrated on a countable subset of itself Q. Let $(\mathcal{U}_n : n < \omega)$ be a sequence of open covers of Z. Let $(U_{2n} \in \mathcal{U}_{2n} : n < \omega)$ cover Q and then choose $(U_{2n+1} \in \mathcal{U}_{2n+1} : n < \omega)$ to cover the countably many elements of $Z \setminus \bigcup_{n<\omega} U_{2n}$.

So assume $\mathfrak{b} > \omega_1$.

Suppose there is an uncountable strong measure zero set. Then by standard arguments there exists an $X \subseteq 2^\omega$ with $|X| = \omega_1$ such that for every $f \in \omega^\omega$ there exists $(s_n \in 2^{f(n)} : n < \omega)$ such that for every $x \in X$ there are infinitely many n with $s_n \subseteq x$.

Claim. X has property C''.
Proof
Let $(\mathcal{U}_n : n < \omega)$ be open covers of X. Without loss we may assume each element of each $\mathcal{U}_n$ is of the form $[s]$ for some $s \in 2^{<\omega}$. Since $|X| < \mathfrak{b}$ we can find finite $A_n \subseteq 2^{<\omega}$ so that $s \in A_n$ implies $[s] \in \mathcal{U}_n$ and for each $x \in X$ for all but finitely many n there exists $s \in A_n$ with $s \subseteq x$. Let $f : \omega \to \omega$ be such that $f(n) > \max\{|s| : s \in A_n\}$. Using strong measure zero of X choose $s_n \in 2^{f(n)}$ so that every element of X is in infinitely many $[s_n]$. Define $t_n \in A_n$ as follows. If there exists $t \in A_n$ with $t \subseteq s_n$ then let t_n be such. If there isn't, choose t_n arbitrarily. We claim that $\{[t_n] : n < \omega\}$ covers X. For any $x \in X$ for all but finitely many n we have that there exists $t \in A_n$ with $t \subseteq x$. But for infinitely many n we have that $s_n \subseteq x$. Since $|s_n| > |t_n|$ it must be the case that $t_n \subseteq x$ for infinitely many t_n.

This proves the Claim and the Proposition.
QED

Define X is a strong γ-set iff there exists an increasing sequence of integers $(k_n : n < \omega)$ so that for every sequence $(\mathcal{U}_n : n < \omega)$ where $\mathcal{U}_n$ is a k_n-cover of X (i.e., covers every k_n element subset of X) there exists a γ-cover of the form $(U_n \in \mathcal{U}_n : n < \omega)$. These were first defined in Galvin and Miller [4]. Tsaban [12] has shown that an equivalent definition results if we always require $k_n = n$.

Theorem 9 *In the Cohen real model, i.e., ω_2 Cohen reals added to a model of CH, every strong γ-set is countable but there is an uncountable γ-set.*

Proof

First we construct an uncountable γ-set. This proof is a modification of the construction from Just, Miller, Scheepers, and Szeptycki [7] section 5.

Without loss of generality we may rearrange the generic set of ω_2 Cohen reals to have order type $\omega_2 + \omega_1$ and assume that

$$N = M[x_\alpha \subseteq \omega : \alpha < \omega_1]$$

where M is determined by the first ω_2 Cohen reals. Note that M fails to satisfy CH. Construct $y_\alpha \in [\omega]^\omega$ descending mod finite so that $(y_\beta : \beta < \alpha) \in M[x_\beta : \beta < \alpha]$ as follows:

At stage $\alpha + 1$ let

$$y_{\alpha+1} = x_{\alpha+1} \cap y_\alpha$$

This is infinite because $x_{\alpha+1}$ is Cohen generic over y_α At limit stages choose $y_\alpha \in M[x_\beta : \beta < \alpha]$ so that $y_\alpha \subseteq^* y_\beta$ all $\beta < \alpha$.

The choice of y_α can be made in some canonical way so that the sequence of names $(\overset{\circ}{y}_\alpha : \alpha < \omega_1)$ is in M. For $y \in [\omega]^\omega$ let $[y]^{*\omega} = \{x \in [\omega]^\omega \ : \ x \subseteq^* y\}$.

Claim. Suppose $(\mathcal{U}_n : n < \omega) \in M[x_\beta : \beta \le \alpha]$ is a family of ω-covers of

$$[\omega]^{<\omega} \cup \{y_\beta : \beta \le \alpha\}$$

Then there exists a sequence $(U_n \in \mathcal{U}_n : n < \omega)$ in $M[x_\beta : \beta \le \alpha + 1]$ which is a γ-cover of

$$[\omega]^{<\omega} \cup \{y_\beta : \beta \le \alpha\} \cup [y_{\alpha+1}]^{*\omega}$$

Proof

Let $\bigcup_n F_n = [\omega]^{<\omega} \cup \{y_\beta : \beta \le \alpha\}$ be an increasing union of finite sets and define $\mathcal{V}_n = \{U \in \mathcal{U}_n : F_n \subseteq U\}$ and note that they are ω-covers. Next inductively define $\mathcal{W}_n$ by $\mathcal{W}_0 = \mathcal{V}_0$ and

$$\mathcal{W}_{n+1} = \{U \cap V : U \in \mathcal{V}_n, V \in \mathcal{W}_n\}$$

and note that they are ω-covers which refine each other. Working in the ground model $M[x_\beta : \beta \le \alpha]$ construct an increasing sequence k_n and $U_n \in \mathcal{W}_n$ so that

$$\{x \subseteq \omega : x \cap [k_n, k_{n+1}) = \emptyset\} \subseteq U_n$$

this can be done since $\mathcal{W}_n$ is an ω-cover of $[\omega]^{<\omega}$. Now since $x_{\alpha+1}$ is Cohen real the following set will be infinite:

$$A = \{n < \omega : x_{\alpha+1} \cap [k_n, k_{n+1}) = \emptyset\}$$

The same or larger set will work for $y_{\alpha+1}$ and so $(U_n : n \in A)$ will be a γ-cover of $[y_{\alpha+1}]^{*\omega}$. The refining conditions on $\mathcal{W}_n$ means we can fill it in on the complement of A and the choice of $\mathcal{V}_n$ means it is a γ-cover of the rest.

QED

The Claim shows that $[\omega]^{<\omega} \cup \{y_\alpha : \alpha < \omega_1\}$ is a γ-set.

Next we show that there are no uncountable strong γ-sets. Suppose for contradiction that $X \subseteq 2^\omega$ is an uncountable strong γ-set witnessed by $(k_n : n < \omega)$ in the model N. By the usual ccc arguments we may suppose that $X, (k_n : n < \omega) \in M$ where $M \subseteq N$ is some model of CH. Let $u \in N \cap \omega^\omega$ be Cohen generic over M and $v \in N \cap \omega^\omega$ Cohen generic over $M[u]$ so that if we let

$$\mathcal{U}_n = \{[s] : s \in 2^{u(n)}\}$$

then (using that N thinks X is strong γ) there exists

$$(\mathcal{V}_n \in [\mathcal{U}_n]^{\leq k_n} : n < \omega) \in M[u, v]$$

so that $\forall x \in X \forall^\infty n \;\; x \in \cup \mathcal{V}_n$. Let $\mathbb{P}$ denote Cohen forcing and since it is countable there must be some $(p, q) \in \mathbb{P} \times \mathbb{P}$ and $N < \omega$ such that

$$(p, q) \Vdash (\overset{\circ}{\mathcal{V}}_n \in [\overset{\circ}{\mathcal{U}}_n]^{\leq k_n} : n < \omega)$$

and

$$Y = \{x \in X : (p, q) \Vdash \forall n > N \;\; x \in \cup \overset{\circ}{\mathcal{V}}_n\}$$

is uncountable. Fix $n > N, |p|$. Now since Y is uncountable there exist some level $l < \omega$ with

$$|\{x \restriction l : x \in Y\}| > k_n$$

Let $r \supseteq p$ be an extension with $r(n) = l$. But this is a contradiction since

- $(r, q) \Vdash$ "$\mathcal{V}_n \subseteq 2^l$ and $|\mathcal{V}_n| \leq k_n$", and
- $(r, q) \Vdash$ "$x \in \mathcal{V}_n$" for every $x \in Y$ and so $(r, q) \Vdash$ "$\{x \restriction l : x \in Y\} \subseteq \mathcal{V}_n$"

QED

Remark. T. Bartoszyński has shown that in the iterated superperfect real model every strong γ-set is countable. Superperfect forcing is also called rational perfect set forcing, see Miller [9]. The principle $\diamondsuit(\mathfrak{b})$ (see Džamonja, Hrušák, and Moore [3]) implies that there is an uncountable γ-set. Since $\diamondsuit(\mathfrak{b})$ holds in the iterated superperfect real model, we get another model for the consistency of strong γ-BC but not γ-BC.

Remark. Tsaban and Weiss [13] have shown that the following are equivalent:

1. Every set of reals of strong measure zero is countable (Borel conjecture).
2. Every set of reals with the property $S_1(\Omega, \Omega)$ is countable.

References

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Appendix

This is not intended for publication but only for the electronic version.

Theorem 10 *(T. Bartoszynski) In the iterated superperfect forcing model, every strong γ-set is countable.*

Proof
This model is obtained by the countable support iteration of length ω_2 of superperfect forcing over a model of CH.

First we consider one-step. Let f be superperfect generic over M a model of set theory. Define $(\mathcal{U}_n : n < \omega)$ by

$$\mathcal{U}_n = \{[s] : s \in 2^{f(n)}\}.$$

Claim. Let $g \in \omega^\omega \cap M$ and $(\mathcal{V}_n \in [\mathcal{U}_n]^{<g(n)} : n < \omega) \in M[f]$. Then

$$M[f] \models |\{x \in M \cap 2^\omega \ : \ \forall^\infty n \ \ x \in \cup \mathcal{V}_n)| \leq \omega.$$

Proof
For p a superperfect tree, define $s \in \text{splitnode}(p)$ iff $\exists^\infty n \ sn \in p$. Superperfect trees are those trees in which the split nodes are dense. Suppose

$$p \Vdash (\overset{\circ}{\mathcal{V}}_n \in [\mathcal{U}_n]^{<g(n)} : n < \omega)$$

By the usual fusion arguments we can obtain a superperfect tree $q \leq p$ and $(K_s \subseteq 2^\omega \ : \ s \in \text{splitnode}(q))$ so that

1. $|K_s| < g(|s|)$ for each $s \in \text{splitnode}(q)$
2. for each $s \in \text{splitnode}(q)$ and $x \in 2^\omega \setminus K_s$ for all but finitely many $n \in \text{split}(q, s)$
$$q_{sn} \Vdash x \notin \overset{\circ}{\mathcal{V}}_{|s|}$$

It follows that

$$q \Vdash \text{``}M \cap (\cup_{m<\omega} \cap_{n>m} \cup \overset{\circ}{\mathcal{V}}_n) \subseteq \cup \{K_s \ : \ s \in \text{splitnode}(q)\}\text{''}$$

QED

Now suppose for contradiction that X is an uncountable strong γ-set in the model $M[f_\alpha : \alpha < \omega_2]$. By the ω_2 chain condition and a Lowenheim-Skolem argument there must be an $\alpha_0 < \omega_2$

with $X, (k_n : n < \omega) \in M[f_\alpha : \alpha < \alpha_0]$ such that

$$M[f_\alpha : \alpha < \alpha_0] \models X \text{ is a strong } \gamma\text{-set with witness } (k_n : n < \omega)$$

Denote $M[f_\alpha : \alpha < \alpha_0]$ as M_0. Now using f_{α_0} (the next superperfect real) Let $\mathcal{U}_n = \{[s] \ : \ s \in 2^{f_{\alpha_0}(n)}\}$. By the one step argument for any $g \in M_0 \cap \omega^\omega$

$$M_0[f_{\alpha_0}] \models \forall(\mathcal{V}_n \in [\mathcal{U}_n]^{g(n)} |\{x \in X \cap 2^\omega \ : \ \forall^\infty n \ \ x \in \cup \mathcal{V}_n)| \leq \omega.$$

Denote $M_0[f_{\alpha_0}]$ as M_1. Our final model $M_2 = M[f_\alpha : \alpha < \omega_2]$ satisfies the Laver property over the intermediate models. . This means for any $f \in M_2 \cap \omega^\omega$ such that there exists $h \in M_1 \cap \omega^\omega$ which bounds f, i.e., $f(n) < h(n)$ all n, there exists $(H_n : n < \omega) \in M_1$ with $|H_n| \leq 2^n$ and $f(n) \in H_n$ for all n. The reason this is true is that the Laver property holds in the one-step superperfect model by essentially the same argument as for Laver forcing. It also holds in the iteration by either the same argument Laver employed or by the general fact that it is preserved by countable support iteration of proper forcings (see Bartoszynski and Judah; **Set theory. On the structure of the real line.** A K Peters, Ltd., Wellesley, MA, 1995.).

But now we get a contradiction. Let $\mathcal{U}_n^*$ to be the family of k_n unions of elements of $\mathcal{U}_n$. Since M_2 thinks that X is a strong γ-set there is a γ-cover of X of the form $(V_n \in \mathcal{U}_n^* : n < \omega)$. But by the Laver property this means there exists $(\mathcal{V}_n \in [\mathcal{U}_n]^{k_n 2^n} : n < \omega) \in M_1$ with $V_n \subseteq \cup \mathcal{V}_n$. But this is a contradiction for $g(n) = k_n 2^n$ and $M_1 = M_0[f_{\alpha_0}]$.
QED

Next we show that there is an uncountable γ-set in the superperfect model. We construct it using the principle $\Diamond(\mathfrak{b})$. This is stronger than $\mathfrak{b} = \omega_1$ and is defined in Dzamonja, Hrusak, and Moore [3]. They prove that it holds in any model of $\mathfrak{b} = \omega_1$ which is obtained by the ω_2-iteration with countable support of proper Borel orders which are reasonably homogeneous. Hence, $\Diamond(\mathfrak{b})$ is true in the iterated superperfect set forcing model.

I do not know if $\mathfrak{b} = \omega_1$ is enough to construct an uncountable γ-set.

Define $\Diamond(\mathfrak{b})$: For every $F : 2^{<\omega_1} \to \omega^\omega$ such that each $F \restriction 2^\alpha$ is Borel for $\alpha < \omega_1$ there exists $g : \omega_1 \to \omega^\omega$ so that for every $f \in 2^{\omega_1}$ such $\exists^\infty n \ F(f \restriction \delta)(n) < g(\delta)(n)$ for stationarily many $\delta < \omega_1$.

Theorem 11 $\diamondsuit(\mathfrak{b})$ *implies there is an uncountable* γ*-set.*

Proof

Let $H_\delta : ([\omega]^\omega)^\delta \to [\omega]^\omega)$ be Borel so that for any $(x_\alpha : \alpha < \delta)$ if and $\alpha < \beta$ implies $x_\beta \subseteq^* x_\alpha$, then for $y = H(x_\alpha : \alpha < \delta)$ we have that $y \subseteq^* x_\alpha$ for every $\alpha < \delta$. By using the first ω-coordinates to code a countable family of open sets we may assume that the domain of F is sets of the form $(\mathcal{U}_n : n < \omega), (x_\alpha \subseteq \omega : \alpha < \delta)$ where the $\mathcal{U}_n$ are families of open subsets of 2^ω and we are to define

$$F((\mathcal{U}_n : n < \omega), (x_\alpha : \alpha < \delta)) = h \in \omega^\omega$$

Suppose

1. $x_\alpha \in [\omega]^\omega$ for each $\alpha < \delta$,
2. $x_\alpha \subseteq^* x_\beta$ for each $\beta < \alpha < \delta$, and
3. $\mathcal{U}_n$ is an ω-cover of $[\omega]^{<\omega} \cup \{x_\alpha : \alpha < \delta\}$ for each n.

(If any of these fail to be true, just define h to be the constant zero function.)

Let $\{\delta_i : i < \omega\} = \delta$ be some previously chosen enumeration of δ and define for each n

$$\mathcal{V}_n = \{U \in \mathcal{U}_n : \{x_{\delta_i} : i < n\} \subseteq U\}$$

It is easy to check that each $\mathcal{V}_n$ is an ω-cover of $[\omega]^{<\omega} \cup \{x_\alpha : \alpha < \delta\}$. Also choosing an element of each will automatically γ-cover $\{x_\alpha : \alpha < \delta\}$. Next define inductively $\mathcal{W}_n$ as follows:

1. $\mathcal{W}_0 = \mathcal{V}_0$,
2. $\mathcal{W}_{n+1} = \{U \cap V : U \in \mathcal{W}_n, V \in \mathcal{V}_n\}$

It is easy to check that the intersections of elements of two ω-covers is an ω-cover, so by induction each $\mathcal{W}_n$ is an ω-cover of $[\omega]^{<\omega} \cup \{x_\alpha : \alpha < \delta\}$. Since $\mathcal{W}_{n+1}$ is a refinement of $\mathcal{W}_n$, if for some $A \in [\omega]^\omega$ we have $(U_n \in \mathcal{W}_n : n \in A)$ is a γ-cover, then we can choose U_n for $n \notin A$ by looking forward to the next element of A so that $(U_n \in \mathcal{W}_n : n \in \omega)$ is a γ-cover.

Apply H to get $H(x_\alpha : \alpha < \delta) = \{k_n : n < \omega\}$ (Note that this does not depend on the covers $\mathcal{U}_n$.) Construct an infinite $B \subseteq \omega$ so that for every successive pair of elements of B, say $n < m$, there exists $U_n \in \mathcal{W}_n$ so that

$$\{x \subseteq \omega : x \cap [k_n, k_m) = \emptyset\} \subseteq U_n$$

This only uses that $\mathcal{W}_n$ is an ω-cover of $[\omega]^{<\omega}$: choose U to cover $[k_n]^{<\omega}$ and then using that U is open make sure that k_m is sufficiently large. Now we make sure that $h \in \omega^\omega$ is such that h eventually dominates the enumeration function of $B \setminus N$ for each $N < \omega$. We leave to the reader the details of showing that h can be obtained using a Borel function on $(\mathcal{U}_n : n < \omega), (x_\alpha : \alpha < \delta)$. But note the following: Suppose $g \in \omega^\omega$ has the property that $\exists^\infty n\ g(n) > h(n)$, then there must be infinitely many i so that there exists $n < m$ elements of B so that $g(i) \leq n < m \leq g(i+1)$. Otherwise the enumeration function of some $B \setminus N$ would dominate g which is impossible.

Applying $\diamondsuit(\mathfrak{b})$ to our function F we get a $g : \omega \to \omega^\omega$. Construct our γ-set $X = [\omega]^{<\omega} \cup \{x_\alpha : \alpha < \omega_1\}$ as follows:

Given $\{x_\alpha : \alpha < \delta\}$ and descending sequence in $\subseteq^*$ apply H to get $H(x_\alpha : \alpha < \delta) = \{k_n : n < \omega\}$. Let $g = g(\delta) \in \omega^\omega$ and put $x_\delta = \{k_{g(n)} : n < \omega\}$. Now we verify that X is a γ-set. Suppose that $(\mathcal{U}_n : n < \omega)$ are open ω-covers of X. By the definition of $\diamondsuit(\mathfrak{b})$ there are stationarily many $\delta < \omega_1$ such that

$$F((\mathcal{U}_n : n < \omega), (x_\alpha : \alpha < \delta)) = h \in \omega^\omega$$

and if $g = g(\delta)$, then $\exists^\infty n\ g(n) > h(n)$. Note that $x_\delta = \{k_{g(i)} : n < \omega\}$. So as we have remarked there are infinitely many i (say $i \in C$) so that there exists elements of $n_i < m_i$ of B with $g(i) < n_i < m_i < g(i+1)$. The way the elements of B were construct means that there exists $U_{n_i} \in \mathcal{W}_{n_i}$ such that

$$\{x \subseteq \omega : x \cap [k_{n_i}, k_{m_{i+1}}) = \emptyset\} \subseteq U_{n_i}$$

But this means that $(U_{n_i} : i \in C)$ is a γ-cover of $\{x : x \subseteq^* x_\delta\}$. But the construction of $(\mathcal{W}_n : n \in \omega)$ guarantees that we can define them on all n so that $(U_n \in \mathcal{W}_n : n < \omega)$ is an γ-cover of

$$[\omega]^{<\omega} \cup \{x_\alpha : \alpha < \delta\} \cup \{x : x \subseteq^* x_\delta\}$$

which includes X.
QED

Corollary 12 *In the iterated superperfect model we have (strong γ)- BC and not(γ-BC)*