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Dear Prof. Guy

Here is a solution to a problem of Louis Funar which is on p280 of Math Monthly April 1986. It is asked if for an arbitrary function $f : \mathbb{R} \to \mathbb{R}$ do there exist functions $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ with g bijective and h injective such that $f = g + h$. The answer is no.

$$f(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Suppose g and h existed. Then

$$f(g^{-1}(x)) = g(g^{-1}(x)) + h(g^{-1}(x))$$

let $i = f \circ g^{-1}$ and $k = h \circ g^{-1}$

Then $i(x) = \begin{cases} 0 & x \neq a \\ 1 & x = a \end{cases}$ where $a = g(0)$

$k(x)$ is injective, and

$i(x) = x + k(x)$ all x.

Hence

$$0 = x + k(x) \quad \text{if } x \neq a$$
$$1 = a + k(a)$$

But then $k(a) = 1 - a = k(a-1)$

so k is not injective.

$\boxtimes$

The following is a theorem:

$\forall f : \mathbb{R} \to \mathbb{R} \;\; \exists\, g : \mathbb{R} \to \mathbb{R} \;\; \exists\, h : \mathbb{R} \to \mathbb{R}$

bijections such that for all <u>but finitely</u> many $x \in \mathbb{R}$ $f(x) = g(x) + h(x)$.

So it is "almost" yes.

Here is a sketch of the proof:

Let $F = f(\mathbb{R})$. i.e. range of f.

Lemma 1 : Suppose G is a subgroup of $(\mathbb{R}, +)$, $F \subseteq G$, and $|F| = |G|$.
and $\forall x \in F$ $f^{-1}\{x\} \cap G \neq \emptyset$.
Then $\exists h : G \to G$ $\exists g : G \to G$ bijections and $f \restriction G = h + g$.

proof

$|F|$ is the cardinality of F say κ.

Then $F = \{x_\alpha : \alpha < \kappa\}$ and $G = \{y_\alpha : \alpha < \kappa\}$.

h, g are constructed by transfinite induction and a back-and-forth argument.

So we construct h_α, g_α for $\alpha < \kappa$ one-to-one partial functions from G to G such that

1. $\alpha < \beta \to (h_\alpha \subseteq h_\beta$ and $g_\alpha \subseteq g_\beta)$
2. $y_\alpha \in \mathrm{dom}(h_{\alpha+1}) = \mathrm{dom}(g_{\alpha+1})$
3. $f \restriction \mathrm{dom}(h_\alpha) = h_\alpha + g_\alpha$
4. $y_\alpha \in \mathrm{range}(h_{\alpha+1}) \cap \mathrm{range}(g_{\alpha+1})$

5. $|h_\alpha| = |g_\alpha| \leq |\alpha| + \omega$.

Details of construction:

Let $h_0 = g_0 =$ empty function

For limit ordinals λ

$$h_\lambda = \bigcup_{\alpha<\lambda} h_\alpha \quad \text{and} \quad g_\lambda = \bigcup_{\alpha<\lambda} g_\alpha$$

For successor ordinals $\alpha+1$ we're given h_α, g_α. 1st satisfy 2.

If $y_\alpha \notin \mathrm{dom}(h_\alpha)$ then find

$z \in G \setminus \mathrm{range}(h_\alpha)$ so that

$f(y_\alpha) - z \in G \setminus \mathrm{range}(g_\alpha)$.

Then let
$$h^* = h_\alpha \cup \{<y_\alpha, z>\}$$
$$g^* = g_\alpha \cup \{<y_\alpha, f(y_\alpha) - z>\}$$

hence $f(y_\alpha) = h^*(y_\alpha) + g^*(y_\alpha)$

To make sure $y_\alpha \in \mathrm{range}(h_{\alpha+1})$

choose $z \in G$ so that

$z \notin \mathrm{dom}(h^*)$ and $f(z) - y_\alpha \notin \mathrm{range}(g^*)$

(z can be found because $|F| = \kappa$)

Now extend h^*, g^* by $h^{**}(z) = y_\alpha$ and

$g^{**}(z) = f(z) - y_\alpha$. Repeat this argument to put $y_\alpha \in \text{range}(g_{\alpha+1})$

$\boxed{\times}$

<u>Lemma 2.</u> Suppose G is a subgroup of $\mathbb{R}$ with $\text{range}(f) \subseteq G$ and if $|f^{-1}(a)| \leq |G|$ then $f^{-1}(a) \subseteq G$.

Then $\exists h: \mathbb{R}\setminus G \to \mathbb{R}\setminus G$ $\exists g\ \mathbb{R}\setminus G \to \mathbb{R}\setminus G$ bijections such that

$$f \restriction \mathbb{R}\setminus G = h + g.$$

<u>proof:</u>

Define $x \sim y$ iff $x - y \in G$ (i.e the usual cosets decomposition)

Let $F_0 = \{a \in \mathbb{R} \mid f^{-1}(a) \not\subseteq G\}$

and let $H_a = f^{-1}(a) \setminus G$ for each $a \in F_0$.

By assumption $\langle H_a : a \in F_0 \rangle$ is partition of $\mathbb{R}\setminus G$ into sets each of cardinality greater than G.

Let $\langle K_a : a \in F_0 \rangle$ be a partition of $\mathbb{R} \setminus G$ into sets which are $\sim$ invariant, i.e. $(x \sim y \in K_a) \to x \in K_a$, and $|K_a| = |H_a|$. This is possible since each $\sim$ equivalence class has the same cardinality as G. Now let $\pi : \mathbb{R} \setminus G \to \mathbb{R} \setminus G$ be a bijection such that $\pi(K_a) = H_a$ for each $a \in F_0$. It is enough to find $h, g : \mathbb{R} \setminus G \to \mathbb{R} \setminus G$ bijections such that

$$k = f \circ \pi \neq h + g.$$

Since then

$$f \restriction \mathbb{R} \setminus G = h \circ \pi^{-1} + g \circ \pi^{-1}$$

Note that $k^{-1}(a) = K_a$ since

$$k(x)=a \leftrightarrow f(\pi(x)) = a \leftrightarrow \pi(x) \in H_a \leftrightarrow x \in K_a.$$

Let $h(x) = -x$, $g(x) = x+a$ if $x \in K_a$.

Then $k(x) = h(x) + g(x)$, h is clearly a bijection of $\mathbb{R} \setminus G$ and g is also since for each $a \in F_0$ $g \restriction K_a$ is a bijection of K_a into K_a (since K_a is $\sim$ invariant) $\square$

Lemma 3 Given $f : \mathbb{R} \to \mathbb{R}$, $F = \text{range}(f)$

There is a subgroup G of $\mathbb{R}$ such that

(1) $F \subseteq G$, $|G| = |F| + \omega$ (i.e. $|G|$ smallest inf card $\geq |F|$)

(2) $\forall x \in F$ $f^{-1}(x) \cap G \neq \phi$

(3) $\forall x \in F$ $|f^{-1}(x)| \leq |G| \longrightarrow f^{-1}(x) \subseteq G$

infact if $f^{-1}(x)$ inf. then $f^{-1}(x) \cap G$ is inf.

proof:

Let G_0 be the subgroup of $\mathbb{R}$ generated by $F \cup \bigcup\{x_a : a \in F\}$ where $x_a \subseteq f^{-1}(a)$ is nonempty for every a and countably infinite if $f^{-1}(a)$ is infinite. Then $|G_0| = |F| + \omega$.

Let $X = \bigcup\{ f^{-1}(a) \mid |f^{-1}(a)| \leq |G_0| \}$

then $|X| = |G_0|$

and let G be the group generated by $G_0 \cup X$.

$\boxtimes$

Combining the Lemmas we see that the only case left is if $F = \text{range}(f)$ is finite (except in this case we get $\forall x \in \mathbb{R} \quad f(x) = g(x) + h(x)$)

So now assume F is finite. Since we need only prove: for all but finitely many x $f(x) = g(x) + h(x)$, we may assume that for every $a \in F$ $f^{-1}(a)$ is infinite. This is the only time we need alter f so if range(f) infinite or $\forall a \in$ range(f) $f^{-1}(a)$ infinite then $\exists g, h$ bijections $\forall x \in \mathbb{R}$ $f(x) = g(x) + h(x)$.

Let G be a countable subgroup of $\mathbb{R}$ given by Lemma 3. And let H be a countable subgroup of $\mathbb{R}$ with $G \subseteq H$ and $|H/G| \geq |F|$.

Since H satisfies Lemma 2 it suffices to show there exists bijections

$$g : H \to H$$
$$h : H \to H$$

such that $f \upharpoonright H = g + h$. But this is similar to the proof of Lemma 2, namely: find

$\langle K_a : a \in T \rangle$ partition H

K_a inf and $\sim$ invariant
$x \sim y$ $(x - y \in G)$

$$\pi : H \to H$$
$$\pi(K_a) = (f^{-1}(a) \cap H)$$

$$k = f \circ \pi = h + g$$

where $h(x) = -x$
$g(x) = x + a$ if $x \in K_a$.

etc. $\square$

I think this theorem (every function is 'almost' the sum of 2 bijections) will generalise to any group which is not finitely generated. Its false for $(\mathbb{Z},+)$.

Could you give me your opinion if it would be appropriate to write these results up for possible publication in the Math Monthly?

Sincerely

Arnold W. Miller

P.S. I am sending a copy of this letter to Louis Funar.