It is worth noting that (b) generalizes: if $-1 = \boxed{n}$ in K and t is chosen so that $2^t \leqslant n < 2^{t+1}$, then $-1 = \boxed{2^t}$ in K. In particular the level of a field, if finite, is a power of two. This generalization fails for rings in general, however. For example -1 is a sum of three squares, but not a sum of two, in the ring of integers modulo 12.

The implication "$\Leftarrow$" in (c) is trivial, and "$\Rightarrow$" is nearly so: $-1 = (a + b\sqrt{-m})^2$ entails $a^2 - mb^2 = -1$ and $ab = 0$; but $b \neq 0$ since $-1 \neq \boxed{1}$ in $\mathbb{Q}$; hence $a = 0$, and then $mb^2 = 1$ forces $m = 1$ since m is squarefree.

This completes the proof of Lemma 8, and we now use it to show that Theorem 5 follows from Theorem 1 (Gauss' Theorem), or from its special case Corollary 2. From Lemma 8 we know that $s(\mathbb{Q}(\sqrt{-m})) = 1$ for $m = 1$ and $S(\mathbb{Q}(\sqrt{-m})) = 2$ or 4 for all $m > 1$. With Lemma 7, Corollary 2 gives $s(\mathbb{Q}(\sqrt{-m})) = 4 \Leftrightarrow m \equiv 7 \pmod 8$, and this secures Theorem 5.

The proof of the equivalence of Theorem 1 and Theorem 5 is now complete, and one may ask for the moral of the story. The interpretation in **[R]** is that Theorem 5, which first appears (as far as I know) as Theorem 7 in **[FGS]**, gives a new proof of Gauss' Theorem. Alternatively, one can start with Gauss and view the equivalence as providing a very simple demonstration of Theorem 5. From a slightly loftier (?) point of view, one might use the equivalence to put Theorem 5 in perspective: Gauss' Theorem, however one approaches it, is a substantial result, and the same may therefore be said of Theorem 5 which computes the levels of quadratic number fields.

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UNSOLVED PROBLEMS

EDITED BY RICHARD GUY

In this department the MONTHLY *presents easily stated unsolved problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada T2N 1N4.*

ANY ANSWERS ANENT THESE ANALYTICAL ENIGMAS?

RICHARD K. GUY

Several unsolved problems have been submitted recently, which are brief to state. Neither the referees nor your editor were able to find relevant references. Here are four examples from analysis.

Alan A. Grometstein, Lincoln Laboratory, Massachusetts Institute of Technology, Lexington, MA 02173-0073, asks about the function

$$f(x, y) = yx^y\{y^x - (y-1)^x\} - xy^x\{x^y - (x-1)^y\}.$$

He notes that it is antisymmetric: $f(x, y) = -f(y, x)$ and $f(x, x) = 0$.

If x, y are integers, $x > y \geqslant 1$, is $f(x, y) > 0$?

He says that there is a good deal of numerical evidence. The restriction to integers may be inessential.

Louis Funar, Department of Mathematics, University of Craiova, Craiova, A.I. Cuza nr. 13, 1100 Romania, asks if, given an arbitrary function $f\colon \mathbb{R} \to \mathbb{R}$,

Do there exist functions $g\colon \mathbb{R} \to \mathbb{R}$ and $h\colon \mathbb{R} \to \mathbb{R}$,

the first one bijective and the second one injective, such that $f = g + h$?

Bogusław Tomaszewski, Department of Mathematics, Oklahoma State University, OK 74078, considers n real numbers $a_1, \ldots, a_n$ such that $\sum_{i=1}^n a_i^2 = 1$. Of the 2^n expressions $|\varepsilon_1 a_1 + \cdots + \varepsilon_n a_n|$ with $\varepsilon_i = \pm 1, 1 \leqslant i \leqslant n$,

Can there be more with value > 1 than with value $\leqslant 1$?

Carl Ponder, Computer Science Division, University of California, Berkeley, CA 94720, defines $\varphi_h(x)$ by the differential equation

$$\frac{d}{dx}\varphi_h(x) = \{\varphi_{h-1}(x)\}^2$$

with boundary conditions $\varphi_0(x) = \varphi_h(0) = 1$, and asks

What is the asymptotic behavior of $\varphi_h(1)$ as $h \to \infty$?

It is not hard to see that $\varphi_h(x)$ is a polynomial in x of degree $2^h - 1$. For example

$$\varphi_1(x) = 1 + x,$$

$$\varphi_2(x) = 1 + x + x^2 + \frac{1}{3}x^3,$$

$$\varphi_3(x) = 1 + x + x^2 + x^3 + \frac{2}{3}x^4 + \frac{1}{3}x^5 + \frac{1}{9}x^6 + \frac{1}{63}x^7.$$

If $\varphi_h(x) = \sum_{i=0}^{2^h-1} a_{h,i}x^i$, then, for $h \geqslant 0$, $a_{h,0} = a_{h,1} = \cdots = a_{h,h} = 1$, and $a_{h,h+1} = a_{h+1,h+3} = 1 - 2^h/(h+1)!$ However, the formulas

$$a_{h,h+3} = 1 - \frac{2^{h-3}}{(h-1)!} - \frac{2^h}{3(h!)} + \frac{5 \cdot 2^h}{(h+3)!} \quad (h \geqslant 2)$$

$$a_{h,h+4} = 1 - \frac{2^{h-4}}{3((h-2)!)} - \frac{2^{h-1}}{3((h-1)!)} - \frac{2^{h-2}}{h!} + \frac{5 \cdot 2^{h-1}}{(h+2)!} + \frac{5 \cdot 2^{h+1}}{(h+4)!} \quad (h \geqslant 3)$$

$$a_{h,h+5} = 1 - \frac{2^{h-7}}{3((h-3)!)} - \frac{2^{h-3}}{3((h-2)!)} - \frac{13 \cdot 2^{h-3}}{9((h-1)!)} - \frac{2^h}{5(h!)} + \frac{5 \cdot 2^{h-3}}{(h+1)!} + \frac{5 \cdot 2^h}{3((h+2)!)} + \frac{5 \cdot 2^h}{(h+3)!} + \frac{119 \cdot 2^h}{(h+5)!} \quad (h \geqslant 4)$$

are increasingly disappointing, though we can say that

$$a_{h,2^h-1} = 1/(2-1)^{2^{h-1}}(2^2-1)^{2^{h-2}}(2^3-1)^{2^{h-3}}\cdots(2^{h-2}-1)^{2^2}(2^{h-1}-1)^2(2^h-1).$$

It is also not hard to show that $0 \leqslant a_{h,i} \leqslant 1$, so we immediately have the bounds

$$h+1 \leqslant \varphi_h(1) \leqslant 2^h.$$

Calculation of the first few values:

$h =$	0	1	2	3	4	5	6	7	8
$\varphi_h(1) =$	1	2	$3\frac{1}{3}$	$5\frac{8}{63}$	7.533	10.747	15.019	20.674	28.131

suggests that, for $h \geqslant 2$, $\varphi_h(1) > (3/2)^h$, but your editor was unable to obtain a convincing proof.

NOTES

EDITED BY SABRA S. ANDERSON, SHELDON AXLER, AND J. ARTHUR SEEBACH, JR.

For instructions about submitting Notes for publication in this department see the inside front cover.

A SIMPLE PROOF OF THE DIRICHLET-JORDAN CONVERGENCE TEST

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Let f be a complex-valued integrable function on the interval $[0, 2\pi]$, and recall that the Fourier coefficients of f are defined by

$$a_k = (1/2\pi)\int_0^{2\pi} f(t)\,e^{-ikt}\,dt,$$

and that the partial sums of the Fourier series of f are defined by

$$S_N f(x) = \sum_{k=-N}^{N} a_k e^{ikx}.$$

The Dirichlet-Jordan convergence test states that if f has bounded variation, then for each x the limit of $S_n f(x)$, as n tends to infinity, exists and is equal to $(f(x^+) + f(x^-))/2$. Here $f(x^+)$ and $f(x^-)$ denote the right and left hand limits of f at x, and as usual we will extend f to be a 2π-periodic function defined on the whole real line.

The original proof of the Dirichlet-Jordan convergence test, as given by Dirichlet for a monotonic function [**1**], and extended by Jordan to functions of bounded variation ([**2**], pp. 264–289), is based upon the second mean value theorem (presented, for example, in [**3**], p. 245). This method is used again by Zygmund ([**4**], pp. 57–58), who also gives another proof using tools from Cesaro summability theory.

The proof presented here is more straightforward and involves slightly less work than the usual proofs. However, the usual proofs have the advantage of giving the extra information that if f is continuous in addition to having bounded variation, then the partial sums $S_n f$ converge uniformly to f, rather than just pointwise.

We begin our proof by recalling that the usual elementary classical computations produce the familiar integral formula for $S_n f(x)$:

$$S_n f(x) = \frac{1}{\pi}\int_0^{\pi/2}\{f(x+2u) + f(x-2u)\}\frac{\sin(2n+1)u}{\sin u}\,du.$$