UNSOLVED PROBLEMS

EDITED BY RICHARD GUY

In this department the MONTHLY *presents easily stated unsolved problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada T2N 1N4.*

Monthly Unsolved Problems, 1969–1987

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Many Unsolved Problems have turned out to be not as unsolved, or not as unsolvable, as expected, resulting in a longer than usual updating article. Readers share my mixed feelings. Some are delighted to discover something new. Some are dismayed that their original work originated a century ago, or simultaneously with a dozen contemporaries.

Regular readers will know that in these updating articles in December of odd-numbered years, brackets contain year and page numbers of references to this MONTHLY, earlier updating articles being [1971, 1113; 1973, 1120; ...; 1983, 683; 1985, 717]. Years in parentheses are of publications listed at the end; items not yet published are labelled (tbp) or (wrc) according as they are likely to be published formally or to remain as written communications.

In order to enhance the usefulness of the references, Math. Reviews numbers are added, and a bracket indicating where the problem originally appeared in the MONTHLY.

A ***d*-dimensional snake** is a simple circuit in the edge-set of a d-dimensional cube that has no chords, i.e., every edge of the cube which joins two vertices of the circuit is an edge of the snake. Such snakes are used to construct "snake-in-the-box" codes. Knut Diemer (1985) has shown that snakes of dimension $d \geqslant 6$ have length at most

$$2^{d-1}(d-2)(d-3)/(d^2-5d+7).$$

This improves the Douglas upper bound mentioned by Klee [1970, 63].

A good survey of problems related to the sequences associated with Thue and Morse-Hedlund [1971, 886], which avoid immediate repetitions of blocks (subwords, factors) has been made by Baker, McNulty and Taylor (tbp). They say that a word U is **avoidable on *n* letters** if there are arbitrarily long words on an n-letter alphabet that avoid U, where W **avoids** U if no block of W is the image of U under a homomorphism of free semigroups without unit. For example, *abcdbcdba* is not squarefree, because *bcd* appears squared, whereas the word *abacabcacbabcabacabc* avoids xx, i.e. is squarefree. If $\mu(U)$ is the least n for which U is avoidable on n letters, they give a linear bound for $\mu(U)$ in terms of the alphabet size of U. There is a word that is avoidable on 4 letters, but not on 3. If U is this word, the number of words of length L on a $\mu(U)$-letter alphabet, that avoid U, has a polynomial bound in L. On the other hand, the bound for xx is exponential. McNulty draws our attention to a paper of Zimin (1984).

D. H. Lehmer conjectured that there is no *composite* value of n such that $\varphi(n)$, Euler's totient function, is a divisor of $n-1$, i.e., that for no value of n is $\varphi(n)$ a *proper* divisor of $n-1$ [1973, 192].

Schinzel noted that if $n = p$ or $2p$, where p is prime, then $\varphi(n)+1$ divides n, and asked if the converse is always true. See B37 in Guy (1981).

Sanford Segal observed that Schnizel's question reduces to Lehmer's, that it arises in group theory, and may have been raised by G. Hajós. See Miech (1966), though it is there attributed to Gordon. For the reduction of Schinzel's question to that of Lehmer, see Cohen (tbp).

Bernardo Recamán [1973, 919; and see 1975, 998] asked several questions about Ulam's sequence, $U_1 = 1$, $U_2 = 2$, and for $n \geqslant 3$, U_n is the least integer expressible *uniquely* as the sum of two distinct earlier members of the sequence. A remark of Eggleton [1973, 920] shows that $U_{n+1} \leqslant U_n + U_{n-2}$. Hence $U_{n+1} < 2U_n$ and it follows [1977, 809] that $\{U_n\}$ is **complete**, i.e. that every positive integer is expressible as the sum of distinct U-numbers. David Zeitlin (see [1977, 815] for reference) conjectured that $\{U_n\}$ is still complete, even after the deletion of one or two members. Robert Stong (wrc) recalls his earlier proofs of this, and of another Zeitlin conjecture, that $\{U_n^*\}$ is complete, where $U_1^* = 1$, $U_2^* = 2$, and, for $n \geqslant 3$, $U_n^* = U_n + U_{n-2}$ (since $U_{n+1}^* = U_{n+1} + U_{n-1} < 2U_n + 2U_{n-2} = 2U_n^*$).

Molnar [1974, 383] asked for determinants with nonunit integer entries whose value was 1, and remained so when the entries were squared. Several solutions were given [1975, 999–1000; 1977, 809] but the following have not previously appeared. The first two are by Don Coppersmith, the next three by Morris Newman, and the last two by Sadao Saito (wrc).

$$\begin{bmatrix} 27 & 26 & 23 \\ 5 & 5 & 4 \\ -6 & -5 & -6 \end{bmatrix}\begin{bmatrix} 119 & 208 & 277 \\ 9 & 14 & 16 \\ 12 & 21 & 28 \end{bmatrix}\quad\begin{bmatrix} 43257 & 7 & 9 \\ 18544 & 3 & 4 \\ 12376 & 2 & 3 \end{bmatrix}\begin{bmatrix} -386723 & -17 & -23 \\ 68242 & 3 & 4 \\ 45 & 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 8n^2-8n & 2n+1 & 4n \\ -4n^2-4n & n+1 & 2n+1 \\ -4n^2-4n+1 & n & 2n-1 \end{bmatrix}\quad\begin{bmatrix} -8n^2-4n+1 & 2n & 2n+2 \\ -8n^2-8n-3 & 2n+1 & 2n+1 \\ -8n^2-4n & 2n & 2n+1 \end{bmatrix}$$

$$\begin{bmatrix} -8n^2-4n+1 & 2n & 4n+4 \\ -8n^2-8n-3 & 2n+1 & 4n+2 \\ -4n^2-2n & n & 2n+1 \end{bmatrix}$$

Richard Macintosh found two examples involving several Fibonacci numbers:

$$\begin{bmatrix} 1167 & 2 & 5 \\ 1698 & 3 & 8 \\ 2866 & 5 & 13 \end{bmatrix}\quad\begin{bmatrix} 610 & 5 & 13 \\ 1054 & 8 & 21 \\ 1665 & 13 & 34 \end{bmatrix}$$

In Conway's game of Sylver Coinage, players alternately name positive integers, subject to their not being the sum (with repetitions allowed) of previously named integers: if you name 1, you lose. The last of the twenty questions [1976, 634] was: can the reader complete the table (of good replies to various positions)? The 1983 reprinting of *Winning Ways* showed this (p. 596) largely done by John Francis, who also found the good replies: 28 to $\{16, 24, 5\}$; 6 and 9 to $\{16, 24, 7\}$; and 7 and 11 to $\{16, 24, 9\}$. Francis Voelkle (wrc) has since made considerable advances, finding the good replies: 10 to $\{16, 24\}$; 10, 15, 16, and 21 to $\{12, 18\}$; and 24 to $\{18, 27\}$. However, we still don't know (questions 4 & 5) if $\{16\}$, $\{18\}$ or $\{27\}$ has any good reply, though Voelkle conjectures that 15 may be a good reply to $\{27\}$. We conjectured (questions 6 & 7) that $\{a, b, c, \dots\}$ was a $\mathscr{P}$-position (previous-player-winning) if $\{2a, 2b, 2c, \dots\}$ or $\{3a, 3b, 3c, \dots\}$ was, but Voelkle finds numerous counterexamples: e.g. $\{8, 14\}$ is a $\mathscr{P}$-position, while $\{4, 7\}$ has the good reply 13; $\{12, 15, 18\}$ is a $\mathscr{P}$-position, but $\{4, 5, 6\}$ is not.

A pair (a, b) is called an **α-pair** if $\{5, a, 2a, b, 2b\}$, with one member in each residue class, mod 5, is a $\mathscr{P}$-position: e.g. (2,3), (7,8), (17,18), (22,23), (39,41). Question 18 asked about the truth of the *α-hypothesis* ¿$|a - b| = 1$ or 2? Voelkle has now found larger differences.

The following table summarizes the work of Hutchings and Voelkle and shows all known good replies to the first few composite numbers, and (in parentheses) the only other candidates:

{4}	6	{12}	8($4n + 2 \geqslant 30$, $3n \geqslant 27$)	{20}	5($2n \geqslant 16$)
{6}	4, 9	{14}	7, 8, 10(26)	{21}	7($3n \geqslant 15$)
{8}	12, 14	{15}	5($3n \geqslant 21$)	{22}	11($2n \geqslant 18$)
{9}	6	{16}	(20, $2n \geqslant 26$)	{24}	($3n \geqslant 15$, $2n \geqslant 20$)
{10}	5, 14(26, 32, 36, 46)	{18}	($2n \geqslant 20$, $3n \geqslant 30$)	{25}	5

Voelkle acknowledges extensive use of the VAX 8600 at the École Polytechnique Fédérale de Lausanne.

We quoted [1985, 718] Tunnell's paper in connexion with the congruent number problem [1980, 43] in which he observed that the conjecture of Birch and Swinnerton-Dyer, together with a result of Waldspurger (1981) leads to a conjectured explicit description of congruent numbers. Kramarz (1986) verifies a version of the conjecture and finds all congruent numbers < 2000.

Ernst Selmer (1986) has produced two volumes on the postage stamp problem [1980, 206]. It is convenient to distinguish the (Frobenius, Sylvester) **coin problem**: find the *largest* number of units that cannot be made up from a given set of coin denominations (which *doesn't* include a unit coin), from the **postage stamp problem**: find the *smallest* number of units that cannot be affixed to an envelope with room only for a given number of stamps, chosen from a given set of denominations (which *does* include a unit stamp). Selmer distinguishes between this, the **local** stamp problem, and the **global** one: given the envelope size and the number of different denominations, choose these denominations to maximize the range of consecutive postages that can be stamped. Selmer's encyclopedic work contains 103 references, but there remains a plethora of unsolved problems, requiring interplay of theory and computation.

Many problems remain open concerning "peeling rinds" from a sequence [1982, 113]. We can now complete the bibliographic details for Gibson and Slater (1984) and for Schwenk (1984), who finds the maximum number of rinds that can be peeled from a sequence of n symbols, each occurring twice, for $8 \leqslant n \leqslant 14$, and conjectures that, for all $n \geqslant 8$, the sequence exemplified, for $n = 10$, by

4 3 1 2 0 0 1 2 3 4 9 8 7 6 5 5 7 6 8 9

gives this maximum number.

The $3x + 1$ problem [1983, 35; 1985, 3] remains a hardy annual. Korec and Znâm (wrc) write $P \prec Y$ to mean that for every positive integer x, there is some $y \in Y$, and i, j such that $f^i(y) = f^j(x)$ where $f(x) = 3x + 1$ (x odd), $= x/2$ (x even). So all we have to show is $P \prec \{1\}$. If $a(m)$ denotes the set of positive integers $\equiv a (\text{mod } m)$, they prove that if p is an odd prime, and 2 is a primitive root of p^2, then $P \prec a(p^n)$ for every pair of positive integers n, a with a prime to p.

Yuri Fradkin (wrc) obtains an equivalent problem by defining $g(x)$ for odd x by $(3x + 1)/2$ $(x = 4k - 1)$, $(3x + 1)/4 (x = 8k + 1)$, and $(x - 1)/4 (x = 8k - 3)$, and defining the set, RO, of **regular odd numbers** by $1 \in$ RO; $4x + 1 \in$ RO if x does; $(4x - 1)/3 \in$ RO if $x \equiv 1 \pmod 3$ does; and $(2x - 1)/3 \in$ RO if $x \equiv 2 \pmod 3$ does. The set RO is similar to those considered by Klarner and Rado (1973, 1974).

Alon and Frankl (1985) prove an old conjecture of Erdős by showing that the number of disjoint pairs in a family of 2^{n+1} subsets of a $2n$-element set is bounded by $(1 + o(1))2^{2n}$. They also verify the conjecture of Daykin and Erdős [1983, 119] and establish the corresponding Erdős-Stone type result by showing that if $\mathscr{F}$ is a family of m distinct subsets of an n-element set, $d(\mathscr{F})$ is the number of disjoint pairs in $\mathscr{F}$, and $d(n, m)$ the maximum of $d(\mathscr{F})$ over all m-element families, then, for $m = 2^{(1/(k+1)+\delta)n}$ and $\delta > 0$, there is a $\beta > 0$ such that

$$d(n, m) < \left(1 - \frac{1}{k}\right)\binom{m}{2} + O(m^{2-\beta\delta^2})$$

We give details of the paper of Giblin and Kingston (1986) which solved Giblin's moving triangle problem [1983, 121].

Yang Yanlin (wrc) of the Beijing Light Industry Institute, proves the conjecture of Borwein and Edelstein [1983, 389] that if A and B are two finite, disjoint sets whose union spans projective $(m + n)$-space, then there is either an A-monochrome m-flat or a B-monochrome n-flat (an affine variety of dimension n spanned by points of B, that contains no points of A).

Rzymowski and Stachura (1986) show that the circle $|z| < rv/(v - 1)$ is the domain of largest area which can be guarded by a defender with destruction radius r and maximum speed 1 against an invader with maximum speed $v > 1$. See Thews [1984, 416].

Roger Nelsen (1987) affirmatively answers Walter Piegorsch's question [1984, 562]: can we generate a bivariate Poisson distribution with a negative correlation? He constructs a bivariate probability function with arbitrary marginal distributions (which need not be members of the same family) and *any* required correlation between the theoretical minimum and maximum values. His techniques are elementary and can be adapted to simulation studies requiring samples from discrete bivariate distributions.

The note of Broadie and Cottle (1984) on the simplexity of the 5-cube [1984, 628] has appeared.

Forcade and Pollington (wrc), using computer time at Bellcore, NJ, found the counterexample 195 to their conjecture with Lamoreaux [1986, 119], and suspect that 255 may be another. These have three distinct odd prime factors and make a good example of the Strong Law of Small Numbers (1988), parallel to the converse of Fermat's little theorem (Carmichael numbers) and to the size of the coefficients in cyclotomic polynomials.

Bob Guralnick (wrc) answers Feuer's question [1986, 120] negatively with the example, in $G = S_n (n \geqslant 6)$,

$$a = A = (12)(34) \qquad b = (34)(56) \qquad B = (13)(24)$$

so that $\langle A, B\rangle \cong \langle a, b\rangle$ is the Klein 4-group, and either $f(a, b) = f(A, B) = 1$ or $f(a, b)$, $f(A, B)$ are products of two transpositions, and hence are conjugate in G. But $\langle A, B\rangle$, $\langle a, b\rangle$ are not conjugate in G since they do not have the same orbit sizes. He believes that this has been known since the turn of the century, and that variations on the problem have applications to number theory and geometry, as well as group theory.

There have been several small rumblings concerning Hofstadter's sequence [1986, 186], but they are mainly variations on a theme, rather than answers to the original problem. Thanks to John Robertson for noting the misprints in two of the tables: for $k = 7, 8$; $Q(2^k + 1) = 63$, 143; $Q(3.2^k + 1) = 135$, 278; $f_k = 9, -8$ and $\Delta f_k (= e_k?) = -1, -17$. We look forward to a paper of Golomb (1988).

For the Mahler-Popken problem [1986, 188]: find the least number, $f(n)$, of ones needed to represent n, using only $+$ and $\times$ (and parentheses), Isbell and Myerson suggested that there might be examples of shape $n = (3x + 1)^2 + 6$ which required $f(n)$ to be as large as $2(f(x) + 4) + 5 = 2f(x) + 13$. However, John Selfridge (wrc) notes that

$$\begin{aligned} f(n) &= f(3(x(3x + 2) + 2) + 1) \\ &\leqslant 3 + \big(f(x) + (3 + f(x) + 2 + 2)\big) + 1 = 2f(x) + 11. \end{aligned}$$

He also gives a negative answer to the second question on p. 189, by observing that $f(2^7) = 14 = f(3^3 5)$, while $3^3 5$ is greater than 2^7, but not of the form $2^{7-3c}3^{2c}$.

Craig Bailey, Joel Brenner, Kārlis Čerāns (1987), Miklós Laczkovich, François Sigrist, as well as Stanley Rabinowitz and Jack Arrow, all send proofs of Grometstein's inequality [1986, 279].

$$yx^y\{y^x - (y-1)^x\} - xy^x\{x^y - (x-1)^y\} > 0$$

for all real $x > y > 1$ (not $y \geqslant 1$ as originally printed).

Z. Z. Uoiea of Grouse Creek, UT, and a score of others, from as far afield as Novosibirsk, observed that the characteristic function of a single point, say $f(x) =$

$0(x \neq 0)$, $f(0) = 1$, is a counterexample to Funar's conjecture [1986, 280]. However, many went on to show how nearly it was true. Dan Velleman, of Amherst, proved that if $f\colon \mathbb{R} \to \mathbb{R}$ can be written as the sum of a bijection and an injection, then it can be written as the sum of two bijections, and asked what can be said if we don't assume the axiom of choice. Eric Milner, of Calgary, proved that if S is an infinite abelian group, then any map $f\colon S \to S$ whose range $f[S]$ has the same cardinality as S is expressible as the sum of a bijection and an injection. Arnold Miller, of Wisconsin, showed that the conjecture is true if we ignore at most finitely many values of x.

Carl Ponder [1986, 280] asked for the asymptotic behavior of $\varphi_h(1)$, where $\varphi_h(x)$ is the (polynomial of degree $2^h - 1$) solution of the differential equation

$$\frac{d}{dx}\varphi_h(x) = \{\varphi_{h-1}(x)\}^2$$

with boundary conditions $\varphi_0(x) = \varphi_h(0) = 1$. James B. Shearer (wrc) obtains the quite good bounds

$$c_1\lambda_1^h < \varphi_h(1) < c_2\lambda_2^h$$

with $\lambda_1 = \max_\alpha((\alpha+1)/2)^{1/\alpha} \approx 1.26107$, $\lambda_2 = 2\ln 2 \approx 1.38629$, $c_1 = 1$ and $c_2 = 2.12$. His methods can be used to improve the upper bound slightly, and to prove the existence of the limit of $\{\varphi_h(1)\}^{1/h}$.

Ih-Ching Hsu [1986, 371] asked for the general solution of the functional equation

$$F(x, y) + F(\varphi(x), \psi(y)) = F(x, \psi(y)) + F(\varphi(x), y) \tag{1}$$

(a) where φ, ψ are given functions, (b) where φ, ψ are given involutions, and (c) where $\varphi(t) = \psi(t) = 1 - t$. This generated correspondence from Roger Howe, Yale; John Snygg, Upsala College; and Dan Velleman, Amherst; and papers from Kouong Law, Longwood College; John S. Lew, IBM, Yorktown Heights; Richard Rice, Seattle; and Mario Taboada, Minnesota.

Taboada answered (c) with $F(x, y) = A(x - 1/2, y - 1/2) + B(x, y - 1/2)$ where A is even in x and odd in y, and B is even in y. Velleman gave a similar solution and added conditions for the solution to be continuous, and to be differentiable, and noted that his method could be used to solve (b). This was done by Snygg. A version of (c) with $\varphi(t) = a - t$, $\psi(t) = b - t$ was solved by Kouong Law with a power series in $x - a/2$, $y - b/2$.

Howe showed that any solution had the form $F_1 + F_2$, where $F_1(x, y) = F_1(\varphi(x), y)$ and $F_2(x, y) = F_2(x, \psi(y))$; that if φ, ψ were involutions, or, more generally, of finite order, then the problem could be analyzed in terms of the representation theory of finite groups; that the solution was purely set-theoretic and didn't concern the topological structure of $\mathbb{R}$, so the general solution might well be discontinuous; that there would be situations in which the general *continuous* solution had the form

$$F(x, y) = f(x) + g(y) \tag{2}$$

noted by Hsu; and that there was a connexion with D'Alembert's solution of the wave equation in one dimension.

Rice used the same equivalence relation as Lew (see next paragraph), calling the classes **φ-orbits**. Then, given φ, K, the equation $F(\varphi(x), y) = F(x, y) + K(x, y)$ has a solution just if K sums to zero on cycles of φ-orbits. He also gave the general solution to (1).

Lew gave the most complete treatment. If φ: $X \to X$ is an arbitrary mapping on an arbitrary set, define the equivalence relation E, on X, by $x_1 E x_2$ just if $\varphi^m(x_1) = \varphi^n(x_2)$ for some integers $m, n \geqslant 0$. Let C, C' be φ-orbits of X and D, D' be ψ-orbits of Y, then the F-values on $C \times D$ do not affect the F-values on any disjoint $C' \times D'$. If $r(C)$, $s(D)$ are representative points of C, D, then you can assign F-values arbitrarily on every set $[C \times s(D)] \cup [r(C) \times D]$, and extend this, by (1), to a solution. Every solution of (1) has this form. Here, use of r, s assumes the axiom of choice, but in particular cases (e.g., the solutions of Kouong Law, Taboada, and Velleman) the representatives can be chosen constructively. Lew gave conditions under which (2) is the most general *continuous* solution. Examples are (1) $X = Y =$ the complex numbers, with φ, ψ non-linear polynomials, or $az + b$ with $|a| \neq 1$; (2) $X = Y =$ the Riemann sphere, with φ, ψ rational functions whose numerator and denominator are not both linear polynomials; (3) $X = Y =$ the reals, with φ, ψ continuous functions such that the fixed points of $\varphi(\varphi(x))$ and $\psi(\psi(y))$ are nonempty *countable* sets.

Myerson [1986, 457] asked how small a sum of five Nth roots of unity can be. Dean Hickerson (wrc) finds infinitely many N for which $f(5, N) < 8\pi/\sqrt{5}N^2$, and shows that for all sufficiently large N, $f(5, N) \leqslant 8\pi N^{-4/3}$.

Two correspondents suggest that David Dowe's question [1986, 627] "Are Maxwell's equations logically consistent?" makes little sense. They observe that it is equivalent to the question "Is mathematics, say *ZFC*, consistent?" and that there is no "weak" or "strong" notion of consistency in mathematics, nor in applied mathematics. I apologize that, in rewording the question, I may have misinterpreted the referee's remarks. Dowe himself adds the question "When, and in what sense, can a physical theory be said to be logically complete?" Meanwhile, I recall a classical article of Wightman (1976), on Hilbert's sixth problem, which contains 136 references, at least a few of which may be relevant.

Pambuccian [1986, 627] defined $a(n)$ to be the smallest integer a for which there is an integer b, $0 < b < a$, $(a, b) = 1$, with all members of the arithmetic progression $a + b, 2a + b, \ldots, na + b$ composite. He conjectured that $a(n)$ was always prime, but Erdős thought not. No surprise that Erdős was right and I was wrong, though I was right to suggest that a computer might settle our differences. Andy Odlyzko (wrc) make some of the earliest and most extensive calculations, among several others, and exhibited $a(135) = 8207 = 29 \times 283$, with $b = 3251$; and $a(150) = 12311 = 13 \times 947$, with $b = 6779$.

Noam Elkies (and independently John Leech and Ian Macdonald) noted that there are generally 2^n spheres touching all $n + 1$ hyperfaces of an n-dimensional simplex, not $n + 2$ as stated in Hatada's problem [1986, 628]. In the regular, 3-dimensional case, three are at infinity. The intended $n+1$ spheres are those which touch n hyperplanes on the same side as the insphere does, and one hyperplane on the opposite side. Hatada let $f(n)$ be the minimum ratio of the content of the simplex formed by these $n+1$ excentres, to the content of the original simplex, and conjectured that $f(n) = 2^n/(n-1)^n$ for $n \geqslant 2$, and that this minimum is attained just when the simplex is regular. Elkies and Macdonald each show that $f(n)$ can be made as small as you like for $n \geqslant 3$, and that Hatada's value is realized for $n = 3$ for just those tetrahedra whose faces split into two pairs with equal area-sum.

Bencsath and Mezei (wrc) relate Corley's problem [1986, 628] to the "hard spheres" problem of statistical mechanics, which leads to work on simulation and approximation. See Barker and Henderson (1976) and Mezei and Beveridge (1986). They later sent an extended bibliography, available from the present writer.

George Andrews [1986, 708], in studying Ramanujan's "lost" notebook, came across the striking q-series

$$1 + \sum_{n=1}^{\infty} \frac{q^{n(n+1)/2}}{(1+q)(1+q^2)\cdots(1+q^n)} = \sum_{n=0}^{\infty} S(n)q^n$$
$$= 1 + q - q^2 + 2q^3 + \cdots + 4q^{45} + \cdots + 6q^{1609} + \cdots + 8q^{3288} + \cdots$$

about which he made two conjectures:

1. $\limsup |S(n)| = +\infty$?
2. $S(n) = 0$ for infinitely many n?

Dean Hickerson studied a table of values of $S(n)$ and conjectured a "closed form" for $S(n)$ which, if valid, would imply both conjectures. Here are his observations:

Consider the diophantine equation

$$x^2 - 6y^2 = m \tag{3}$$

for positive or negative integers $m \equiv 1 \pmod{24}$ and call two solutions (x, y) and (x', y') **equivalent** if

$$x + y\sqrt{6} = \pm(5 + 2\sqrt{6})^r(x' + y'\sqrt{6})$$

for some integer r. It's easy to show, by induction on $|r|$, that if (x, y) and (x', y') are equivalent, then $x + 3y \equiv \pm(x' + 3y') \pmod{12}$.

Let $T(m)$ be the excess of the number of inequivalent solutions of (3) with $x + 3y \equiv \pm 1 \pmod{12}$ over the number with $x + 3y \equiv \pm 5 \pmod{12}$. Then Hickerson conjectured that

$$¿\, S(n) = T(24n+1)\,?$$

Andrews and Hickerson (tbp) have since proved this conjecture, which raises some further questions:

Is there a partition-theoretic interpretation of $T(m)$ for $m < 0$? Freeman Dyson, at the Ramanujan Centenary Conference, conjectured what it was, and Andrews and Hickerson proved it.

Are there similar expressions in other quadratic fields? Hickerson has found $T(m)$ in terms of $Q(\sqrt{2})$ and $Q(\sqrt{3})$.

Ramanujan would have liked this.

Jim Lawrence (wrc) notes that Shapiro's conjecture [1987, 46] on polyhedral cones is Theorem 8.9(b) of McMullen and Schneider (1983). In collaboration with Jon Spingarn he also gives a quite short direct proof.

My indebtedness to numerous correspondents is obvious. So too, is the usefulness of a clearinghouse for information on problems which are not always quite as unsolved as we thought.

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Edgar Allan Poe on Probability

Nothing, for example, is more difficult than to convince the merely general reader that the fact of sixes having been thrown twice in succession by a player at dice, is sufficient cause for betting the largest odds that sixes will not be thrown in the third attempt. A suggestion to this effect is usually rejected by the intellect at once. It does not appear that the two throws which have been completed, and which lie now absolutely in the Past, can have influence upon the throw which exists only in the Future. The chance for throwing sixes seems to be precisely as it was at any ordinary time—that is to say, subject only to the influence of the various other throws which may be made by the dice. And this is a reflection which appears so exceedingly obvious that attempts to controvert it are received more frequently with a derisive smile than with anything like respectful attention. The error here involved—a gross error redolent of mischief—I cannot pretend to expose within the limits assigned me at present; and with the philosophical it needs no exposure. It may be sufficient here to say that it forms one of an infinite series of mistakes which arise in the path of Reason through her propensity for seeking truth *in detail*.

The Mystery of Marie Roget

[Contributed by Gerald Weinstein of City University of New York]