# Countable subgroups of Euclidean space

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In his paper [1], Konstantinos Beros proved a number of results about compactly generated subgroups of Polish groups. Such a group is $K_\sigma$, the countable union of compact sets. He notes that the group of rationals under addition with the discrete topology is an example of a Polish group which is $K_\sigma$ (since it is countable) but not compactly generated (since compact subsets are finite).

Beros showed that for any Polish group $G$, every $K_\sigma$ subgroup of $G$ is compactly generated iff every countable subgroup of $G$ is compactly generated. He showed that any countable subgroup of $\mathbb{Z}^\omega$ (infinite product of the integers) is compactly generated and more generally, for any Polish group $G$, if every countable subgroup of $G$ is finitely generated, then every countable subgroup of $G^\omega$ is compactly generated.

In unpublished work Beros asked the question of whether "finitely generated" may be replaced by "compactly generated" in this last result. He conjectured that the reals $\mathbb{R}$ under addition might be an example such that every countable subgroup of $\mathbb{R}$ is compactly generated but not every countable subgroup of $\mathbb{R}^\omega$ is compactly generated. We prove (Theorem 4) that this is not true. The general question remains open.

In the course of our proof we came up with some interesting countable subgroups. We show (Theorem 6) that there is a dense subgroup of the plane which meets every line in a discrete set. Furthermore for each $n$ there is a dense subgroup of Euclidean space $\mathbb{R}^n$ which meets every $n-1$ dimensional subspace in a discrete set and a dense subgroup of $\mathbb{R}^\omega$ which meets every finite dimensional subspace of $\mathbb{R}^\omega$ in a discrete set.

**Theorem 1** *Every countable subgroup $G$ of $\mathbb{R}$ is compactly generated.*

Proof
If $G$ has a smallest positive element, then this generates $G$. Otherwise let $x_n \in G$ be positive and converge to zero. Let $G = \{g_n \ : \ n < \omega\}$. For each $n$ choose $k_n \in \mathbb{Z}$ so that $|g_n - k_n x_n| \leq x_n$. Let

$$C = \{0\} \cup \{x_n,\ g_n - k_n x_n \ : \ n < \omega\}$$

then $C$ is a sequence converging to zero, so it is compact. Also

$$g_n = (g_n - k_n x_n) + k_n x_n$$

so it generates $G$.
QED

**Theorem 2** *For $0 < m < \omega$ every countable subgroup $G$ of $\mathbb{R}^m$ is compactly generated.*

Proof
For any $\epsilon > 0$ let

$$V_\epsilon = \mathrm{span}_{\mathbb{R}}(\{u \in G \ : \ ||u|| < \epsilon\})$$

where here the span is taken respect to the field $\mathbb{R}$. Note that for $0 < \epsilon_1 < \epsilon_2$ that $V_{\epsilon_1} \subseteq V_{\epsilon_2}$. And since they are finite dimensional vector spaces there exists $\epsilon_0 > 0$ such that $V_\epsilon = V_{\epsilon_0}$ whenever $0 < \epsilon < \epsilon_0$. Let $V_0 = V_{\epsilon_0}$. Let $\epsilon_0 > \epsilon_1 > \cdots$ descend to zero. For each $n$ let $B_n \subseteq \{u \in G \ : \ ||u|| < \epsilon_n\}$ be a basis for $V_{\epsilon_n} = V_0$. Then $|B_n| = \dim(V_0)$ and the sequence $(B_n)_{n<\omega}$ "converges" to the zero vector $\vec{0}$. Hence $\{\vec{0}\} \cup \bigcup_{n<\omega} B_n$ is a compact subset of $G$. Let

$$G_0 = \mathrm{span}_{\mathbb{Z}}(\bigcup_{n<\omega} B_n)$$

where $\mathrm{span}_{\mathbb{Z}}(X)$ is the set of all linear combinations from $X$ with coefficients in $\mathbb{Z}$ or equivalently the group generated by $X$.

**Claim.** $G_0$ is dense in $V_0$.
Proof
Let $|B_n| = k_0 = \dim(V_0)$. Suppose $v \in V_0$ and we are given $\epsilon_n$. Then $v = \sum_{u \in B_n} \alpha_u u$ for some $\alpha_n \in \mathbb{R}$. Choose $n_u \in \mathbb{Z}$ with $n_u \leq \alpha_u < n_u + 1$. Let $w = \sum_{u \in B_n} n_u u_n$ Then

$$||v - w|| = || \sum_{u \in B_n} (\alpha_u - n_u) u|| \leq \sum_{u \in B_n} |\alpha_u - n_u| \cdot ||u|| \leq |B_n| \epsilon_n = k_0 \epsilon_n$$

Since $k_0 \epsilon_n \to 0$ as $n \to \infty$ we are done.
QED

Next we tackle the "discrete part" of $G$.

**Claim.** There exists a finite $F \subseteq G$ such that $G \subseteq \text{span}_{\mathbb{Z}}(F) + V_0$.
Proof
First note that for any $u, v \in G$, if $||u - v|| < \epsilon_0$, then $u - v \in V_0$ and hence $u + V_0 = v + V_0$. Now let $F_0 \subseteq G$ be finite so that

$$\text{span}_{\mathbb{R}}(F_0) = \text{span}_{\mathbb{R}}(G)$$

As we saw above for any $u \in G$ there exists $v \in \text{span}_{\mathbb{Z}}(F_0)$ such that

$$||u - v|| \leq N_0 =^{def} \sum_{u \in F_0} ||u||$$

If no such $F$ exists then we may construct an infinite sequence $u_n \in G$ such that

$$u_n \notin \text{span}_{\mathbb{Z}}(F_0 \cup \{u_i \ : \ i < n\}) + V_0 \text{ and } ||u_n|| \leq N_0.$$

But by compactness of the closed ball of radius $N_0$ there would have to be $i < j$ with $||u_j - u_i|| < \epsilon_0$ and hence $u_j \in u_i + V_0$ which is a contradiction.
QED

Now let

$$C = \{\vec{0}\} \cup F \cup \bigcup_{n<\omega} B_n$$

and note that $C$ is a compact subset of $G$ and $\text{span}_{\mathbb{Z}}(C)$ is dense in $G$. To finish we can repeat the argument for $\mathbb{R}$:

Let $G = \{g_n \ : \ n < \omega\}$. Choose $u_n \in \text{span}_{\mathbb{Z}}(C)$ with $||g_n - u_n|| < \frac{1}{2^n}$. Then $C \cup \{g_n - u_n \ : \ n < \omega\}$ is compact and generates $G$.
QED

**Question 3** *Do there exist Polish groups* $\Gamma_1, \Gamma_2$ *such that both have the property that every countable subgroup is compactly generated but the same is not true for* $\Gamma_1 \times \Gamma_2$*?*

Let $S$ be the real line with the Sorgenfrey topology, i.e., half-open intervals $[x, y)$ are basic open sets. Then $(S, +)$ is not a topological group; addition is continuous but not subtraction. However countable subgroups are compactly generated since a descending sequence with its limit point is compact. However $\{(x, -x) : x \in S\}$ is discrete in $S \times S$, so the subgroup, so $\{(x, -x) : x \in \mathbb{Q}\}$ is not compactly generated.

**Theorem 4** *Every countable subgroup $G \subseteq \mathbb{R}^\omega$ is compactly generated.*

Proof

Define $||x||_N = \sup\{|x(i)| : i < N\}$. So this is just the sup-norm but restricted to the first $N$-coordinates, i.e., $||x||_N = ||x \restriction N||$. An open neighborhood basis for $x$ is given by the sets $\{y : ||y - x||_N < \epsilon\}$ for $N < \omega$ and reals $\epsilon > 0$. Similarly we put

$$G \restriction n =^{def} \{u \restriction n \;:\; u \in\}.$$

For any $n$ by applying the proof of Theorem 2 we may find $\epsilon_n > 0$ such that if

$$V_n =^{def} \mathrm{span}_{\mathbb{R}}(\{u \restriction n \;:\; u \in G \text{ and } ||u||_n < \epsilon_n\}) \subseteq \mathbb{R}^n$$

then for any $\delta$ with $0 < \delta \leq \epsilon_n$

$$V_n = \mathrm{span}_{\mathbb{R}}(\{u \restriction n \;:\; u \in G \text{ and } ||u||_n < \delta\}).$$

Without loss we may assume that $\epsilon_n < \frac{1}{2^n}$. Take $B_n \subseteq G$ so that

$$|B_n| = \dim(V_n) \leq n \text{ and } V_n = \mathrm{span}_{\mathbb{R}}(\{u \restriction n \;:\; u \in B_n\}).$$

By the argument of the first claim of Theorem 2, we have that for every $u$ with $u \restriction n \in V_n$ there exists $v \in \mathrm{span}_{\mathbb{Z}}(B_n)$ with

$$||u - v||_n < n \cdot \epsilon_n < \frac{n}{2^n}$$

By the argument of the second claim there exists a finite $F_n \subseteq G$ such that

$$G \restriction n \subseteq \mathrm{span}_{\mathbb{Z}}(F_n \restriction n) + V_n.$$

Hence for every $u \in G$ there exists $v \in \mathrm{span}_{\mathbb{Z}}(F_n \cup B_n)$ with $||u - v||_n < \frac{n}{2^n}$.

Apply this to each $u \in F_{n+1}$. So we may choose $\hat{u} \in \mathrm{span}_{\mathbb{Z}}(F_n \cup B_n)$ with $||u - \hat{u}||_n < \frac{n}{2^n}$. Put

$$H_{n+1} =^{def} \{(u - \hat{u}) \;:\; u \in F_{n+1}\}$$

and put

$$C = \{\vec{0}\} \cup \bigcup_{n<\omega} (B_n \cup H_n).$$

Then $C$ is compact, since for every $n$ if $u \in B_n$, then we have that $||u||_n < \frac{1}{2^n}$ and if $v \in H_{n+1}$, then $||v||_n < \frac{n}{2^n}$. By induction we have that $F_n \subseteq \text{span}_{\mathbb{Z}}(C)$ for each $n$, since if we assume $F_n \subseteq \text{span}_{\mathbb{Z}}(C)$, then for $u \in F_{n+1}$ we have that $\hat{u} \in \text{span}_{\mathbb{Z}}(C)$ and so $u = (u - \hat{u}) + \hat{u}$ is in $\text{span}_{\mathbb{Z}}(C)$. It follows that $\text{span}_{\mathbb{Z}}(C)$ is dense in $G$ since $\bigcup_{n<\omega}(B_n \cup F_n)$ is dense in $G$. By the argument used in Theorems 1 and 2 we get that $G$ is compactly generated.
QED

**Question 5** *For countable subgroups of $\mathbb{R}^\omega$ the generating compact set can always be taken to be a convergent sequence. We don't know if this is more generally true.*

The motivation for the following example was a doomed attempt to prove Theorem 2 for the plane by looking at one dimensional subspaces and considering multiple cases.

**Theorem 6** *There exists a dense subgroup $G \subseteq \mathbb{R}^2$ such that for every line $L$ in the plane, $G \cap L$ is discrete.*

Proof
Let $M$ be a countable transitive model of a large finite fragment of ZFC. Working in $M$ let $\mathbb{P}$ be a countable family of nonempty open subsets of the plane which is a basis for the topology. Forcing with $\mathbb{P}$ produces a generic point $p \in \mathbb{R}^2$.

The following facts are well-known.

1. If $p$ is $\mathbb{P}$-generic over $M$ then $p$ is not in any closed nowhere dense subset of $\mathbb{R}^2$ coded in $M$.

2. If $(p, q)$ are $\mathbb{P}^2$-generic over $M$, then $p$ is $\mathbb{P}$-generic over $M$ and $q$ is $\mathbb{P}$-generic over $M[p]$.

3. If $(p, q)$ are $\mathbb{P}^2$-generic over $M$, then $p + q$ is $\mathbb{P}$-generic over $M$.

4. If $p$ is $\mathbb{P}$-generic over $M$ and $q \in M \cap \mathbb{R}^2$, then $p + q$ is $\mathbb{P}$-generic over $M$. And if $\alpha \in M \cap \mathbb{R}$ is nonzero, then $\alpha p$ is $\mathbb{P}$-generic over $M$.

Now let $\sum_{n<\omega}\mathbb{P}$ be the countable direct sum of copies of $\mathbb{P}$. Let $(p_n : n<\omega)$ be $\sum_{n<\omega}\mathbb{P}$-generic over $M$ and let $G = \text{span}_{\mathbb{Z}}(\{p_n : n<\omega\})$.

It is easy to see that the $p_i$ are algebraically independent vectors over the field of rationals (or even the field $M\cap\mathbb{R}$) so for any $u\in G$ there is a unique finite $F\subseteq\omega$ such that let $u=\sum_{i\in F} n_i p_i$ and each $n_i\neq 0$ for $i\in F$. Let $F=\text{supp}(u)$ (the support of $u$).

Suppose $L$ is any line in the plane thru the origin and let

$$L^+ = L\setminus\{\vec{0}\}.$$

Note that $L$ is determined by any element of $L^+$.

**Claim.** If $u,v\in L^+\cap G$, then $\text{supp}(u)=\text{supp}(v)$.
Proof
Let $\text{supp}(u)=F_0$ and $\text{supp}(v)=F_1$. Suppose for contradiction that $F_1\setminus F_0$ is nonempty. Let $i\in F_1\setminus F_0$ and put $F=F_1\setminus\{i\}$. Then

$$v = k_i p_i + \sum_{j\in F} k_j p_j.$$

It follows from the well-known facts above that $p_i$ is $\mathbb{P}$-generic over the model $N =^{def} M[(p_j : j\in F_0\cup F)]$. Also since $q =^{def} \sum_{j\in F} k_j p_j \in N$ and $k_i\in\mathbb{Z}$ we know that $v = k_i p_i + q$ is $\mathbb{P}$-generic over $N$. On the other hand the line $L$ is coded in $N$ since $u$ is in this model. But $L$ is a closed nowhere dense subset of the plane so by genericity $v\notin L$.
QED

Let $F_0$ be the common support of all $u\in L^+\cap G$. Fix any $i_0\in F_0$. Choose $u\in L^+\cap G$ such and $u=\sum_{i\in F_0} n_i p_i$ and $n_{i_0}>0$ is the minimal possible positive coefficient of $p_{i_0}$ and for any point in $L^+\cap G$.

**Claim.** $L\cap G = \{nu : n\in\mathbb{Z}\}$.
Proof
Let $v=\sum_{i\in F_0} m_i p_i$ be in $L^+\cap G$. Then $m_{i_0}$ must be divisible by $n_{i_0}$ otherwise $n_{i_0}$ would not have been minimal. Say $n\cdot n_{i_0} = m_{i_0}$ for some $n\in\mathbb{Z}$. Then note that $v-n\cdot u$ is in $L\cap G$ and its support is a subset of $F_0\setminus\{i_0\}$. Hence $v-n\cdot u=\vec{0}$ and we are done.
QED

Now suppose that $L$ is a line not containing $\vec{0}$. If $L \cap G$ is nonempty take $u_0 \in L \cap G$. Let $L_0 = -u_0 + L$. Since $L_0$ is line thru the origin there exists $u_1 \in G$ such that $G \cap L_0 = \{nu_1 \ : \ n \in \mathbb{Z}\}$. Then

$$G \cap L = \{u_0 + nu_1 \ : \ n \in \mathbb{Z}\}.$$

QED

**Remark 7** .

1. *Note that dense-in-itself is not the same as dense, for example,* $\mathbb{Z} \times \mathbb{Q}$ *is dense-in-itself but not dense in* $\mathbb{R}^2$.

2. *Finitely generated does not imply discrete. The additive subgroup of* $\mathbb{R}$ *generated by* $\{1, \sqrt{2}\}$ *is dense.*

3. *A circle* $C$ *is closed nowhere dense subset of the plane and determined by any three points, hence* $C \cap G \subseteq span_{\mathbb{Z}}(F)$ *for some finite* $F \subseteq G$. *Also the graph* $Q$ *of a polynomial of degree* $n$ *is determined by any* $n+1$ *points so there will be a finite* $F \subseteq G$ *with* $G \cap Q \subseteq span_{\mathbb{Z}}(F)$.

4. *One can get a group* $G \subseteq \mathbb{R}^2$ *which is continuum-dense and meets every line in a discrete set. To see this construct a sequence of perfect sets* $P_n$ *such that every nonempty open set contains one of them and such that for every finite* $\{p_i : i < n\} \subseteq \cup_n P_n$ *of distinct points the tuple* $(p_i : i < n)$ *is* $\mathbb{P}^n$*-generic over* $M$. *Then* $G = span_{\mathbb{Z}}(\cup_n P_n)$ *meets every line discretely.*

**Theorem 8** *There exists a dense subgroup* $G \subseteq \mathbb{R}^N$ *such that* $G \cap V$ *is discrete for every* $V \subseteq \mathbb{R}^N$ *a vector space over* $\mathbb{R}$ *of dimension* $N-1$.

Proof

In the model $M$ we let $\mathbb{P}$ be the nonempty open subsets of $\mathbb{R}^N$. Take $(p_i : i < \omega)$ be $\sum_{n<\omega} \mathbb{P}$-generic over $M$ and put $G = \text{span}_{\mathbb{Z}}(\{p_n : n < \omega\})$.

Let $\sum_{n<\omega} \mathbb{Z}$ be the sequences $x \in \mathbb{Z}^\omega$ such that $x(n) = 0$ except for finitely many $n$. Let $h : \sum_{n<\omega} \mathbb{Z} \to \mathbb{R}^N$ be the homomorphism defined by

$$h(x) = \sum_{n<\omega} x(n) p_n$$

and note that this is a finite sum, the coefficients are integers and the $p_n$ are generic vectors in $\mathbb{R}^N$.

Clearly the generic vectors are linearly independent over the rationals so the map $h$ is a group isomorphism $h : \sum_{n<\omega} \mathbb{Z} \to G$.

Let $C \subseteq \sum_{n<\omega} \mathbb{Z}$ be $h^{-1}(G \cap V)$. As in the proof of Theorem 4.2 Beros [1], let

$$C_n = \{x \in C \ : \ \forall i < n \ \ x(i) = 0\}.$$

If $C_{n+1}$ is a proper subgroup of $C_n$ choose $k_n \in \mathbb{Z}$ non-zero which divides every $y(n)$ for $y \in C_n$. Otherwise, if $C_{n+1} = C_n$ put $k_n = 0$. For each $n$ choose $x_n \in C_n$ with $x_n(n) = k_n$.

**Claim.** If $k_n \neq 0$, then $h(x_n) \notin \mathrm{span}_{\mathbb{R}}(\{h(x_i) : i > n\})$.
Proof
Suppose for contradiction that for some $m$

$$h(x_n) \in \mathrm{span}_{\mathbb{R}}(\{h(x_{n+i}) : i = 1 \dots m\}).$$

Let $V_0 = \mathrm{span}_{\mathbb{R}}(\{h(x_{n+i}) \ : \ i = 1 \dots m\})$ and note that $V_0 \subseteq V$ and so $\dim(V_0) \leq \dim(V) < N$. Hence $V_0$ is closed and nowhere dense in $\mathbb{R}^N$.

Now $\langle h(x_{n+i}) : i = 1 \dots m\rangle \in M[\langle p_i : i > n\rangle]$ and hence $V_0$ is a closed subset of $\mathbb{R}^N$ coded in $M[\langle p_i : i > n\rangle]$. But

$$h(x_n) = k_n p_n + \sum_{i>n} x_n(i) p_i$$

and so as we saw in the last proof $h(x_n)$ is $\mathbb{P}$-generic over $M[\langle p_i : i > n\rangle]$. But this means that it avoids the closed nowhere dense set $V_0$ coded in $M[\langle p_i : i > n\rangle]$. Contradiction.
QED

**Claim.** $|\{n : k_n \neq 0\}| \leq \dim(V)$.
Proof
Suppose not and let $Q \subseteq \{n : k_n \neq 0\}$ be finite with $|Q| > \dim(V)$. Then for each $n \in Q$ we have that

$$h(x_n) \notin \mathrm{span}_{\mathbb{R}}(\{h(x_m) \ : \ m \in Q \text{ and } m > n\}).$$

It follows that $\{h(x_n) : n \in Q\}$ is a linearly independent subset of $V$ of size greater than $\dim(V)$, which is a contradiction.

QED

Hence $Q =^{def} \{n : k_n \neq 0\}$ is finite. This means that

$$C = \text{span}_{\mathbb{Z}}(\{x_n : n \in Q\})$$

by a process resembling Gaussian elimination. Knowing that $Q$ is finite shows that the algorithm terminates.

It follows that $V \cap G = \text{span}_{\mathbb{Z}}(\{h(x_n) : n \in Q\})$ and $\{h(x_n) : n \in Q\}$ is a basis for $\text{span}_{\mathbb{R}}(V \cap G)$. Extend $\{h(x_n) : n \in Q\}$ to a basis $B$ for $\mathbb{R}^N$. Take a bijective linear transformation $j : \mathbb{R}^N \to \mathbb{R}^N$ determined by mapping the standard basis of $\mathbb{R}^N$ to $B$. It follows that $j(\mathbb{Z}^{|Q|} \times \{\vec{0}\}) = V \cap G$ and so $V \cap G$ is discrete in $\mathbb{R}^N$.
QED

**Remark 9** *For $\mathbb{R}^\omega$ using this technique we get a dense generically generated group $G \subseteq \mathbb{R}^\omega$ such that $V \cap G$ is discrete for every finite dimensional subspace $V \subseteq \mathbb{R}^\omega$.*

Although we are enamored with the generic sets argument of Theorem 8, it also possible to give an elementary inductive construction. We haven't verified it in general but we did check the proof for the plane. As long as we are at it, we might as well construct a subgroup of $\mathbb{R}^2$ which is maximal with respect to linear discreteness, although this will require a subgroup of cardinality the continuum.

**Theorem 10** *Let $\mathcal{L}$ be the family of all lines in the plane containing the origin. There exists a subgroup $G \subseteq \mathbb{R}^2$ such that $G \cap L$ is an infinite discrete set for every $L \in \mathcal{L}$.*

Proof
Define $G \subseteq \mathbb{R}^2$ is linearly discrete iff $G \cap L$ is discrete for every $L \in \mathcal{L}$.
Define $\mathcal{L}_G = \{L \in \mathcal{L} \ : \ |G \cap L| \geq 2\}$.

**Lemma 11** *For any linearly discrete subgroup $G \subseteq \mathbb{R}^2$ with $|G| < |\mathbb{R}|$ and any $L_0 \in \mathcal{L} \setminus \mathcal{L}_G$ there exists $p \in L_0 \setminus \{\vec{0}\}$ such that $G + \mathbb{Z}p$ (the group generated by $G$ and $p$) is linearly discrete and $(G+\mathbb{Z}p) \cap L = G \cap L$ for every $L \in \mathcal{L}_G$.*

Proof
Suppose $np + q \in L$ where $n \in \mathbb{Z}$ is nonzero, $L \in \mathcal{L}_G$, and $q \in G$. Then $p + \frac{q}{n} \in L$ since $L$ contains the origin. But the lines $L_0$ and $L - \frac{q}{n}$ meet in at most one point. It follows that at most $|G|$ many points of $L_0$ are ruled out so we may choose $p \in L_0 \setminus \{\vec{0}\}$ such $(G + \mathbb{Z}p) \cap L = G \cap L$ for every $L \in \mathcal{L}_G$.

This implies that $G + \mathbb{Z}p$ is linearly discrete by the following argument. We only need to worry about new lines $L \in \mathcal{L}$ which are not in $\mathcal{L}_G$. Suppose $np+q \in L$ where $L \in \mathcal{L} \setminus \mathcal{L}_G$, $n \in \mathbb{Z}$ nonzero, and $q \in G$. Choose such a point on $L$ with minimal $n > 0$. We claim that $L \cap (G + p\mathbb{Z}) = \{knp + kq\,;\ k \in \mathbb{Z}\}$.

Then for any $k \in \mathbb{Z}$ we have that $knp+kq \in L$. Any point on $L \cap (G+p\mathbb{Z})$ has the form Let $mp + q'$ be an arbitrary point of $p\mathbb{Z} + G$. If it is in $L$ it must be that $n$ divides $m$ otherwise $0 < m - kn < n$ for some $k \in \mathbb{Z}$ and

$$(mp + q') - (knp + kq) = (m - kn)p + (q' - kq) \in L$$

shows that $n$ was not minimal. Hence for some $k \in \mathbb{Z}$ we have that $kn = m$. It must be that $kq = q'$ since otherwise $q' - kq \in L$ is a nontrivial element of $G$ in $L$, contradicting $L \notin \mathcal{L}_G$.
QED

Using the Lemma it is an easy inductive construction of an and increasing sequence of subgroups $G_\alpha \subseteq \mathbb{R}^2$ for $\alpha < |\mathbb{R}|$ such that $|G_\alpha| = |\alpha + \omega|$ is linearly discrete and for each line $L \in \mathcal{L}$ there is an $\alpha$ with $|G_\alpha \cap L| \geq 2$.
QED

**Question 12** *For a line $L$ in the plane not containing the origin, if $G$ meets $L$ at all, then taking $u \in G \cap L$ and considering $L - u \in \mathcal{L}$ shows us that $G$ meets $L$ in an infinite discrete set. I don't know if we can construct such a $G$ which meets every line. I don't even know how to meet all of the horizontal lines.*

Jan Pachl points out that every locally compact compactly generated Abelian group is a product of a compact Abelian group and a finite number of copies of $\mathbb{R}$ and $\mathbb{Z}$ (see Hewitt and Ross [2] 9.8). Also, every locally compact Abelian group is a product of $\mathbb{R}^k$ and a locally compact subgroup that has a compact open subgroup (see Hofmann and Morris [3] 7.57). Every $K_\sigma$ topological group is a closed subgroup of a compactly generated group (see Pestov [4]).

## References


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