Notre Dame Journal of Formal Logic
Volume 24, Number 1, January 1983

On the Borel Classification of the Isomorphism Class of a Countable Model

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Introduction For ρ, a countable similarity type, let X_ρ be the space of structures of similarity type ρ whose universe is ω (see [13], Section 3). For any element $\mathcal{A}$ of X_ρ, let $[\mathcal{A}]$ be the set of all elements of X_ρ which are isomorphic to $\mathcal{A}$. Scott [10] showed that $[\mathcal{A}]$ is a Borel subset of X_ρ. In fact, he showed that for any such $\mathcal{A}$ there is a sentence θ of $L_{\omega_1\omega}$ such that $[\mathcal{A}]$ is exactly the set of elements of X_ρ which are models of θ (see [1], Ch. VII, for a good write-up of Scott sentences).

In [13] Vaught considerably strengthened Scott's result. There is a natural hierarchy of formulas of $L_{\omega_1\omega}$. Let $\Pi^0_0 = \Sigma^0_0$ be the quantifier-free first-order formulas. For any $\alpha \geqslant 1$ the Π^0_α formulas are those of the form:

$$\bigwedge_{n<\omega} \forall x_1 \forall x_2 \ldots \forall x_n \theta_n$$

where each θ_n is $\Sigma^0_{\beta_n}$ for some $\beta_n < \alpha$. The Σ^0_α formulas are those of the form:

$$\bigvee_{n<\omega} \exists x_1 \exists x_2 \ldots \exists x_n \theta_n$$

where each θ_n is $\Pi^0_{\beta_n}$ for some $\beta_n < \alpha$. A set $B \subseteq X_\rho$ is called invariant iff it is closed under isomorphism. Vaught showed that for every Π^0_α invariant set B there is a Π^0_α sentence θ such that B is the set of models of θ, and similarly for Σ^0_α.

*Research was partially supported by the NSF.

Received September 21, 1981; revised July 7, 1982

This result was extended by D. E. Miller [8] to the classical Hausdorff difference hierarchy. He proves that for any $\alpha \geqslant 1$ if B is an invariant $\Delta^0_{\alpha+1}$ set, then there exists $\beta < \omega_1$ and invariant decreasing Π^0_α sets C_δ for $\delta < \beta$ such that:

$$B = \cup \{C_\delta - C_{\delta+1}: \delta \text{ even} < \beta\}.$$

He also showed that if an invariant set is the difference of Π^0_α sets ($\Pi^0_\alpha \wedge \Sigma^0_\alpha$), then it is the difference of invariant Π^0_α sets.

Note that $[\mathcal{A}]$ is a minimal invariant set. From Miller's Theorems we see that if $[\mathcal{A}]$ is $\Delta^0_{\alpha+1}$, then it is the difference of two invariant Π^0_α sets. If it is not properly the difference of two Π^0_α sets, then it is the union of an invariant Π^0_α and an invariant Σ^0_α, and so it is either Π^0_α or Σ^0_α. If α is a limit ordinal and $[\mathcal{A}]$ is Σ^0_α, then it is Π^0_β for some $\beta < \alpha$. This follows immediately from Vaught's theorem by considering the form of a Σ^0_α sentence. For the same reason (for α limit) $[\mathcal{A}]$ cannot be properly the difference of two Π^0_α sets.

In Section 1 we show that the isomorphism class of a countable model in a countable similarity type containing no operation symbols cannot be properly Σ^0_2 (Σ^0_1 is also impossible). In Section 2 we show how Wadge games may be used to classify the Borel class of the isomorphism class of some common models. In Section 3 we calculate the Borel class of the isomorphism class of each countable ordinal. In Section 4 we give examples of isomorphism classes properly in each Borel class not ruled out by the results above except for $\Sigma^0_{\lambda+1}$ for λ an infinite limit ordinal. This case is open. In Section 5 we give an example of an $\aleph_0$-categorical theory whose (only) model has an isomorphism class which is properly Π^0_ω.

The theorems in the first four sections appeared in [6] and the result in Section 5 was announced in [7].

1 No isomorphism class is properly Σ^0_2 In this section we prove that if $\mathcal{A}$ is a model in a countable similarity type with no operation symbols and $[\mathcal{A}]$ is Σ^0_2 then it is Δ^0_2. D. E. Miller [8] has shown that no $[\mathcal{A}]$ is properly Σ^0_2 in the topology generated by first-order logic. However, I do not know how to deduce either result from the other.

Theorem 1 *No* $[\mathcal{A}]$ *is properly* Σ^0_2.

Proof: Suppose that $[\mathcal{A}]$ is Σ^0_2, then by Vaught's Theorem there is a Σ^0_2 sentence θ such that $[\mathcal{A}]$ is the set of models of θ. Note that θ has the form:

$$\bigvee_{n<\omega} \exists x_1 \exists x_2 \ldots \exists x_n \bigwedge_{m<\omega} \psi_{n,m}(x_1, x_2, \ldots, x_n),$$

where each $\psi_{n,m}$ is a universal first-order formula. Since $\mathcal{A}$ models one of the disjuncts, we can sssume that θ has the form:

$$\exists x_1 \exists x_2 \ldots \exists x_n \bigwedge_{m<\omega} \psi_m(x_1, x_2, \ldots, x_n).$$

Lemma 1.1 $\mathcal{A}$ *is saturated.*

Proof: Note that from the form of θ, if $\mathcal{B}$ is any (first-order) elementary extension of $\mathcal{A}$, then $\mathcal{B}$ models θ and therefore $\mathcal{B}$ is isomorphic to $\mathcal{A}$. From this it follows that every type in $Th(\mathcal{A})$ is realized in $\mathcal{A}$ (i.e., $\mathcal{A}$ is weakly saturated). Therefore there is a countable saturated model of $Th(\mathcal{A})$, and since $\mathcal{A}$ is an elementary substructure of it, we have that it is isomorphic to $\mathcal{A}$.

Remark: In fact, it is not hard to show (see [8], Section 3) that $Th(\mathcal{A})$ is $\aleph_0$-categorical.

Next I will show that we may assume, without loss of generality, some simplifications in the properties of the ψ_m.

Lemma 1.2 *There exists $\psi_m(\vec{x})$ such that:*

(1) *$\psi_m(x_1, x_2, \ldots, x_n)$ are universal first-order formulas with only $x_1, x_2, \ldots, x_n$ free,*
(2) *$[\mathcal{A}]$ is the set of models of $\exists \vec{x} \bigwedge_{m<\omega} \psi_m(\vec{x})$,*
(3) *for any m, $\psi_{m+1}(\vec{x}) \rightarrow \psi_m(\vec{x})$,*
(4) *for any m, $\psi_m(\vec{x}) \rightarrow \bigwedge_{i \neq j}(x_i \neq x_j)$, (i.e., irreflexive)*
(5) *for any m and permutation σ of $\{1, 2, \ldots, n\}$,*

$$\psi_m(x_1, x_2, \ldots, x_n) \rightarrow \psi_m(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$$

(i.e., symmetric).

Proof: To obtain (3) just replace the ψ_m by $\bigwedge_{i<m} \psi_i$. To get (4) just look at the cardinality of a witness in $\mathcal{A}$. To get (5) replace $\psi_m(x_1, x_2, \ldots, x_n)$ by

$$\bigvee_{\sigma} \psi_m(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$$

where the disjunct is taken over all permutations of $\{1, 2, \ldots, n\}$. Note that (2) is retained since by (3) it is enough to satisfy infinitely many ψ_m, so the same σ must be used infinitely often.

Let $\psi(\vec{x})$ be the infinite conjunction of the $\psi_m(\vec{x})$. Let $D \in [A]^n$ be arbitrary such that $\mathcal{A} \models \psi(D)$. Since $\mathcal{A}$ is saturated we can find $B \subseteq A - D$ a set of D-indiscernibles of order type the rationals. By D-indiscernible I mean that they remain indiscernible even if constant symbols are added for elements of D. Consider for any $A_0 \subseteq D$, $\mathcal{A} \upharpoonright (A_0 \cup B)$ (i.e., the substructure of $\mathcal{A}$ with universe $A_0 \cup B$). In this model B is certainly still A_0-indiscernible with respect to quantifier-free formulas. In fact, B is A_0-indiscernible in $\mathcal{A} \upharpoonright (A_0 \cup B)$ even with respect to universal formulas. This is because any witness for an existential formula is from $A_0 \cup B$ and B has order type the rationals. Choose $A_0 \subseteq D$ of minimal cardinality, say $n_0 \leqslant n$, such that for all $X \in [B]^{n-n_0}$, $\mathcal{A} \upharpoonright (A_0 \cup B) \models \psi(A_0 \cup X)$. Clearly $\mathcal{A} \upharpoonright (A_0 \cup B)$ is isomorphic to $\mathcal{A}$, so we may assume $A_0 \cup B = A$.

Lemma 1.3 *For every $A_1 \in [A]^{n_0}$, $A_0 = A_1$ iff $\forall X \in [A - A_1]^{n-n_0}$ $\mathcal{A} \models \psi(A_1 \cup X)$.*

Proof: Left to right follows by the definition of A_0. To prove right to left, suppose $A_0 \neq A_1$ and let $A_2 = A_0 \cap A_1$, and let the cardinality of A_2 be $n_2 < n_0$. Since B is A_0-indiscernible with respect to universal formula it is easy to show

$\forall X \in [B]^{n-n_2}$ $\mathfrak{A} \models \psi(A_2 \cup X)$. Since universals go down, the same is true in $\mathfrak{A} \restriction (A_2 \cup B)$ which contradicts the minimality of A_0.

Lemma 1.4 *Suppose f is any bijection from A to A which is the identity on A_0, then f is an automorphism of $\mathfrak{A}$.*

Proof: Recall that $A = A_0 \cup B$ where the elements of B are A_0-indiscernible with respect to universal formula. I claim that B is *totally* A_0-indiscernible with respect to quantifier-free formulas. Suppose ρ is quantifier free with some parameters from A_0.

Claim *If $\vec{b} < c_1 < c_2 < \vec{d}$ are from B, then*

$$\models \rho(\vec{b}, c_1, c_2, \vec{d}) \longleftrightarrow \rho(\vec{b}, c_2, c_1, \vec{d}).$$

To prove the claim, suppose

$$\models \rho(\vec{b}, c_1, c_2, \vec{d}) \wedge \neg \rho(\vec{b}, c_2, c_1, \vec{d}).$$

Choose $C = \{c_n : n < \omega\} \subseteq B$ such that $\vec{b} < c_n < c_{n+1} < \vec{d}$. Now consider the model $\mathfrak{A} \restriction (A_0 \cup \{\vec{b}, \vec{d}\} \cup C)$. This model is isomorphic to $\mathfrak{A}$. But $\rho(\vec{b}, x, y, \vec{d})$ defines a linear order of order type ω on a cofinite subset of the universe, which contradicts the fact that $\mathfrak{A}$ is saturated.

Suppose $b_1 < b_2 < b_3 < \ldots < b_m$ are elements of B and let P be all permutations σ of $\{1, 2, \ldots, m\}$ such that for all $\rho(x_1, x_2, \ldots, x_m)$ quantifier free with parameters from A_0,

$$\mathfrak{A} \models \rho(b_1, b_2, \ldots, b_m) \longleftrightarrow \rho(b_{\sigma(1)}, b_{\sigma(2)}, \ldots, b_{\sigma(m)}).$$

Note that P is closed under composition and by the claim contains all two cycles. It follows that P is the set of all permutations. The total A_0-indiscernibility of B implies the Lemma.

Note that Lemma 1.3 implies that A_0 is definable ($L_{\omega_1\omega}$) in $\mathfrak{A}$. Next we simplify this. For any $k < \omega$, let $\tau_k(x_1, x_2, \ldots, x_{n_0})$ say that $\vec{x}$ are distinct and for $\vec{y}$ a sequence of $n - n_0$ distinct elements disjoint from $\vec{x}$, $\psi_k(\vec{x} \cup \vec{y})$. Note that from Lemma 1.3 we have that for all $K \in [A]^{n_0}$, $K = A_0$ iff $\mathfrak{A} \models \bigwedge_{k<\omega} \tau_k(K)$.

Lemma 1.5 There exists $N < \omega$ such that for all $H \in [A]^N$ and for all $K \in [H]^{n_0}$:

$$K = A_0 \text{ iff } \mathfrak{A} \restriction H \models \text{“}\tau_N(K)\text{”}.$$

Proof: Consider the first-order theory T consisting of

(a) for each $k < \omega$ “there are at least k elements”,
(b) for each $k < \omega$, $\tau_k(b_1, b_2, \ldots, b_{n_0})$,
(c) for each $k < \omega$, $\tau_k(c_1, c_2, \ldots, c_{n_0})$, and
(d) $\{b_1, b_2, \ldots, b_{n_0}\} \neq \{c_1, c_2, \ldots, c_{n_0}\}$.

Any countable model of T will be isomorphic to $\mathfrak{A}$ (when reduced to the language without the new constant symbols c_i and b_i for $i = 1, 2, \ldots, n_0$). Since A_0 is uniquely defined, T is inconsistent. Hence for all sufficiently large $N < \omega$ for all $H \in [A]^N$ there is at most one $K \in [H]^{n_0}$ such that $\mathfrak{A} \restriction H \models$ “$\tau_N(K)$”.

Clearly, since τ_N is universal, if $A_0 \subseteq H$, then $\mathcal{A} \upharpoonright H \models$ "$\tau_N(A_0)$". Now suppose $\mathcal{A} \upharpoonright H \models$ "$\tau_N(K)$" and $K \neq A_0$. Therefore, part of K lies in B, thus if $N > 2n_0$, we can find a bijection f from A into A which fixes A_0, sends H into H, and such that $f(K) \neq K$. By Lemma 1.4 f is an automorphism of $\mathcal{A}$ and thus an automorphism of $\mathcal{A} \upharpoonright H$. But then

$$\mathcal{A} \upharpoonright H \models \text{"}\tau_N(K) \wedge \tau_N(f(K))\text{"},$$

a contradiction. This proves Lemma 1.5.

Let $\theta(H, K)$ be the quantifier-free formula which says that $H \in [A]^N$, $K \in [H]^{n_0}$, and

$$\mathcal{A} \upharpoonright H \models \text{"}\tau_N(K)\text{"}.$$

Then the conjunction of the Σ_1^0 sentence:

$$\exists H \exists K\, \theta(H, K)$$

and the Π_1^0 sentence:

$$\forall H \forall K(\theta(H, K) \rightarrow \bigwedge\!\!\!\bigwedge_{m<\omega} \tau_m(K))$$

characterizes the isomorphism class of $\mathcal{A}$. Thus $[\mathcal{A}]$ is the difference of two Π_1^0 sets.

Remark: I have also been able to show that proper Σ_2^0 isomorphism classes are impossible in the language which consists of a single unary operation symbol. The most general case is open. In a language without operation symbols every consistent Σ_2^0 sentence has finite models. This, of course, is not true if the language contains operation symbols, e.g., "f is one-to-one and not onto". However it is easily shown that a counterexample (in the general case) must have finite models.

*2 **Using Wadge games*** In this section we show how to calculate the Borel class of the isomorphism class of some common models using Wadge games.

If X and Y are topological spaces and $A \subseteq X$, $B \subseteq Y$, we say that $A \leqslant_W B$ iff there exists a continuous map f: $X \rightarrow Y$ such that $f^{-1}(B) = A$. Wadge noted that in common spaces such as ω^ω, 2^ω, etc., the map f could be described very conveniently as the winning strategy of a particular two-person infinite game. For a good reference to Wadge games, see Van Wesep [14]. For simplicity let ρ be a finite similarity type with relation symbols only. Let $A \subseteq \omega^\omega$ and $B \subseteq X_\rho$ and consider the following game $G(A, B)$. Player I and player II alternate and make infinitely many moves. On the k^{th} play, player I plays some $n_k \in \omega$ and player II plays some $\mathcal{A}_k$ with universe a finite subset of ω and such that $\mathcal{A}_{k-1}$ is a substructure of $\mathcal{A}_k$. We demand from player II that $\omega = \cup \{|\mathcal{A}_k|: k < \omega\}$. At the end of infinitely many plays, player I has written down $f = (n_k: k < \omega) \in \omega^\omega$, and player II has written down $\mathcal{A} = \bigcup_{n<\omega} \mathcal{A}_n \in X_\rho$. Player II wins this particular play iff ($f \in A$ iff $\mathcal{A} \in B$). Player II wins the game $G(A, B)$ iff he has a winning strategy (i.e., a function which tells him what to play at each point in the game and which wins against all plays of player I). A winning strategy for player II determines a continuous map which shows that $A \leqslant_W B$. And conversely, every continuous map witnessing $A \leqslant_W B$ determines a winning strategy for player II.

Let Γ be any of the Borel classes Σ^0_α, Π^0_α, $\Pi^0_\alpha \wedge \Sigma^0_\alpha$ (i.e., the difference of two Π^0_α sets), or $\Pi^0_\alpha \vee \Sigma^0_\alpha$ (i.e., the dual of $\Pi^0_\alpha \wedge \Sigma^0_\alpha$). Let Γ^D, the dual of Γ, be the set of complements of the elements of Γ. Each of these Γ is closed under continuous preimage (i.e., if $A \leqslant_W B \in \Gamma$, then $A \in \Gamma$). Also, each of them is nonselfdual (i.e., $\Gamma \neq \Gamma^D$). Thus, to show that some $B \subseteq X_\rho$ is not in Γ^D it suffices to show that for every $A \in \omega^\omega \cap \Gamma$, player II has winning strategy in $G(A, B)$.

Example 1: Π^0_1. Suppose ρ is the similarity type containing one unary relation symbol P. In this case $X_\rho = 2^\omega$. If $\mathcal{A} \models \forall x P(x)$, then $[\mathcal{A}]$ is a single point of X_ρ. Since no point of X_ρ is isolated it cannot be Σ^0_1 (i.e., open).

Example 2: $\Pi^0_1 \wedge \Sigma^0_1$. In the same space let $\mathcal{A}$ model the sentence:

$$\exists x\, P(x) \wedge \forall y \forall z (P(y) \wedge P(z)) \rightarrow y = z$$

(i.e., $\exists ! x P(x)$).
Suppose that $A = 0 \cap C$ where $0 \subseteq \omega^\omega$ is open and $C \subseteq \omega^\omega$ is closed. Let us give a winning strategy for player II in $G(A, [\mathcal{A}])$. Let $\omega^{<\omega}$ be a set of all finite sequences of elements of ω. Because $0 \subseteq \omega^\omega$ is open there exists $\bar{0} \subseteq \omega^{<\omega}$ such that:

$$0 = \{f \in \omega^\omega | \exists n\, f \restriction n \in \bar{0}\}.$$

Also, since $C \subseteq \omega^\omega$ is closed there exists $\bar{C} \subseteq \omega^{<\omega}$ such that

$$C = \{f \in \omega^\omega | \forall n\, f \restriction n \in \bar{C}\}.$$

Now we describe player II's winning strategy. Player II plays $\mathcal{A}_n \models$ "$\forall x \neg P(x)$" until Player I plays $f \restriction n \in \bar{0}$ (if this never happens he continues to play such $\mathcal{A}_n$ forever). At that point, he plays $\mathcal{A}_n \models$ "$\exists ! x P(x)$". He continues to play models of $\exists ! x P(x)$ unless player I plays $f \restriction m \notin \bar{C}$. At that point player II plays $\mathcal{A}_m \models$ "$\exists x \exists y\, (x \neq y \wedge P(x) \wedge P(y))$".

Example 3: Π^0_2, η. Let ρ be the similarity type with one binary relation (so $X_\rho = 2^{\omega \times \omega}$). Let η be a dense linear order without end points. It is easily seen that $[\eta]$ is Π^0_2. To see that it is not Σ^0_2 let's use Wadge games. Suppose $A = \bigcap_{n<\omega} 0_n$ is any Π^0_2 subset of ω^ω (each 0_n open). The strategy for player II is to wait to fill in gaps until player I has put $f \restriction m \in \bar{0}_n$ for some new n. That is, he plays $\mathcal{A}_{m+1} \supseteq \mathcal{A}_m$ an end extension (i.e., for every $x \in A_m$ and $y \in A_{m+1} - A_m$, $x < y$) unless there exists n such that $f \restriction m \in \bigcap_{i \leqslant n} \bar{0}_i$ but $f \restriction (m-1) \notin \bigcap_{i \leqslant n} \bar{0}_i$ and then he plays $\mathcal{A}_{m+1} \supseteq \mathcal{A}_m$ so that for all $x, y \in A_m$ there exists $z \in A_{m+1}$ $x < z < y$ and there exists $z_0, z_1 \in A_{m+1}$ $z_0 < x$ and $y < z_1$.

Example 4: $\Pi^0_2 \wedge \Sigma^0_2$, $1 + \eta + 1$. Let $1 + \eta + 1$ be the countable dense linear order with endpoints. It is easily seen to be $\Pi^0_2 \wedge \Sigma^0_2$. Let "$\exists^\infty n$" abbreviate "there exist infinitely many n" and let "$\forall^\infty n$" abbreviate "for all but finitely many n". It is easily seen that every Π^0_2 subset of ω^ω is Wadge reducible to the set of models of "$\exists^\infty nP(n)$". Thus every $\Pi_2^0 \wedge \Sigma_2^0$ set is reducible to the models of "$\exists^\infty nP(n) \wedge \forall^\infty nQ(n)$". To see that the set of models of "$\exists^\infty nP(n) \wedge \forall^\infty nQ(n)$" reduces to the isomorphism class of $1 + \eta + 1$ is easy. Use $P(n)$ to fill in gaps as in Example 3 and use $Q(n)$ to pick the endpoints. That is, whenever $\neg Q(n)$ appears arrange things so that there exists $x_0, x_1 \in A_{n+1}$ such that for all $y \in A_n$, $x_0 < y < x_1$.

Example 5: Π_3^0, ω. It is easy to see that the isomorphism class of ω (i.e., the order type of ω) is Π_3^0. It is also easy to see that every Π_3^0 set is reducible to the models of "$\forall n \forall^\infty m C(n, m)$". To reduce the models of "$\forall n \forall^\infty m C(n, m)$" to the isomorphism class of ω one strategy for player II can be roughly described as follows: Imagine that he first plays a copy of ω. In each interval $[n, n + 1]$ he plays $a_{i+1} < a_i$ for each i such that $\neg C(n, i)$. Thus if there exist infinitely i such that $\neg C(n, i)$ a copy of ω^* is jammed between $[n, n + 1]$, otherwise $[n, n + 1]$ is finite.

Remarks: Some other structures are also easy to do using Wadge games. For example, (ω, S) (where S is the successor function) has an isomorphism class which is complete $\Pi_2^0 \wedge \Sigma_2^0$. A slightly more difficult argument (see [6]) can be used to show that the model consisting of ω many copies of (ω, S) and ω many copies of (Z, S) (Z is the integers) has an isomorphism class which is complete Π_4^0. This example was motivated by the fact that every finite valency structure has an isomorphism class which is Π_4^0. (For the definition of finite valency structure, see [5].) This shows Π_4^0 is best possible.

*3 **Ehrenfeucht games or back and forth properties*** Here we review some material which is well known. In this section let $\mathcal{A}$ and $\mathcal{B}$ be countable models in the same similarity type. Define for α an ordinal, $\mathcal{A} \underset{\alpha}{\rightarrow} \mathcal{B}$ by induction on α. Define $\mathcal{A} \underset{0}{\rightarrow} \mathcal{B}$ iff $\mathcal{A}$ and $\mathcal{B}$ model the same quantifier-free sentences. Define $\mathcal{A} \underset{\alpha}{\rightarrow} \mathcal{B}$ iff for all $\beta < \alpha$ and $\vec{a} \in A^{<\omega}$ there exists $\vec{b} \in B^{<\omega}$ such that $(\mathcal{B}, \vec{b}) \underset{\beta}{\rightarrow} (\mathcal{A}, \vec{a})$.

Lemma 3.1 *$\mathcal{A} \underset{\alpha}{\rightarrow} \mathcal{B}$ iff every Σ_α^0 sentence true in $\mathcal{A}$ is true in $\mathcal{B}$.*

Proof: Note that the right-hand side is equivalent to every Π_α^0 sentence true in $\mathcal{B}$ is true in $\mathcal{A}$. The proof is by induction. If

$$\bigvee_{n<\omega} \exists \vec{x}_n \theta_n(\vec{x}_n)$$

is true in $\mathcal{A}$ where each θ_n is $\Pi_{\beta_n}^0$ for some $\beta_n < \alpha$, then choose $\vec{a} \in A^{<\omega}$ and n such that

$$\theta_n(\vec{a})$$

is true in $\mathcal{A}$. Let $\vec{b} \in B^{<\omega}$ be such that $(\mathcal{B}, \vec{b}) \underset{\beta_n}{\rightarrow} (\mathcal{A}, \vec{a})$. By inductive hypothesis, $\theta_k(\vec{b})$ is true in $\mathcal{B}$. To prove the converse, suppose every Σ_α^0 sentence true in $\mathcal{A}$ is true in $\mathcal{B}$. Given $\vec{a} \in A^n$ and $\beta < \alpha$, it is enough to find $\vec{b} \in B^n$ such that every Σ_β^0 sentence true in $(\mathcal{B}, \vec{b})$ is true in $(\mathcal{A}, \vec{a})$. If not, then for all $\vec{b} \in B^n$ there exists a Σ_β^0 formula $\theta_{\vec{b}}(\vec{x})$ such that:

$$(\mathcal{B}, \vec{b}) \models \theta_{\vec{b}}(\vec{b}),$$

but

$$(\mathcal{A}, \vec{a}) \models \neg\theta_{\vec{b}}(\vec{a}).$$

But then $\mathcal{A}$ models the Σ^0_α formula:

$$\exists \bar{x} \bigwedge_{\vec{b} \in B^n} \neg\theta_{\vec{b}}(\bar{x})$$

but $\mathcal{B}$ does not.

Lemma 3.2 Suppose $\mathcal{A}$ is not isomorphic to $\mathcal{B}$ and $\mathcal{A} \underset{\alpha}{\rightarrow} \mathcal{B}$, then $[\mathcal{A}]$ is not Σ^0_α and $[\mathcal{B}]$ is not Π^0_α.

Proof: This is an immediate Corollary of Lemma 3.1 and Vaught's Theorem.

Next, we prove some facts about ordinals which are basically due to Ehrenfeucht [2] (see also Karp [4]). For any ordinal α, we use α to stand for the model in the language of one binary relation $<$ of order type α. We need to strengthen the notion $\mathcal{A} \underset{\delta}{\rightarrow} \mathcal{B}$ by using the idea of a ($\delta-$) elementary substructure.

Define $\mathcal{A} \precsim_\delta \mathcal{B}$ iff $\mathcal{A}$ is a substructure of $\mathcal{B}$ and for all $\vec{a} \in A^{<\omega}$ and $\beta < \delta$, $(\mathcal{B}, \vec{a}) \underset{\beta}{\rightarrow} (\mathcal{A}, \vec{a})$. We will say that α is δ oblivious iff for any $\gamma_1 \geqslant \gamma_0 \geqslant 1$, $\alpha \cdot \gamma_0 \precsim_\delta \alpha \cdot \gamma_1$.

Lemma 3.3 *If α is δ oblivious, then $\alpha \cdot \omega$ is $\delta + 2$ oblivious.*

Proof: See the proof of Theorem 12 in [2].

Lemma 3.4 *For λ limit and $n < \omega$, $\omega^{\lambda+n}$ is $\lambda + 2n + 1$ oblivious.*

Proof: First note that ω is 3 oblivious (i.e., if $\alpha \leqslant \beta$ are limit ordinals, then $\alpha \precsim_3 \beta$). Use the identity for the first move from α to β and use the fact that α is a limit on the second move from β to α. For $n = 0$ and λ a limit we want to prove that if $1 \leqslant \gamma_0 \leqslant \gamma_1$ and $\vec{a} \in \omega^\lambda \cdot \gamma_0$, then $(\omega^\lambda \cdot \gamma_1, \vec{a}) \underset{\lambda}{\rightarrow} (\omega^\lambda \cdot \gamma_0, \vec{a})$. But $\mathcal{A} \underset{\lambda}{\rightarrow} \mathcal{B}$ iff for all $\beta < \lambda$ $\mathcal{A} \underset{\beta}{\rightarrow} \mathcal{B}$. By induction if $\beta < \lambda$ and $\beta + \alpha = \lambda$, then $\omega^\beta \cdot (\omega^\alpha \cdot \gamma_0) \precsim \omega^\beta \cdot (\omega^\alpha \cdot \gamma_1)$. Thus $(\omega^\lambda \cdot \gamma_1, \vec{a}) = (\omega^\beta \cdot (\omega^\alpha \cdot \gamma_1), \vec{a}) \underset{\beta}{\rightarrow} (\omega^\beta \cdot (\omega^\alpha \cdot \gamma_0), \vec{a}) = (\omega^\lambda \cdot \gamma_0, \vec{a})$.

Using Lemma 3.4 it is easy to compute the Borel class in which the isomorphism class of each countable ordinal lies.

Lemma 3.5 *If $\gamma = \omega^{\lambda+m} \cdot n + \delta$ where λ is a limit, $n, m < \omega$, and $\delta < \omega^{\lambda+m}$, then if $n = 1$, then $[\gamma]$ is $\Pi^0_{\lambda+2m+1}$ but not $\Sigma^0_{\lambda+2m+1}$, if $n > 1$, then $[\gamma]$ is $\Pi^0_{\lambda+2m+1} \wedge \Sigma^0_{\lambda+2m+1}$ but not $\Pi^0_{\lambda+2m+1} \vee \Sigma^0_{\lambda+2m+1}$.*

Proof: The computation of the upper bound on complexity is left to the reader.

For example, to see that $[\omega]$ is Π^0_3 say that it is a linear order with no greatest element (Π^0_2) and every element has finitely many predecessors. To see that $[\omega + \omega]$ is $\Pi^0_3 \wedge \Sigma^0_3$ say that it is a linear order with no greatest element (Π^0_2), there exists a nonzero limit point (Σ^0_3), and for all $x < y$ either x has finitely many predecessors or y has finitely many predecessors greater than x (Π^0_3). Now we verify the lower bound.

Since $\omega^{\lambda+m}$ is $\lambda + 2m$ oblivious, we have that:

$$(\omega^{\lambda+m} + \delta) \xrightarrow[\lambda+2m+1]{} (\omega^{\lambda+m}\cdot n + \delta) \xrightarrow[\lambda+2m+1]{} (\omega^{\lambda+m}\cdot(n+1) + \delta).$$

Here we are using the fact that if $\alpha \underset{\gamma}{\rightarrow} \beta$, then $(\alpha + \delta) \underset{\gamma}{\rightarrow} (\beta + \delta)$. From the first arrow we get that $\omega^{\lambda+m} + \delta$ cannot be $\Sigma^0_{\lambda+2m+1}$ and for $n > 1$, $\omega^{\lambda+m}\cdot n + \delta$ cannot be $\Pi^0_{\lambda+2m+1}$. From the second arrow we get that $\omega^{\lambda+m}\cdot n + \delta$ cannot be $\Sigma^0_{\lambda+2m+1}$.

Remarks: For any countable ordinal α let $WO(\alpha)$ be the subset of $2^{\omega\times\omega}$ of all well ordering ω of type less than α. Stern, in [11] and [12], showed, for example, that for any limit λ and $n < \omega$, $WO(\omega^{\lambda+n})$ is $\Sigma^0_{\lambda+2n}$ but not $\Pi^0_{\lambda+2n}$. His argument used a variant of Steel's forcing. He also calculated the Borel class of the set of well-founded trees of rank less than α. This characterization had been found earlier by Garland [3] using continuous reducibility (and not forcing). I don't know how to use continuous reducibility to do the ordinal case. Also, the use of forcing allowed Stern to prove more. He showed that assuming $MA + \neg CH$ any Borel set which is the union of $\aleph_1$ Σ^0_α sets must be Σ^0_α. This result can also be proved by using the Vaught transform and Ehrenfeucht's analysis of well-orderings in place of forcing. For example, let us show that the isomorphism class of the order type of ω is not the ω_1 union of Σ^0_3 sets (assuming $MA + \neg CH$). Since $[\omega]$ is a minimal invariant set, it is enough to show that any invariant set which is the ω_1 union of Σ^0_3 sets is the ω_1 union of invariant Σ^0_3 sets. But note that Vaught [13] shows that for any Σ^0_3 set B, B^Δ is an invariant Σ^0_3 set. Also under $MA + \neg CH$ it is easy to see that:

$$\left(\bigcup_{\alpha<\omega_1} B_\alpha\right)^\Delta = \bigcup_{\alpha<\omega_1} B^\Delta_\alpha.$$

Also the fact that $WO(\omega + \omega)$ is not Π^0_3 can be proved using Ehrenfeucht games. This was proved in the appendix of [12] using forcing. It is not hard to see that $WO(\omega + \omega)$ is $\Pi^0_3 \wedge \Sigma^0_3$. Games can be used to show that for any Σ^0_3 sentence θ true in $(\omega + \omega^*, <)$ there is an $n < \omega$ such that θ is true in $(\omega + n, <)$.

4 Some finitely axiomatizable theories and other examples In this section we begin by giving some examples of finitely axiomatizable, first-order, $\aleph_0$-categorical theories.

Construction 1 Given two models $\mathcal{A}$ and $\mathcal{B}$ in the same language and $i = 0, 1, 2$, the model $\mathcal{C}_i$ can be described as follows. Let $\approx$ and $\leqslant$ be two new binary relations. In $\mathcal{C}_i$ $\approx$ is an equivalence relation and $\leqslant$ densely orders (order type η) the $\approx$ equivalence classes. Each equivalence class is isomorphic to either $\mathcal{A}$ or $\mathcal{B}$ and in $\mathcal{C}_i$ exactly i are isomorphic to $\mathcal{A}$. The proofs of the next two claims are left to the reader.

Claim 1 If $\mathcal{A} \underset{n}{\rightarrow} \mathcal{B}$, then $\mathcal{C}_0 \xrightarrow[n+1]{} \mathcal{C}_1 \xrightarrow[n+1]{} \mathcal{C}_2$, and therefore by Lemma 3.2 $\mathcal{C}_0 \notin \Sigma^0_{n+1}$ and $\mathcal{C}_1 \notin \Sigma^0_{n+1} \vee \Pi^0_{n+1}$.

Claim 2 If $[\mathcal{A}]$ and $[\mathcal{B}]$ are Δ^0_{n+1} $(n \geqslant 2)$, finitely axiomatizable, and complete (i.e., no finite models), then $[\mathcal{C}_0]$ is Π^0_{n+1}, finitely axiomatizable, and complete and $[\mathcal{C}_1]$ is $\Pi^0_{n+1} \wedge \Sigma^0_{n+1}$, finitely axiomatizable, and complete.

Now starting with η and $1 + \eta + 1$ (Examples 3 and 4 of Section 2) and noting that $\eta \underset{2}{\rightarrow} 1 + \eta + 1$ we get examples for all the Borel classes Π_n^0 and $\Pi_n^0 \wedge \Sigma_n^0$ for $3 \leqslant n < \omega$.

Construction 2 Given two structures $\mathscr{A}$ and $\mathscr{B}$ in the same language, we construct two models $\mathscr{D}_0$ and $\mathscr{D}_1$ similar to the first construction. The new $\leqslant$ and $\approx$ are the same as in the $\mathscr{C}_i$ and every equivalence class is isomorphic to either $\mathscr{A}$ or $\mathscr{B}$, and both $\mathscr{A}$ and $\mathscr{B}$ are isomorphic to some equivalence class. In addition, every equivalence class isomorphic to $\mathscr{A}$ is $\leqslant$ every equivalence class isomorphic to $\mathscr{B}$. The only difference between $\mathscr{D}_1$ and $\mathscr{D}_0$ is that in $\mathscr{D}_1$ there is a $\leqslant$ greatest class isomorphic to $\mathscr{A}$ and in $\mathscr{D}_0$ there isn't. The following claims are easy to verify.

Claim 1 If $[\mathscr{A}]$ and $[\mathscr{B}]$ are Δ_{n+1}^0 $(n \geqslant 1)$, finitely axiomatizable, and complete, then $[\mathscr{D}_0]$ is Π_{n+2}^0 finitely axiomatizable and complete and $[\mathscr{D}_1]$ is Σ_{n+2}^0 finitely axiomatizable and complete.

Remark: In case $n + 2 = 3$ we should take for $\mathscr{A}$ and $\mathscr{B}$ the one-element and two-element models in the empty language.

Claim 2 If $\mathscr{A} \underset{n}{\rightarrow} \mathscr{B}$ then $\mathscr{D}_0 \underset{n+2}{\longrightarrow} \mathscr{D}_1$, and thus by Lemma 3.2 $\mathscr{D}_1$ is not Π_{n+2}^0.

This construction gives examples for all the Borel classes Σ_n^0 for $3 \leqslant n < \omega$. The ordinals give examples for all Borel classes Π_α^0 and $\Pi_\alpha^0 \wedge \Sigma_\alpha^0$ for odd $\alpha \geqslant 3$. Coupled with these two constructions we get examples for all Borel classes not ruled out, except for Π_λ^0, $\Sigma_{\lambda+1}^0$, $\Sigma_{\lambda+2}^0$ for λ an infinite limit ordinal. We now give examples for Π_λ^0 and $\Sigma_{\lambda+2}^0$. For simplicity let $\lambda = \omega$. The isomorphism class of the model $\mathscr{A}_\omega = (\omega^\omega, <, P)$ where $P = \{\omega^n \colon n < \omega\}$ is easily seen to be Π_ω^0. Let $\mathscr{A}_n = (\omega^{n+1}, <, P)$ where $P = \{\omega^i \colon i \leqslant n\} \cup \{\omega^n \cdot m \colon m < \omega\}$. It is not hard to show that for each n, $\mathscr{A}_n \underset{n}{\rightarrow} \mathscr{A}_\omega$, thus $[\mathscr{A}_\omega]$ is not Π_n^0 for any $n < \omega$.

Construction 3 Let $\mathscr{B}_1$ be the model $(Q, <, c_n)_{n<\omega}$ where Q is the rationals and c_n a sequence strictly increasing to 0. Let $\mathscr{B}_0$ be $\mathscr{B}_1$ minus 0 (i.e., the c_n's have no supremum). Let $\mathscr{B}_1^*$ be obtained from $\mathscr{B}_1$ by replacing each c_n by a copy of $\mathscr{A}_n$ and each other element of Q by a copy of $\mathscr{A}_\omega$. Similarly construct $\mathscr{B}_0^*$. It is easily shown that

$$\mathscr{B}_0^* \underset{\omega+2}{\longrightarrow} \mathscr{B}_1^*.$$

Therefore, $[\mathscr{B}_1^*]$ is not $\Pi_{\omega+2}^0$. On the other hand, direct calculation shows that $[\mathscr{B}_1^*]$ is $\Sigma_{\omega+2}^0$.

5 An $\aleph_0$-categorical theory properly of class Π_ω^0 In this section we give an example of a $\aleph_0$-categorical, first-order theory whose model has an isomorphism class which is Π_ω^0 but not Σ_ω^0. It is a variation of an (unpublished) example of Kueker and Baldwin of a countable, $\aleph_0$-categorical theory which has the property that no finite extension is model complete.

For $N \leqslant \omega$ let T_N be the following universal theory in the language: R, S binary relations and Q_n $n+1$-ary relation for n each $< N$. The axioms of T_N say:

(1) R and S are symmetric and irreflexive,
(2) for $n < N$, Q_n is symmetric and irreflexive, and

(3) for $n + 1 < N$, nothing in Q_{n+1} is totally connected (by R if n even, S if n odd) to anything in Q_n.

More formally by (3) for n even I mean:

$$\forall \bar{x}\ \forall \bar{y}(Q_n(\bar{x}) \wedge Q_{n+1}(\bar{y})) \to \bigvee_{\substack{i<n \\ j<n+1}} \neg R(x_i, y_j);$$

and for n odd the same sentence with S in place of R.

Lemma 5.1 *T_N has the amalgamation and joint embedding properties.*

Proof: What these properties say is that any two models of T_N can be embedded into a third model (joint embedding) or amalgamated over a common submodel (amalgamation). The proof is trivial since (3) can be made true simply by making $Q_n(\bar{x})$ fail for all $\bar{x}$ which are new.

Lemma 5.2 *Suppose $\mathcal{A} \models T_N$, $\vec{a} \in A^{n+1}$, $n + 1 < N$ (n even), and $\mathcal{A} \models \neg Q_{n+1}(\vec{a})$, then there exists $\mathcal{B} \models T_N, \mathcal{B} \supseteq \mathcal{A}$, and $\mathcal{B} \models \exists \vec{b}\, Q_n(\vec{b}) \wedge \bigwedge_{\substack{i<n \\ j<n+1}} R(b_i, a_j)$. (Similarly for n odd with S in place of R.)*

Proof: Let $B = A \cup \{b_0, b_1, \ldots, b_n\}$; for $k \neq n$ let $Q_k^B = Q_k^A$; let $Q_n^B = Q_n^A \cup \{\vec{b}' : \vec{b}'$ is a permutation of $\vec{b}\}$; let $S^B = S^A$, and let $R^B = R^A \cup \{(a_i, b_j), (b_j, a_i): i < n + 1, j < n\}$. Here is where we needed both R and S, since Q_{n-1} might hold on some subset of $\vec{a}$.

Let $\mathcal{A}_N$ be the universal homogeneous countable model of T_N. That is, every finite model of T_N is isomorphic to a substructure of $\mathcal{A}_N$ and every isomorphism of finite substructures of $\mathcal{A}_N$ extends to an automorphism of $\mathcal{A}_N$. The Theory of $\mathcal{A}_N$ is $\aleph_0$-categorical. For any $k \leqslant N$ let $\mathcal{A}_N^k$ be the reduct of $\mathcal{A}_N$ to the language $R, S, Q_i: i < k$. By Lemma 5.2 it is easy to see that every Q_i is definable in $\mathcal{A}_N^1$, thus the theory of $\mathcal{A}_N^1$ is also $\aleph_0$-categorical. We will show that $[\mathcal{A}_\omega^1]$ is not Π_n^0 for any $n < \omega$. First note that for $n < m < \omega$, $\mathcal{A}_n^1$ is not isomorphic to $\mathcal{A}_m^1$. This is because $\mathcal{A}_m^1$ satisfies $\exists \vec{x}\, Q_n(\vec{x})$ but $\mathcal{A}_n^1$ does not (we mean here the definition of Q_n from $\{R, S, Q_0\}$). This in turn is proved like Lemma 5.2. Define $\mathcal{A} \equiv_n \mathcal{B}$ iff $\mathcal{A} \underset{n}{\to} \mathcal{B}$ and $\mathcal{B} \underset{n}{\to} \mathcal{A}$.

Lemma 5.3 *If $k + 1 \leqslant \min(N, N')$, then if $(\mathcal{A}_N^{k+1}, \vec{a}) \equiv_0 (\mathcal{A}_{N'}^{k+1}, \vec{a}')$, then $(\mathcal{A}_N^k, \vec{a}) \equiv_1 (\mathcal{A}_{N'}^k, \vec{a}')$.*

Proof: Let $\vec{b} \in A_N^{<\omega}$. Construct a model C in the language $\{R, S, Q_n: n < N'\}$ as follows. Let $C = \vec{a}' \cup \vec{b}$, let $\mathcal{C}^k$ be isomorphic to $(\vec{a} \cup \vec{b}, R, S, Q_n: n < k)$ via the given map taking $\vec{a}'$ to $\vec{a}$ and the identity on $\vec{b}$, and for n with $k \leqslant n < N'$ let $Q_n^{\mathcal{C}} = (\vec{a}')^{n+1} \cap Q_n^{\mathcal{A}_{N'}}$.

Claim $\mathcal{C} \models T_{N'}$.

We only need to check (3). Suppose $\mathcal{C} \models$ "$Q_n(\vec{c}) \wedge Q_{n+1}(\vec{d}) \wedge \vec{c}$ and $\vec{d}$ are totally connected by R (if n even, S if n odd)". By construction of C^k it cannot be that $n + 1 < k$. If $n \geqslant k$ then both $\vec{c}$ and $\vec{d}$ are subsets of $\vec{a}'$ and again there is no problem. The remaining case is $n + 1 = k$. From the construction we have that $\vec{d}$ must be contained in $\vec{a}'$ and so Q_k holds on its image in $\vec{a}$, since by assumption $(\vec{a}', R, S, Q_n, n \leqslant k)$ is isomorphic to $(\vec{a}, R, S, Q_n, n \leqslant k)$. But Q_{k-1} holds on the image of $\vec{c}$ in $\vec{a} \cup \vec{b}$, a contradiction. This proves the claim.

But now since $\mathcal{A}_{N'}$ is universal homogeneous, we know there exists $\vec{b}'$ in $\mathcal{A}_{N'}$ such that $\mathcal{C}$ is isomorphic to $(\vec{a}' \cup \vec{b}', R, S, Q_n: n < N')$ (extending the identity on $\vec{a}'$). Therefore, $(\mathcal{A}^k_N, \vec{a}, \vec{b}) \Xi_0 (\mathcal{A}^k_{N'}, \vec{a}', \vec{b}')$.

Lemma 5.4 *If* $k + 1 \leqslant \min(N, N')$, *then if* $(\mathcal{A}^{k+1}_N, \vec{a}) \Xi_i (\mathcal{A}^{k+1}_{N'}, \vec{a}')$, *then* $(\mathcal{A}^k_N, \vec{a}) \Xi_{i+1} (\mathcal{A}^k_{N'}, \vec{a}')$.

Proof: Play the game for i steps, then get through one more step by dropping Q_k.

Lemma 5.5 *If* $k + 1 = \min(N, N')$, *then* $\mathcal{A}^1_N \Xi_k \mathcal{A}^1_{N'}$.

Proof: Immediate from Lemma 5.4.

This lemma gives immediately that $[\mathcal{A}^1_\omega]$ is not Π^0_k for any $k < \omega$.

Remark: The proper generalization to admissible $\lambda > \omega$ is given by the Λ-self-hyp-characterizable models of [9].

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