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Abstract. Let κ_B be the least cardinal for which the Baire category theorem fails for the real line $\boldsymbol{R}$. Thus κ_B is the least κ such that the real line can be covered by κ many nowhere dense sets. It is shown that κ_B cannot have countable cofinality. On the other hand it is consistent that the corresponding cardinal for 2^{ω_1} be $\aleph_\omega$. Similar questions are considered for the ideal of measure zero sets, other ω_1 saturated ideals, and the ideal of zero-dimensional subsets of $\boldsymbol{R}^{\omega_1}$.

A set is nowhere dense iff its closure has no interior. A set is meager (equivalently first category) iff it is the countable union of nowhere dense sets. Thus κ_B is the least cardinal κ such that there are κ many meager subsets of $\boldsymbol{R}$ whose union is all of $\boldsymbol{R}$. Similar cardinals have been considered by the author (1979). If we let κ_U be the least cardinal κ such that there exists a nonmeager set of reals of cardinality κ, then it is easy to see that κ_U cannot have countable cofinality. On the other hand in Solovay's random real model κ_U is the cardinality of the continuum (Martin-Solovay (1970)). Thus κ_U may be any cardinal of cofinality greater than ω. If we let κ_A be the least κ such that there are κ many meager sets of reals whose union is not meager, then it is easily seen that κ_A is a regular cardinal. Martin and Solovay (1970) have shown that Martin's Axiom implies that κ_A is the cardinality of the continuum and so, by results of Solovay and Tennenbaum (1971), κ_A may be any regular uncountable cardinal. For any cardinal κ of uncountable cofinality it is consistent that $\kappa_B = \kappa$. In fact this holds in the Cohen real model of Solovay and Cohen (see Solovay (1972).)

Other consistency results are given by Hechler (1973). In the author's paper (1979) it is shown that it is consistent that $\kappa_A = \omega_1$ and $\kappa_B = \kappa_U = \omega_2$. For any topological space X define $\kappa_B(X)$ to be the least cardinal κ such that X can be covered by κ many nowhere dense (in X) subsets of X. This is well defined for any X in which points are nowhere dense. Stepanek and Vopenka (1967), see also Kulpa and Szymanski (1977), showed that for any nowhere separable metric space X the cardinal $\kappa_B(X)$ is ω_1. Fleissner and Kunen (1978) and Broughan (1977) have noted that for any metric space X without isolated points, $\kappa_B(X) \leq \kappa_B$. Balcar, Pelant, and Simon (1979) have studied $\kappa_B(\beta\boldsymbol{N} - \boldsymbol{N})$, using $n(X)$ for Novak number in place of $\kappa_B(X)$. Fremlin and Shelah (1978) and Hechler (1974) have asked if κ_B can be $\aleph_\omega$. In §1 we show that it cannot be.

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§1. Baire category on the real line. In this section we prove the following theorem.

THEOREM 1. *κ_B cannot have countable cofinality.*

PROOF. The following lemma is well known.

LEMMA 1. *For any separable, completely metrizable space X without isolated points $\kappa_B(X) = \kappa_B$.*

PROOF. There exists a meager F_σ set $Y \subseteq X$ such that $X - Y$ is homeomorphic to the irrationals. To get Y take the countable union of the boundaries of some basis for X together with a countable dense subset of what is left. $X - Y$ is a G_δ set in X and hence completely metrizable. It is easily seen to be zero dimensional and to have the property that no compact clopen set is nonempty, and so it is homeomorphic to the irrationals (see Kuratowski (1966) or Hoffman-Jorgensen and Topsoe (1980)). The lemma follows since if W is comeager in Z, then $\kappa_B(W) = \kappa_B(Z)$. □

For $\boldsymbol{P}$ a partial order define $D \subseteq \boldsymbol{P}$ to be dense iff every element of $\boldsymbol{P}$ can be extended to an element of D:

$$\forall p \in \boldsymbol{P} \exists q \in D, \qquad q \leq p.$$

Define $G \subseteq \boldsymbol{P}$ to be a $\boldsymbol{P}$-filter iff G is upward closed and every two elements of G are compatible in G.

$$q \geq p \in G \Rightarrow q \in G; \quad p, \quad q \in G \Rightarrow \exists r \in G((r \leq p) \wedge (r \leq q)).$$

For F a family of dense subsets of $\boldsymbol{P}$ define G to be $\boldsymbol{P}$-generic over F iff G is a $\boldsymbol{P}$-filter and every member of F intersects G. Let MA_κ (ctble) be the following weak version of Martin's Axiom:

For any countable partial order $\boldsymbol{P}$ and family F of dense subsets of $\boldsymbol{P}$, if $|F| < \kappa$, then there exists a $\boldsymbol{P}$-generic filter G over F.

The following lemma is well known. It says that forcing with countable partial orders is the same as Cohen forcing.

LEMMA 2. *κ_B is the greatest κ such that* MA_κ (ctble) *holds.*

PROOF. To see that $\mathrm{MA}_{\kappa_B^+}$(ctble) fails consider the partial order $\boldsymbol{P}$ of open intervals with rational endpoints ordered by inclusion. Note that a dense subset of $\boldsymbol{P}$ corresponds to a family of intervals whose union is a dense open subset of $\boldsymbol{R}$. One way to see that MA_{κ_B}(ctble) holds is to translate the partial order statement into one about complete boolean algebras, as was done for example in §2 of Martin and Solovay (1970), and then note that any two complete separable atomless boolean algebras are isomorphic.

Alternatively a direct proof can be given along the following lines. The irrationals are homeomorphic to the space ω^ω. The topology on ω^ω is generated by sets of the form $N_s = \{g \in \omega^\omega : s \subseteq g\}$ for $s \in \omega^{<\omega}$, where $\omega^{<\omega}$ is the set of finite sequences of elements of ω. Dense open subsets $D \subseteq \omega^\omega$ correspond to dense subsets of $\omega^{<\omega}$, i.e. $\{s \in \omega^{<\omega} : N_s \subseteq D\}$. Also $\omega^{<\omega}$-generic filters correspond to elements of ω^ω. Therefore $\mathrm{MA}_{\kappa_B}(\omega^{<\omega})$ holds. Let $\boldsymbol{P}$ be any countable partial order. If there is a condition $p \in \boldsymbol{P}$ such that every two extensions of p are compatible, then $\{q \in \boldsymbol{P}: q \leq p \text{ or } p \leq q\}$ is a $\boldsymbol{P}$-filter meeting every dense subset of $\boldsymbol{P}$. If there is no such p, then for every element of $\boldsymbol{P}$ there exists an infinite, maximal set of incompatible extensions. Since $\boldsymbol{P}$ is countable this allows one to inductively construct an order preserving embedding of $\omega^{<\omega}$ onto a dense subset of $\boldsymbol{P}$ (see exercise C4, Chapter 7 of Kunen (1980)). Thus by a well-known theorem of Solovay (1970) we are done. □

A subset of 2^ω (2 is the two point discrete space $\{0, 1\}$) is perfect iff it is closed, nonempty, and has no isolated points; or equivalently it is homeomorphic to 2^ω. From now on assume that the cofinality of κ_B is ω, and let C_α for $\alpha < \kappa_B$ be a family of closed nowhere dense subsets of 2^ω such that $2^\omega = \bigcup\{C_\alpha | \alpha < \kappa_B\}$. For $P \subseteq 2^\omega$ we say that P is good iff P is perfect and for every $\alpha < \kappa_B$ the set C_α is nowhere dense (in the relative topology) in P.

LEMMA 3. *If Q is good and $\beta < \kappa_B$ then there is a good $P \subseteq Q$ such that for all $\alpha < \beta$ the set $C_\alpha \cap P$ is empty.*

Before proving this lemma let us indicate the proof of the theorem. Let κ_n for $n < \omega$ be cofinal in κ_B. Construct $P_{n+1} \subseteq P_n$ a sequence of good sets with $P_0 = 2^\omega$ and $P_{n+1} \cap C_\alpha = \varnothing$ for every $n < \omega$ and $\alpha < \kappa_n$. But then $\bigcap\{P_n: n < \omega\} \neq \varnothing$, a contradiction.

To prove Lemma 3 we may without loss of generality assume $Q = 2^\omega$.

Claim. There exists a countable dense set $H \subseteq 2^\omega$ such that $C_\alpha \cap H = \varnothing$ for all $\alpha < \beta$ and $C_\alpha \cap H$ is finite for all $\alpha < \kappa_B$.

PROOF. Let $2^{<\omega} = \{s_n: n < \omega\}$ and let κ_n for $n < \omega$ be increasing and cofinal in κ with $\kappa_0 > \beta$. Choose $x_n \in N_{s_n} - \bigcup\{C_\alpha | \alpha < \kappa_n\}$ (this set is nonempty since each N_{s_n} is homeomorphic to 2^ω and each C_α is nowhere dense in N_{s_n}). Let $H = \{x_n | n < \omega\}$. □

Let $\boldsymbol{P} = \{(X, n) \mid n < \omega \text{ and } H \in [X]^{<\omega}\}$ ($[X]^{<\omega}$ is the set of finite subsets of X), and define $(X, n) \leq (Y, m)$ iff $n \geq m$, $X \supseteq Y$, and for every $x \in X$ there is $y \in Y$ such that $x \restriction m = y \restriction m$. Note that $\boldsymbol{P}$ is countable. We will determine a family F of dense subsets of $\boldsymbol{P}$ such that $|F| = |\beta|$, and, for any $\boldsymbol{P}$-generic filter G over F, if $K = \bigcup\{X: \exists n\, (X, n) \in G\}$ and P is the closure of K, then P will have the desired properties of Lemma 3. For every $n < \omega$ let

$$D^n = \{(X, m): m \geq n \wedge \forall x \in X \exists y \in X (x \neq y \wedge x \restriction n = y \restriction n)\}.$$

Since H is dense it is easily shown that each D^n is dense in $\boldsymbol{P}$. The D^n's guarantee that P will be perfect. If P is not perfect then there is an $x \in K$ and $n < \omega$ such that $N_{x \restriction n} \cap \{x\}$, but this would imply $G \cap D^n = \varnothing$. For each $\alpha < \beta$ let $D_\alpha = \{(X, n): \forall x \in X\, N_{x \restriction n} \cap C_\alpha = \varnothing\}$. Since $H \cap C_\alpha = \varnothing$ it can be shown that D_α is dense in $\boldsymbol{P}$. The set D_α guarantees that $P \cap C_\alpha = \varnothing$. If $(X, n) \in G$, then $P \subseteq \bigcup\{N_{x \restriction n}: x \in X\}$; and so if $G \cap D_\alpha \neq \varnothing$, then $P \cap C_\alpha = \varnothing$. Finally note that for every $\alpha < \kappa_B$, $C_\alpha \cap K$ is finite, since $C_\alpha \cap H$ is finite, and therefore since P is perfect, $P - C_\alpha$ is open dense in P. Letting $F = \{D^n: n < \omega\} \cup \{D_\alpha: \alpha < \beta\}$ finishes the proof of Lemma 3 and therefore Theorem 1.

For any cardinal κ let $(2^\kappa)_\kappa$ be 2^κ with the smallest topology containing the usual topology and closed under less than κ intersections (see Hung and Negrepontis (1973) and Comfort and Negrepontis (1972)). Then for κ strongly inaccessible the proof of Thoerem 1 can be generalized to show that the cofinality of $\kappa_B((2^\kappa)_\kappa)$ is greater than κ. Note that, assuming the continuum hypothesis, $\kappa_B((2^{\omega_1})_{\omega_1}) = \kappa_B(\beta N - N)$. I do not know whether or not the cofinality of $\kappa_B((2^{\omega_1})_{\omega_1})$ can be less than ω_2. This question is also raised by Balcar, Pelant, and Simon (1979).

§2. **Baire category in** 2^{ω_1}**.** In the rest of this paper the formal statement "Con(ZFC) implies Con(ZFC + P)" will be abbreviated by "It is consistent that P".

Since the space 2^{ω_1} is a compact Hausdorff space it cannot be covered by countably many nowhere dense sets. On the other hand $\kappa_B(2^{\omega_1})$ is less than or equal to $\kappa_B(2^\omega)$. This follows from the fact that if $C \subseteq 2^\omega$ is nowhere dense in 2^ω then $C \times 2^{\omega_1 - \omega}$ is nowhere dense in 2^{ω_1}.

THEOREM 2A. *It is consistent that* $\kappa_B(2^{\omega_1}) = \aleph_\omega$.

PROOF. Let M be a countable transitive model of ZFC plus GCH. Working in M let $\boldsymbol{Q}$ be the partial order for adjoining $\aleph_\omega$ Cohen reals:

$$\boldsymbol{Q} = \{p \mid \operatorname{dom}(p) \in [\aleph_\omega]^{<\omega}, \operatorname{range}(p) = \{0, 1\}\}.$$

Let G be a $\boldsymbol{Q}$-generic filter over M. We claim that for any $X \in M[G] \cap [\omega_1]^{\omega_1}$ there is an $n < \omega$ and $Y \in M[G \restriction \aleph_n] \cap [\omega_1]^{\omega_1}$ such that $Y \subseteq X$. This is true since for every $\alpha \in X$ there is a $p \in G$ and an $n < \omega$ such that

$$\operatorname{dom}(p) \subseteq \aleph_n \quad \text{and} \quad p \Vdash \text{“}\alpha \in X\text{”}.$$

Thus there is an $n < \omega$ such that $Y = \{\alpha \mid \exists p \in G \restriction \aleph_n\ p \Vdash \text{“}\alpha \in X\text{”}\}$ is an uncountable set.

In $M[G \restriction \aleph_n]$ the set $[\omega_1]^{\omega_1}$ has cardinality $\aleph_n$ so there are only $\aleph_\omega$ possible Y's. For any infinite $Y \subseteq \omega_1$ the set $\{X \in 2^{\omega_1}: \forall \alpha \in Y\ X(\alpha) = 1\}$ is closed nowhere dense, therefore in $M[G]$ it must be that $\kappa_B(2^{\omega_1}) \leq \aleph_\omega$. To see that $\kappa_B(2^{\omega_1}) \geq \aleph_\omega$ note that given $\aleph_n$ Borel subsets of 2^{ω_1}, there exists in M a set $A \subseteq \aleph_\omega$ with cardinality less than or equal to $\aleph_n + \aleph_1$ such that each of the Borel sets is coded in $M[G \restriction A]$. Choose a one-to-one function $f: \omega_1 \to \aleph_\omega$ in M so that the range of f is disjoint from A. It is easy to check that if $X \in 2^{\omega_1}$ is defined by $X(\alpha) = G(f(\alpha))$ then X is $\boldsymbol{Q}\omega_1$-generic over $M[G \restriction A]$ and thus not in any meager Borel subset of 2^{ω_1} coded in $M[G \restriction A]$. ($\boldsymbol{Q}\omega_1 = \{p \in \boldsymbol{Q}: \operatorname{dom}(p) \subseteq \aleph_1\}$.) □

It is easy to check that in the model of Theorem 2A for every n with $1 \leq n \leq \omega$ it happens that $\kappa_B(2^{\omega_n}) = \aleph_\omega$. The simplest way to see this is to note that $\kappa \leq \lambda$ implies $\kappa_B(2^\lambda) \leq \kappa_B(2^\kappa)$. It is also true that when $\aleph_{\omega_1}$ Cohen reals are added (i.e. force with $\{p \mid \operatorname{dom}(p) \in [\aleph_{\omega_1}]^{<\omega}, \operatorname{range}(p) = \{0, 1\}\}$ over a model of GCH) that

$$\kappa_B(2^\omega) = \kappa_B(2^{\omega_1}) = \kappa_B(2^{\omega_2}) = \aleph_{\omega_1}.$$

The next remark is due to E. van Douwen (letter December, 1979). Consider the following cardinal functions for a space X. The cellularity (or chain condition) of X is denoted $c(X)$. The density of X is denoted $d(X)$. The π-weight of X, $\pi(X)$, is defined to be the minimum cardinality of a π-base for X, where B is a π-base for X iff B is a family of nonempty open sets and for all nonempty open sets U there is a $V \in B$ such that $V \subseteq U$. If X is a compact zero-dimensional space, then $\pi(X)$ is what Boolean algebraists call the density of the clopen algebra of X. The weight of X is denoted $w(X)$. For background see Juhasz (1971). Let $U(\kappa)$ the space of uniform ultrafilters on κ. Then one has

$$\kappa^+ \leq c(U(\kappa)) \leq d(U(\kappa)) \leq \pi(U(\kappa)) \leq w(U(\kappa)) = 2^\kappa.$$

Baumgartner (1976) has shown that adding any number of Cohen reals to a model of $2^{\omega_1} = \omega_2$ (so that $\omega_3 \to (\omega_2)^2_{\omega_1}$) gives a model in which it is impossible to find ω_3 almost disjoint elements of $[\omega_1]^{\omega_1}$ (see also Jech (1978), p. 432). Thus it is consistent to have $c(U(\omega_1)) < 2^{\omega_1}$. Note that in the model of Theorem 2A the following holds:

$$2^\omega = 2^{\omega_1} = \aleph_{\omega+1} \quad \text{and } \exists B \subseteq [\omega_1]^{\omega_1}(|B| = \aleph_\omega \text{ and } \forall X \in [\omega_1]^{\omega_1} \exists Y \in B(Y \subseteq X)).$$

Since $\kappa_B(2^{\omega_1}) = \aleph_\omega$, in fact we have that $\pi(U(\omega_1)) = \aleph_\omega$. Since $c(U(\omega)) = 2^\omega$ we have also that it is consistent that $\pi(U(\omega_1)) < \pi(U(\omega))$. Also we can show that both

$$\omega_2 = d(U(\omega_1)) < \pi(U(\omega_1)) = 2^{\omega_1} \quad \text{and}$$
$$\omega_2 = d(U(\omega_1)) < \pi(U(\omega_1)) < 2^{\omega_1}$$

are consistent. Let M be a model of $2^{\omega_1} = \omega_2$ plus $2^{\omega_2} \geq \kappa$ and, working in M, let $\boldsymbol{Q}_\kappa = \{p \mid \text{dom}(p) \in [\kappa]^{<\omega}, \text{range}(p) = \{0, 1\}\}$. Then I claim that if G is $\boldsymbol{Q}_\kappa$-generic over M, then in $M[G]$ it is true that $d(U(\omega_1)) = \omega_2$. To see this note that in M, since $2^{\omega_2} \geq \kappa$, it is true that $d(2^\kappa) = \omega_2$, and so $\boldsymbol{Q}_\kappa$ is the union of ω_2 centered sets $\boldsymbol{C}_\alpha$ for $\alpha < \omega_2$. For each $X \in [\omega_1]^{\omega_1} \cap M$ and $f \in \omega_2^X \cap M$ define the centered set $A_{X,f} \subseteq [\omega_1]^{\omega_1}$ in $M[G]$ by

$Y \in A_{X,f}$ iff $\exists \langle q_\alpha: \alpha \in X\rangle \in M$, a Δ-system with root in G,
$\forall \alpha \in X(q_\alpha \in Q_{f(\alpha)}$, and $q_\alpha \Vdash$ "$\alpha \in Y$").

Kunen (unpublished) had previously been able to show the consistency of $d(U(\omega_1)) = \omega_2$ plus $2^{\omega_1} = \omega_3$, by an inductive argument in the same model, but his argument did not generalize to $2^{\omega_1} > \omega_3$.

I do not know whether it is possible to have $c(U(\omega_1)) < d(U(\omega_1))$.

The model of Theorem 2A shows that $\kappa_B(2^\omega) > \kappa_B(2^{\omega_1})$ is possible. Our next theorem brings these cardinals down a little.

THEOREM 2B. *It is consistent that* $\kappa_B(2^\omega) = \omega_2$ *and* $\kappa_B(2^{\omega_1}) = \omega_1$.

PROOF. Recall that a family of sets F is a Δ-system iff there is a set R, called the root of F, such that R is the intersection of any two distinct elements of F. A family of partial functions is a Δ-system iff the domains of the functions form a Δ-system and all functions agree on the root. Some versions of the Δ-system lemma seem to be implied by the results of Pondiczery (1944), Sanin (1946), Marczewski (1947), Bochstein (1948), and Mazur (1952) (see Ross and Stone (1964) or Comfort and Negrepontis (1972) for these references). The countable version needed here was first proved by Erdös and Rado, and also independently by Michael with a different proof. For references to these and another proof due to Davies, see Williams (1977). A very popular proof based on stationary sets and the pushing down lemma can be found in the appendix of Shelah (1978). All this history is to excuse my presentation of yet another proof of the Δ-system lemma, this one based on the Lowenheim-Skolem theorem.

LEMMA (CH). *Every family of* ω_2 *countable sets contains a* Δ*-system of cardinality* ω_2.

PROOF. We may assume we are given $F = \langle A_\alpha: \alpha < \omega_2\rangle$ with each $A_\alpha \in [\omega_2]^{\leq\omega}$. Let M be an elementary substructure of $\langle H_{\omega_3}, \varepsilon\rangle$ with $F \in M$, $M^\omega \subseteq M$, and $|M| = \omega_1$. H_κ is the family of sets whose transitive closure is of cardinality less than κ. We obtain M as the union of an ω_1 length elementary chain with the property that $M_\alpha^\omega \subseteq M_{\alpha+1}$. Although M is not transitive it is true that $M \cap \omega_2 = \alpha$ is an ordinal.

If $T = M \cap A_\alpha$, then since A_α is countable and $M^\omega \subseteq M$ it must be that $T \in M$. By elementarity M must model "for unboundedly many $\gamma < \omega_2$, $A_\gamma \cap \gamma = T$." And thus by elementarity the same must hold in H_{ω_3}, and it is easy now to build a Δ-system of cardinality ω_2 and root T. $\square$

The functional version of the Δ-system lemma follows immediately by applying the lemma to the domains of the functions and noting that if R is the root there are only $2^\omega = \omega_1$ possible $f \restriction R$.

In the stationary set proof one gets some extra information, e.g. a Δ-system with a stationary index set. In this proof we get the extra information that only ω_1 roots are needed for all but ω_1 of the sets, e.g.,

$$\forall F \subseteq [\omega_2]^{\leq\omega}\ |F| = \omega_2 \exists H \subseteq [\omega_2]^{\leq\omega}$$
$$|H| \leq \omega_1\ \forall A \in F - H \exists\ \Delta\text{-system } G \subseteq F$$
$$A \in G,\ |G| = \omega_2,\ \text{root of } G \in H.$$

We now prove Theorem 2B. Let M be a model of GCH. Consider forcing with

$$\boldsymbol{P} = \{p \mid \mathrm{dom}(p) \in [\omega_3]^{\leq\omega},\ \mathrm{range}(p) = \omega^{<\omega}\} \quad \text{and}$$
$$p \leq q \text{ iff } \forall\alpha \in \mathrm{dom}(p) \cap \mathrm{dom}(q),\ p(\alpha) \leq q(\alpha).$$

The partial order $\boldsymbol{P}$ adds ω_3 Cohen reals with countable support. Also ω_1^M is collapsed (see exercise E4 of Chapter 8 of Kunen (1980)). For a hint suppose $\langle g_n : n < \omega\rangle$ are the first ω many Cohen reals. For each $n < \omega$ define $h_n \in \omega^\omega$ inductively by $h_n(0) = g_0(n)$ and $h_n(m + 1) = g_{m+1}(h_n(m))$. Now define $z_n \in 2^\omega$ by $z_n(m) = \frac{1}{2}(1 + (-1)^{h_n(m)})$ and show $(2^\omega)^M = \{z_n : n < \omega\}$. By the Δ-system lemma, $\boldsymbol{P}$ has the ω_2 chain condition, and thus $\omega_1^{M[G]} = \omega_2^M$. Since we are adding Cohen reals it is not hard to see that in $M[G]$, $\kappa_B(2^\omega) = \omega_2 = \omega_3^M$. Thus we need to show that $\kappa_B(2^{\omega_1}) = \omega_1 = \omega_2^M$. Work in M. Let $p \Vdash$ "$X: \omega_2^M \to 2$" and choose $q_\alpha \leq p$ for each $\alpha < \omega_2^M$ so that $q_\alpha \Vdash$ "$X(\alpha) = 0$" or $q_\alpha \Vdash$ "$X(\alpha) = 1$". By the Δ-system lemma there is a $Y \in [\omega_2]^{\omega_2}$ such that $\{q_\alpha : \alpha \in Y\}$ is a Δ-system, and thus for any $Z \in [Y]^\omega$, $q_Z = \bigcup\{q_\alpha : \alpha \in Z\}$ is in $\boldsymbol{P}$. It follows that in $M[G]$ for every $X \in 2^{\omega_1}$ there exists $D \in [\omega_1]^\omega \cap M$ such that $X \restriction D \in M$. And so in $M[G]$, $\kappa_B(2^{\omega_1}) = \omega_1$. $\square$

Note that if $\langle S_\alpha : \alpha \leq \omega_2, \mathrm{cf}(\alpha) = \omega\rangle$ is a club sequence in M, then the proof of Theorem 3 shows that it is a club sequence on $\omega_1^{M[G]}$ in $M[G]$. This gives another proof of a result of Shelah (1979) (see also Devlin (1979)) that club is consistent with $\neg$CH.

Curiously, assuming $\diamondsuit_{\omega_1}$, forcing with any product of copies of $2^{<\omega_1}$ with ω_1 supports does not collapse ω_2. The argument is similar to that of Theorem 6.7 of Baumgartner (1976).

Before ending this section let us consider two other properties of category in the space 2^{ω_1}. First note that there are ω_1 nowhere dense sets whose union is not meager. This contrasts with the fact that under MA + $\neg$CH any union of ω_1 nowhere dense subsets of 2^ω is meager. To see why it is true just let C_α be the set of x in 2^{ω_1} such that for all $n < \omega$, $x(\alpha + n) = 1$, then each C_α is nowhere dense, but the union of the C_α cannot be because every meager subset of 2^{ω_1} is contained in a meager subset of countable support (by the countable chain condition of 2^{ω_1}).

Another property we might consider is the following one. Define Unif(X) to be the smallest cardinality of a nonmeager subset of X. Note that $\kappa \le \lambda$ implies Unif(2^κ) $\le$ Unif(2^λ), since the projection of a nonmeager subset of 2^λ onto 2^κ must be nonmeager in 2^κ. Using the fact that we need only worry about meager subsets of countable support, one can show that Unif(2^ω) = Unif(2^{ω_1}). Assuming the continuum hypothesis the density of $(2^{\omega_2})_{\omega_1}$ is ω_1 (see Comfort and Negrepontis (1972)). This says that there are functions $f_\alpha\colon \omega_2 \to 2$ for $\alpha < \omega_1$ such that for any $X \in [\omega_2]^{\le\omega}$ and $g\colon X \to 2$ there is an $\alpha < \omega_1$ such that $f_\alpha \restriction X = g$. Thus CH implies Unif($2^{\omega_2}$) $= \omega_1$. Is it possible to have Unif(2^ω) $<$ Unif(2^{ω_2})?

§3. The ideal of measure zero sets. For any measure space X define $\kappa_M(X)$ to be the least cardinal κ such that X can be written as the union of κ many sets of measure zero (we assume singletons have measure zero). Given any set I the product measure μ on 2^I is determined by requiring that $\mu(N_s) = 2^{-n}$ where dom(s) $\in [I]^n$, range(s) $= \{0, 1\}$, and $N_s = \{x \in 2^I \mid x \restriction \text{dom}(s) = s\}$. It is well known that κ_M is the same cardinal for 2^ω, $\boldsymbol{R}$ with Lebesgue measure, or any other separable nonatomic measure space (see Halmos (1950)). I do not know if $\kappa_M(2^\omega) = \aleph_\omega$ is possible, but the analogue of Theorem 2A can be proved in the case of measure:

THEOREM 3. *It is consistent that* $\kappa_M(2^\omega) = \aleph_{\omega+1}$ *and* $\kappa_M(2^{\omega_1}) = \aleph_\omega$.

PROOF. Let meas(I) denote the boolean algebra of Borel subsets of 2^I modulo the ideal of measure zero sets. A function $G\colon I \to 2$ is called random over M iff the filter generated by G in meas$(I)^M$ is meas$(I)^M$-generic over M (Solovay (1970)). It is well known that if I_1 and I_2 are any two disjoint sets in M, then given any $F\colon I_1 \cup I_2 \to 2$ and letting $F_1 = F \restriction I_1$ and $F_2 = F \restriction I_2$, then F is (meas$(I_1 \cup I_2)^M$) random over M iff F_1 is (meas$(I_1)^M$) random over M and F_2 is (meas$(I_2)^{M[F_1]}$) random over $M[F_1]$ (see (Pincus and Solovay (1977) and Kunen (1975)). It is proved by using Solovay's characterization of randomness (i.e. avoiding measure zero sets coded in the ground model), the absoluteness of some measure-theoretic notions, and Fubini's theorem in M. A sketch of the proof is to let $A = \{\langle F_1, F_2\rangle \mid F = F_1 \cup F_2$ is (meas$(I)^M$) random over $M\}$, and let $B = \{F_2 \mid F_2$ is (meas$(I_2)^{M[F_1]}$) random over $M[F_1]\}$. Note that A has measure one, and for any F_1, the set B has measure one. Suppose there is a Borel set $C \subseteq 2^{I_1} \times 2^{I_2} = 2^I$ coded in M such that C forces "F_2 is not random over $M[F_1]$". By Fubini's theorem there is an F_1 such that the measure of $(C \cap A)_{F_1}$ (cross section) is one. Choose $F_2 \in (C \cap A)_{F_1} \cap B$, but then $\langle F_1, F_2\rangle \in C$ so F_2 cannot be random over $M[F_1]$.

If $\boldsymbol{B}$ is any complete boolean algebra and θ any sentence, then $[\theta]$ is the sup of all $p \in \boldsymbol{B}$ such that $p \Vdash$ "θ".

To obtain the model of Theorem 3 let M be a model of GCH, and let G be (meas($\aleph_\omega$)) random over M. Suppose $B_\alpha \subseteq 2^\omega$ for $\alpha < \aleph_\omega$ are Borel sets of measure zero coded in $M[G]$. Working in M use the countable chain condition of meas($\aleph_\omega$) to obtain countable sets $A_\alpha \subseteq \aleph_\omega$ such that for each α, B_α is coded in $M[G \restriction A_\alpha]$.

Using a diagonal argument obtain an ω sequence $A \subseteq \aleph_\omega$ in M such that for every α the set $A \cap A_\alpha$ is finite. Let $f\colon \omega \to A$ enumerate A, and let $y(n) = G(f(n))$ for all n. It follows from the (above) product lemma for measure algebras that y is random over $M[G \restriction A_\alpha]$ and thus $y \notin B_\alpha$ for all α. This shows that $\kappa_M(2^\omega) > \aleph_\omega$ in $M[G]$, and since $2^\omega = \aleph_{\omega+1}$ we have $\kappa_M(2^\omega) = \aleph_{\omega+1}$.

A simpler but similar argument shows that $\kappa_M(2^{\omega_1}) \geq \aleph_\omega$ in $M[G]$. Next we prove a lemma due to K. Kunen (μ is product measure on 2^I).

LEMMA. *Given $B_i \subseteq 2^I$ for $i < n$ with $\mu(B_i) \geq \frac{3}{4}$; then*

$$\mu(\{y \in 2^I\colon |\{i\colon y \in B_i\}| \geq \tfrac{5}{8}n\}) \geq \tfrac{1}{3}.$$

PROOF. Let μ' be the counting measure on n, i.e. $\mu'(A) = |A|/n$ for any $A \subseteq n$, and let λ be the product of μ' and μ on $n \times 2^I$. Let $B = \bigcup_{i<n}\{i\} \times B_i$, and note that integrating along the first coordinate shows that $\lambda(B) \geq \frac{3}{4}$. For any $y \in 2^I$ let $B^y = \{i\colon (i, y) \in B\} = \{i\colon y \in B_i\}$ and let $S = \{y\colon |B^y| \geq \frac{5}{8}n\} = \{y\colon \mu'(B^y) \geq \frac{5}{8}\}$. Then by integrating along the second coordinate we get that

$$\lambda(B) \leq \frac{5}{8}\mu(2^I - S) + \mu(S) = \frac{5}{8}(1 - \mu(S)) + \mu(S) = \frac{5}{8} + \frac{3}{8}\mu(S),$$

so $\frac{1}{3} \leq \mu(S)$. $\square$

To show that $\kappa_M(2^{\omega_1}) \leq \aleph_\omega$ we show that no $z \in 2^{\omega_1}$ is random over all $M[G \restriction \aleph_n]$ for $n < \omega$. Suppose $[z \in 2^{\omega_1}$ is random over all $M[G \restriction \aleph_n]$ for $n < \omega] = 1$.

Working in M for each $\alpha < \omega_1$ choose a clopen set $C_\alpha \subseteq 2^{\aleph_\omega}$ such that

$$\mu([z(\alpha) = 1] \triangle C_\alpha) \leq \frac{1}{4}.$$

Choose finite sets $F_\alpha \subseteq \aleph_\omega$ so that for every $x, y \in 2^{\aleph_\omega}$ if $x \restriction F_\alpha = y \restriction F_\alpha$, then $x \in C_\alpha$ iff $y \in C_\alpha$. Choose $m < \omega$ and an ω-sequence $A = \{\alpha_n\colon n < \omega\}$ so that for every $\alpha \in A$ the set $F_\alpha \subseteq \aleph_m$. In $M[G \restriction \aleph_m]$ define $y\colon A \to 2$ by $y(\alpha) = 1$ iff $G \in C_\alpha$ (this can be done since $F_\alpha \subseteq \aleph_m$). Since z is random over $M[G \restriction \aleph_m]$ by the law of large numbers,

$$\lim_{n\to\infty} \frac{|\{k < n\colon y(\alpha_k) = z(\alpha_k)\}|}{n} = \frac{1}{2}.$$

This says that a random real equals a ground model real about half the time (see Feller (1950) for the law of large numbers). Choose $n < \omega$ so that if

$$B = [|\{k < n\colon y(\alpha_k) = z(\alpha_k)\}|/n \leq 9/16],$$

then $\mu(B) > \frac{2}{3}$. If $B_k = [y(\alpha_k) = z(\alpha_k)]$, then by our choice of C_{α_k} the set B_k has measure greater than or equal to $\frac{3}{4}$. Hence by the lemma if $S = \{X \in 2^{\aleph_\omega} | \, |\{k < n\colon x \in B_k\}| \geq \frac{5}{8}n\}$, then $\mu(S) \geq \frac{1}{3}$. It follows by absoluteness that

$$S \Vdash \text{“}|\{k < n | y(\alpha_k) = z(\alpha_k)\}|/n \geq 5/8\text{”}.$$

But this contradicts $\mu(S \cap B) > 0$, since B forces the negation. $\square$

§4. **ω_1-saturated ideals.** A countably additive ideal I in the Borel subsets of 2^ω is ω_1-saturated just in case it is impossible to find ω_1 Borel sets B_α such that $B_\alpha \notin I$ and for all $\alpha \neq \beta$ the set $B_\alpha \triangle B_\beta$ is in I. This is equivalent, by a well-known theorem of Tarski, to saying that the Boolean algebra of Borel sets modulo the ideal I has the countable chain condition. Let MA_κ be the statement "for every partial order satisfying the countable chain condition and family F of dense subsets with $|F| \leq \kappa$ there is a filter meeting each dense set in F" (this is equivalent to A_κ of Martin and Solovay (1970)). By the results of Martin and Solovay (1970) the least κ such that MA_κ fails is the same as the least κ such that there is a nontrivial ω_1-saturated σ-ideal I in the Borel subsets of 2^ω such that 2^ω is the union of κ many members of I. Can κ be singular? It is easy to get a model where, for example, $\mathrm{MA}_{\omega_1} + \neg \mathrm{MA}_{\omega_2} + 2^\omega = \omega_3$. To do it start with a model of $2^\omega = \omega_1$, $2^{\omega_1} = \omega_2$, and $2^{\omega_2} = \omega_4$. Then force MA_{ω_1} in an ω_3 iteration using at each step another partial order of cardinality ω_1 (see Burgess (1977) or Solovay and Tennenbaum (1971)). Since in the resulting model $2^{\omega_2} = \omega_4$ we have $\neg \mathrm{MA}_{\omega_2}$. While we cannot answer the main question, by a clever change of quantifier we can prove

THEOREM 4A. *It is consistent that there exists an ω_1-saturated σ-ideal I in the Borel subsets of 2^ω such that $\aleph_\omega$ is the least κ for which there exists a set $J \subseteq I$ of cardinality κ whose union is 2^ω.*

PROOF. We will use Solovay's almost disjoint sets forcing similar to its use to prove Theorem 2.3 of Martin and Solovay (1970). Let X_α for $\alpha < \omega_1$ be ω_1 almost disjoint subsets of ω. Let $\boldsymbol{Q}$ be the usual finite condition for forcing a Cohen generic map from ω_1 into $\{0, 1\}$ (i.e. $\boldsymbol{Q} = \{p \mid \mathrm{dom}(p) \in [\omega_1]^{<\omega}$ and $\mathrm{range}(p) = \{0, 1\}\}$). We now define a partial order $\boldsymbol{P}$ such that forcing with $\boldsymbol{P}$ produces a set $X \subseteq \omega$ such that if $G\colon \omega_1 \to 2$ is defined by $G(\alpha) = 1$ iff $X_\alpha \cap X$ is finite, then G is $\boldsymbol{Q}$-generic. Define

$$\boldsymbol{P} = \{(p, H) \mid p \in Q \text{ and } H \in [\omega]^{<\omega}\}.$$

$\boldsymbol{P}$ is ordered by $(p, H) \leq (q, K)$ iff $p \supseteq q$, $H \supseteq K$, and for every α if $q(\alpha) = 1$, then $X_\alpha \cap H \subseteq K$. Note that if p and q are compatible in Q, then (p, H) and (q, H) are compatible in $\boldsymbol{P}$. It follows that $\boldsymbol{P}$ has the countable chain condition. If G is $\boldsymbol{P}$-generic, then $X = \bigcup\{H \mid \exists p\ (p, H) \in G\}$ has the required property. For $p = (q, H)$ an element of $\boldsymbol{P}$ let C_p be all the $x \in 2^\omega$ which are potentially (from the point of view of p) the characteristic function of some X. More formally:

$$\begin{aligned} x \in C_p \text{ iff } &[\forall n \in H\ (x(n) = 1)] \\ &\wedge [\forall \alpha\{(q(\alpha) = 1) \Rightarrow (\forall n\ (((x(n) = 1) \wedge (n \in X_\alpha)) \Rightarrow (n \in H)))\}]. \end{aligned}$$

Note that each C_p is closed in the usual topology on 2^ω. Let I be the σ-ideal generated by sets of the form $2^\omega - \bigcup\{C_p\colon p \in D\}$ where $D \subseteq \boldsymbol{P}$ is a maximal antichain. To see that I is an ω_1-saturated σ-ideal in the Borel subsets of 2^ω see §2 of Martin and Solovay (1970). If we think of the C_p for $p \in \boldsymbol{P}$ as basic open sets of a new topology on 2^ω, then in fact I is the σ-ideal of meager sets in this new topology. In the case of the ideal of measure zero sets or the ideal of meager sets (in 2^ω) the next two lemmas are well-known results of Solovay (1970). In both lemmas M is the ground model.

LEMMA 1. *For any Borel set $B \subseteq 2^\omega$ coded in M, $M \models$ "$B \in I$" iff $B \in I$.*

PROOF. The statement "D is a maximal antichain in $\boldsymbol{P}$" is absolute since $\boldsymbol{P} \in M$. For countable sets $D_n \subseteq \boldsymbol{P}$ the statement "$B \subseteq \bigcup_{n<\omega}(2^\omega - \bigcup\{C_p \mid p \in D_n\})$"

is Π^1_1 in a code for B and a code for the sequence of D_n's. It follows by Π^1_1 absoluteness that if $M \models$ "$B \in I$", then B really is an element of I. Conversely if $M \models$ "$B \notin I$", then there is a $p \in P$ such that $M \models$ "$C_p - B \in I$", and so for this p the set $C_p - B \in I$ and thus $B \notin I$. □

Given $y \in 2^\omega$ define $G = \{p | y \in C_p\}$, and conversely, given a nontrivial P-filter G define y to be the characteristic function of $\bigcup\{H: \exists p\ (p, H) \in G\}$.

LEMMA 2. *G is P-generic over M iff y is not in any Borel set in I coded in M.*

PROOF. Suppose we have a maximal antichain $D \subseteq P$ which is in M. If $p \in G \cap D$, then $y \in C_p$ and therefore $y \notin (2^\omega - \bigcup\{C_p: p \in D\})$. Conversely if $G \cap D = \emptyset$, then for every $p \in D$, $y \notin C_p$ and so $y \in (2^\omega - \bigcup\{C_p: p \in D\})$ an element of I. □

Next we prove Theorem 4A. Working in M let P^* be the direct sum of $\aleph_\omega$ copies of P, i.e. $P^* = \{p \mid \text{dom}(p) \in [\aleph_\omega]^{<\omega}$ and $\text{range}(p) = P\}$ with the obvious ordering. We claim that if M is a model of GCH and G is P^*-generic over M, then in $M[G]$ the least κ such that κ many sets from I cover 2^ω is $\aleph_\omega$. For any set $A \subseteq \aleph_\omega$ in M let $G_A = \{p \in G \mid \text{dom}(p) \subseteq A\}$. Suppose that in $M[G]$ we consider a family of $\aleph_n$ Borel sets B_α contained in I. By the countable chain condition there exists a set $A \subseteq \aleph_\omega$ in M of cardinality less than or equal to $\aleph_n$ such that each of the sets B_α is coded in $M[G_A]$. If α is chosen to be an element $\aleph_\omega - A$, then by Solovay's product lemma (1970) the filter $G_{\{\alpha\}}$ is P-generic over $M[G_A]$. It follows from Lemmas 1 and 2 that $2^\omega \neq \bigcup\{B_\alpha | \alpha < \aleph_n\}$. Let $\{B_\alpha | \alpha < \aleph_\omega\}$ be all the sets in I coded in any $M[G_{\aleph_n}]$ for $n < \omega$. By Lemma 2 if this family does not cover 2^ω, then there exists G P-generic over $M[G_{\aleph_n}]$ for all $n < \omega$. This implies that there exists G Q-generic over $M[G_{\aleph_n}]$ for all $n < \omega$. But the proof of Theorem 2A shows that in fact for any $X \in [\omega_1]^{\omega_1}$ there exists $n < \omega$ such that $[X]^{\omega_1} \cap M[G_{\aleph_n}]$ is nonempty. □

It is well known that MA_κ is equivalent to the property that no compact Hausdorff space X with the countable chain condition can be the union of κ nowhere dense sets (see Rudin (1977)). One weakening of MA_κ is the property MA_κ (σ-centered) which says that no compact separable Hausdorff space X can be the union of κ many nowhere dense sets. The partial order equivalence of MA_κ (σ-centered) is gotten by replacing the countable chain condition on P by the stronger requirement that P is the countable union of centered subsets ($q \subseteq P$ is centered iff for every $A \in [Q]^{<\omega}$ there is $p \in P$ such that for every $q \in A$, $p \leq q$) (see Kunen and Tall (1979)). One consequence of MA_κ (σ-centered) (due to Solovay) is the property $P(\kappa)$. The property $P(\kappa)$ holds iff for every family $A \subseteq [\omega]^\omega$ of cardinality less than or equal to κ with each finite subset of A having infinite intersection there exists a set $X \in [\omega]^\omega$ such that for all $Y \in A$ the set $Y - X$ is finite (see Rudin (1977)). Recently M. Bell (1979) has proved the surprising result that $P(\kappa)$ iff MA_κ (σ-centered). Thus by the next theorem the least κ such that MA_κ (σ-centered) fails cannot have countable cofinality.

THEOREM 4B. *The least κ such that $P(\kappa)$ fails cannot have countable cofinality.*

PROOF. Let $Y_\alpha \in [\omega]^\omega$ for $\alpha < \kappa$ have the infinite-finite intersection property, and let κ_n for $n < \omega$ be cofinal in κ. For each $n < \omega$ choose $X_n \in [\omega]^\omega$ so that for each $\alpha < \kappa_n$, $X_n - Y_\alpha$ is finite. For each $\alpha < \kappa$ find $f_\alpha \in \omega^\omega$ so that for all but finitely many n the set $(X_n - f_\alpha(n))$ is included in Y_α. By $P(\kappa_n)$ (a result of E. van

Douwen, see remark, p. 498 of Rudin (1977)) there exists $g_n \in \omega^\omega$ which eventually dominates each f_α for $\alpha < \kappa_n$ (g eventually dominates f means that for all but finitely many n, $g(n) > f(n)$). Let $h \in \omega^\omega$ eventually dominate each g_n for $n < \omega$. Inductively choose $x_n > x_{n-1}$ so that $x_n \in X_n - h(n)$. If $X = \{x_n: n < \omega\}$ then for any α the function h eventually dominates f_α and so $X - Y_\alpha$ is finite. □

§5. **The ideal generated by zero-dimensional sets.** Let I denote in this section the closed unit interval. A topological space is zero dimensional iff it has a clopen basis. A classical theorem of Hurewicz (1928) states that I^ω (the Hilbert cube) is not the union of countably many zero-dimensional subsets (see also Kuratowski (1966)). Hurewicz (1932) used this theorem to prove that CH is equivalent to the existence of an uncountable $X \subseteq I^\omega$ such that every uncountable $Y \subseteq X$ has infinite dimension (infinite dimension, for our purposes, just means not a finite union of zero-dimensional sets). The natural question of how many zero-dimensional sets are necessary to cover I^ω is answered by the following easy proposition.

PROPOSITION. *I^ω is the union of ω_1 zero-dimensional sets.*

PROOF. Choose a set $\{d_\alpha: \alpha < \omega_1\} \subseteq I$ which is ω_1 dense, i.e. for any $x < y$ in I there are uncountably many α such that $x < d_\alpha < y$. For any $\alpha < \omega_1$ let E_α be the zero-dimensional set $I - \{d_\beta | \beta > \alpha\}$. If $Z_\alpha = E_\alpha^\omega$, then it is easy to see that $I^\omega = \bigcup\{Z_\alpha: \alpha < \omega_1\}$. □

One might consider the seemingly weaker property that there is an $X \subseteq I^\omega$ of cardinality 2^ω such that every subset of X of cardinality 2^ω has infinite dimension. But this too is equivalent to CH. By the proposition, if the cofinality of 2^ω is greater than ω_1 such an X cannot exist, since some $Z_\alpha \cap X$ must have cardinality 2^ω. On the other hand if 2^ω has cofinality ω_1 and $\neg$CH, then it is easy to alter the proposition to obtain I^ω as a strictly increasing union of ω_2 zero-dimensional sets, say Z_α, and thus again some $Z_\alpha \cap X$ has cardinality 2^ω.

Henderson (1967) showed that there is an infinite-dimensional compact metric space in which every non-zero-dimensional closed set has infinite dimension. Fedorčuk (1975) showed that assuming CH there is an infinite compact Hausdorff space in which every zero-dimensional closed set is finite.

Consider the space I^{ω_1}. Since I^ω is homeomorphic to a subspace of it, I^{ω_1} cannot be covered by countably many zero-dimensional sets. Let $(I^{\omega_1})_\delta$ be I^{ω_1} with the topology generated by the G_δ subsets of I^{ω_1}. Thus the basic open subsets of $(I^{\omega_1})_\delta$ are of the form $N_s = \{x \in I^{\omega_1} | s \subseteq x\}$ for some $s \in I^{<\omega_1}$.

THEOREM 5. *Every zero-dimensional subset of I^{ω_1} is nowhere dense in $(I^{\omega_1})_\delta$.*

Let κ_Z be the least cardinal κ such that I^{ω_1} is the union of κ many zero-dimensional sets.

COROLLARY A. $\kappa_Z > \omega_1$.

COROLLARY B. $\neg$CH *implies* $\kappa_Z = \omega_2$.

COROLLARY C. *TFAAC* (*the following are all consistent*)

(1) $\omega_2 < \kappa_Z = 2^{\omega_1}$;

(2) $2^\omega = \omega_1 > \omega_2 = \kappa_Z < 2^{\omega_1}$;

(3) $\omega_2 < \kappa_Z < 2^{\omega_1}$.

Let κ_B be $\kappa_B((I^{\omega_1})_\delta)$, and consider the partial order

$$P = \{p \mid \mathrm{dom}(p) \in \omega_1, \mathrm{range}(p) = I\}.$$

Then κ_B is the greatest cardinal κ such that for every family of dense subsets of $\boldsymbol{P}$ of cardinality less than κ there is a generic filter. Theorem 5 says that $\kappa_B \leq \kappa_Z$. Corollary A follows since you can always meet ω_1 dense subsets of $\boldsymbol{P}$. Corollary B is true since assuming $\neg$CH forcing with $\boldsymbol{P}$ collapses ω_2. To prove Corollary C let M be a model of GCH, let γ be any cardinal of M of cofinality greater than ω_1, and force γ many Cohen subsets of ω_1 with countable support (i.e. force with $Q = \{p | \text{dom}(p) \in [\gamma]^{<\omega}, \text{range}(p) = \{0, 1\}\}$). In the resulting model $M[G]$ we have $\kappa_B = 2^{\omega_1} = \gamma$. The easiest way to get (2) is to add to $M[G]$ a family $\{f_\alpha: \alpha < \omega_2\} \subseteq \omega_1^{\omega_1}$ such that for every $g \in \omega_1^{\omega_1}$ there is an $\alpha < \omega_2$ such that for every $\beta < \omega_1$, $g(\beta) < f_\alpha(\beta)$. This can be done with an ω_2 iteration of the order $\boldsymbol{D} = \{(f, \alpha) | f \in \omega_1^{\omega_1}, \alpha < \omega_1\}$ where $(f, \alpha) \leq (g, \beta)$ iff $\alpha \geq \beta$, $f \restriction \beta = g \restriction \beta$, and for all $\gamma, f(\gamma) \geq g(\gamma)$. To see that $\kappa_Z = \omega_2$ let $I = \{x_\alpha | \alpha < \omega_1\}$ and for each $\alpha < \omega_2$ let $Z_\alpha = \{x \in I^{\omega_1} |$ if the βth coordinate of x is X_γ, then $\gamma > f_\alpha(\beta)\}$. Then Z_α is zero dimensional and $I^{\omega_1} = \bigcup\{Z_\alpha | \alpha < \omega_2\}$. To get (3) just do this iteration a little longer.

Next we prove Theorem 5. Suppose $Z \subseteq I^{\omega_1}$ is zero dimensional. We must show that for every $s \in I^{<\omega_1}$ there is a $t \supseteq s$ such that $N_t \cap Z$ is empty. We may as well assume s is the empty sequence, and also that Z is dense in the usual topology on I^{ω_1}. Since Z is zero dimensional it has a nontrivial clopen set, so let U and V be open subsets of I^{ω_1} such that $U \cap V \cap Z$ is empty, $U \cap Z$ and $V \cap Z$ are nonempty, and $Z \subseteq U \cup V$. Now since Z is dense in I^{ω_1} it follows that U and V are disjoint nonempty open sets in I^{ω_1}. Let $W = \text{int}(\text{cl}(U))$, and note that $C = \text{cl}(W) - W$ is disjoint from Z and nonempty since $U \subseteq W$ and $W \cap V = \emptyset$. To find $N_t \subseteq C$ it is enough to see that C has countable support, i.e. there is an $\alpha < \omega_1$ such that for any $x, y \in I^{\omega_1}$ if $x \restriction \alpha = y \restriction \alpha$, then $x \in C$ iff $y \in C$. To see this it is enough to see that W has countable support. To see this it is enough to see that the closure of U has countable support. To see this note that I^{ω_1} satisfies the countable chain condition and hence if H is a maximal family of disjoint basic open sets contained in U, then $\text{cl}(U) = \text{cl}(\bigcup H)$. Of course all of this is immediate from Bochstein's theorem (see Ross and Stone (1964)) which says that open disjoint subsets of I^{ω_1} can be separated by countably supported open sets. □

R. Pol (letter December, 1979) remarks that another proof of Theorem 5 can be given using some results in Pol and Puzio-Pol (1976), particularly Proposition 3 on p. 66.

We end this section with some open problems.

Can κ_Z be strictly greater than κ_B?

Is "$2^\omega = \omega_1 + 2^{\omega_1} = \omega_2 + 2^{\omega_2} = \omega_5 + I^{\omega_2}$ the union of ω_3 zero-dimensional sets" consistent?

Is I^ω with the box topology (i.e. any $\Pi_{n<\omega} U_n$ is open where $U_n \subseteq I$ is open) the union of countably many zero-dimensional sets?

NOTE ADDED IN PROOF, SEPTEMBER 1981. Recently D. Fremlin has shown that the least κ such that MA_κ fails cannot have countable cofinality. However, K. Kunen has shown that it is consistent that it be $\aleph_{\omega_1}$.

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