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Covering 2^ω with ω_1 Disjoint Closed Sets

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Dedicated to Professor S. C. Kleene on the occasion of his 70th birthday

Abstract: It is shown that 2^ω is the ω_1 union of meager sets does not imply 2^ω is the ω_1 union of disjoint non-empty closed sets and the latter does not imply CH.

In HAUSDORFF (1934) he showed that 2^ω is the ω_1 union of strictly increasing G_δ sets. It follows that 2^ω is the ω_1 union of disjoint non-empty $F_{\sigma\delta}$ sets. FREMLIN and SHELAH (1980) proved the following theorem.

Theorem 1. *The following are equivalent.*
(1) 2^ω *is the* ω_1 *union of strictly increasing* F_σ *sets.*
(2) 2^ω *is the* ω_1 *union of meager sets.*
(3) 2^ω *is the* ω_1 *union of disjoint non-empty* G_δ *sets.*

Proof.
(3)⇒(2) see FREMLIN and SHELAH (1980).
(2)⇒(1) Every meager set is contained in a meager F_σ set.
(1)⇒(3) Cover 2^ω with closed sets C_α for $\alpha<\omega_1$ so that no countable subcollection covers. Note that $C_\alpha - \bigcup\{C_\beta\colon \beta<\alpha\}$ are disjoint G_δ sets.

Theorem 2 (Luzin, see KURATOWSKI (1958a, p.348)). *Every* F_σ ($G_{\delta\sigma}$) *set in* 2^ω *can be written as the disjoint countable union of closed* (G_δ) *sets.*

Thus the only remaining case of disjoint ω_1 coverings of 2^ω by Borel sets is:

(C) 2^ω is the ω_1 union of non-empty disjoint closed sets.

Remark. By a theorem of Sierpinski (see Kuratowski (1958b, p.173)) the open unit interval cannot be written as the disjoint countable union of closed (in the closed unit interval) sets. Nevertheless (C) is equivalent to the same statement with 2^ω replaced by any uncountable Polish space.

Theorem 3. *2^ω can be partitioned into ω_1 disjoint non-empty closed sets iff some uncountable Polish space can be iff all uncountable Polish spaces can be.*

Proof: If some uncountable Polish space can be partioned, then ω^ω can be, since every such space is the continuous image of ω^ω. Suppose $\omega^\omega = \bigcup\{C_\alpha : \alpha<\omega_1\}$ where the C_α are nonempty disjoint closed sets. By the proof of Lemma 7 we may assume each C_α is nowhere dense. It is easy to build $P \subseteq \omega^\omega$ compact perfect so that $\exists C_{\alpha_n}$ for $n<\omega$ such that each C_{α_n} is nowhere dense in P and $\bigcup\{C_{\alpha_n}: n<\omega\}$ is dense in P. P cannot be covered by countably many of the C_α's since then $P\cap\bigcup\{C_{\alpha_n}: n<\omega\}$ would be a dense meager (in P) G_δ set. Hence we conclude 2^ω can be partitioned. Next we show the unit interval $[0,1]$ can be partitioned. Assume $2^\omega = \bigcup\{C_\alpha: \alpha<\omega_1\}$ where the C_α are disjoint nowhere dense closed sets. By a back and forth argument it is not hard to show that for any two dense countable subsets of 2^ω there is a homeomorphism of 2^ω taking one to the other. Let E be $\{x\in 2^\omega : \exists n \forall m>n\ x(m)=1 \text{ or } \forall m>n\ x(m)=0\}$. We may assume that for every $\alpha<\omega_1$ $|C_\alpha \cap E| \leqslant 1$. Define the map F from 2^ω to $[0,1]$ by

$$F(x) = \sum\left\{\frac{x(n)}{2^{n+1}} : n<\omega\right\}.$$

Let $D_\alpha = F''C_\alpha$. Hence by lumping together the distinct pairs of D_α's which intersect we partition $[0,1]$. Now let X be any uncountable Polish space, we may assume X has no isolated points. Embed X into $[0,1]^\omega$, and if some projection of X contains an interval, then decompose that interval and pull the decomposition back to X. Hence we may assume X is zero dimensional. Thus either X contains a clopen set homeomorphic to 2^ω or it doesn't in which case X is homeomorphic to ω^ω and in either case we are done.

The following theorem was first proved by J. Baumgartner (unpublished) and rediscovered by the author and others.

Theorem 4. (C)$\not\Rightarrow$CH.

Proof. Let M be a model of $\neg$CH. Construct an ω_1 length c.c.c. Solovay and Tennenbaum (1971) extension. For $X\subseteq 2^\omega$ define the partial order $\mathbb{P}(X)$. Conditions are finite consistent sets of sentences of the form "$[s]\cap C_n = \varnothing$" or "$x\in C_n$" where $n<\omega, x\in X, s\in 2^{<\omega}$. Then $F=\bigcup\{C_n: n<\omega\}$ will be a meager (in fact measure zero) F_σ set covering X. (See Miller (1979) for similar arguments.) Iterate ω_1 times to get M_α for $\alpha\leqslant\omega_1$ so that $M_{\alpha+1}$ is gotten by forcing with $\mathbb{P}(2^\omega - \bigcup\{F^\beta: \beta<\alpha\})$ in M_α

creating the F_σ set F^α. An easy density argument shows that the F^α's are disjoint. By c.c.c. $M_{\omega_1} \models$ "$2^\omega = \bigcup\{F^\beta\colon \beta<\omega_1\}$." Note that in 2^ω any F_σ set is the countable union of disjoint closed sets. Since $F=\bigcup\{C_n\colon n<\omega\}$ implies

$$F=\bigcup_n C_n - \Big(\bigcup_{m<n} C_m\Big),$$

it is enough to see this for F_σ sets of the form $C\cap G$ where C is closed and G is open, but G is the disjoint union of countably many clopen sets.

Note that in the above model 2^ω is the ω_1 union of measure zero sets. Is this implied by (C)? The answer is no by the following theorem of Stern (1977), also discovered later but independently by K. Kunen.

Theorem 5. (C) *holds in any random real extension of a model of* CH.

Proof. Let $(\mathbb{B},\mu)$ be any measure algebra in the ground model M. Every element of 2^ω in $M^{\mathbb{B}}$ is random with respect to some Borel measure on 2^ω in M. (For any x such that $[\![x\in 2^\omega]\!]=1$ consider the Borel measure $\nu(B)=\mu[\![x\in B]\!]$.) Every Borel measure ν on 2^ω is regular (see Royden (1968, p. 305)), so for any $E\subseteq 2^\omega$ Borel,

$$\nu(E)=\sup\{\nu(C): C\subseteq E \text{ and } C \text{ is closed}\}$$

and for any closed C,

$$\nu(C)=\inf\{\nu(D): C\subseteq D \text{ and } D \text{ is clopen}\}.$$

Since M models CH there are at most ω_1 Borel measures on 2^ω in M, so it is easy to construct disjoint F_σ sets F^α for $\alpha<\omega_1$ so that for every Borel measure ν in M, $\exists\alpha<\omega_1$ so that $\nu(\bigcup\{F^\beta\colon \beta<\alpha\})=1$.

Theorem 6. 2^ω *is the* ω_1 *union of meager sets does not imply* (C).

Proof. Any $C\subseteq 2^\omega$ closed is coded by a tree $T\subseteq 2^{<\omega}$ whose set of infinite branches

$$[T]=\{x\in 2^\omega: \forall n<\omega \quad x\restriction n\in T\}$$

is C. Perfect set forcing (Sacks, 1971) corresponds to forcing with perfect trees $T\subseteq 2^{<\omega}$ (perfects means $\forall s\in T$ there are incompatible extensions of s in T). $T\leqslant S$ iff $T\subseteq S$. Given $C_\alpha:\alpha<\omega_1$ disjoint non-empty closed subsets of 2^ω, $\mathbb{P}$ will be a suborder of perfect set forcing defined as follows:

$$T\in\mathbb{P} \text{ iff } T \text{ is perfect and for every } \alpha<\omega_1, C_\alpha \text{ is meager in } [T].$$

C meager in $[T]$ iff $\forall s\in T\,\exists t\supseteq s\; t\in T$ and $[T_t]\cap C=\varnothing$, where $T_t=\{r\in T: r\subseteq t \text{ or } t\subseteq r\}$. This modification is similar to that of Shelah.

Lemma 7. *$\mathbb{P}$ is not empty.*

Proof. For each $\alpha<\omega_1$ choose $x_\alpha\in C_\alpha$. Let $T=\{s\in 2^{<\omega}$: for uncountably many $\alpha, s\subseteq x_\alpha\}$. Then $T\in\mathbb{P}$.

Just as in perfect set forcing if G is $\mathbb{P}$-generic, then $x=\bigcup\bigcap G$ is an element of 2^ω and $G=\{T\in\mathbb{P}: x\in[T]\}$. Note that for any $\alpha<\omega_1, \Vdash$"$x\notin \bar{C}_\alpha$" is the closed set in the extension with the same code as C_α), because $\forall T\in\mathbb{P}\,\exists t\in T[T_t]\cap C_\alpha=\varnothing$, so $[T_t]\Vdash$"$x\notin\bar{C}_\alpha$."

Starting with M a model of CH an ω_2 iteration with countable support (as was done in LAVER (1976)) will be used to obtain a model N, where on each step some sequence of disjoint non-empty closed sets will be taken care of with the corresponding order $\mathbb{P}$. Provided sufficient care is taken, N will then model $\neg$(C). It will then suffice to show that $N\models$"$2^\omega=\bigcup\{C: C$ is closed nowhere dense and coded in $M\}$." For expository purposes we first show that the above statement holds when $N=M[G]$ for G $\mathbb{P}$-generic over M.

Lemma 8. *Let $T\in\mathbb{P}$ and $F\subseteq[T]$ finite.*

(a) If $T\Vdash$"$W_{i<N}\Theta_i$" *where $N<\omega$, then*

$$\exists S\leqslant T\, F\subseteq[S]\,\exists G\subseteq N\left[\operatorname{card} G=\operatorname{card} F \text{ and } S\Vdash\text{“}\underset{i\in G}{W}\Theta_i\text{”}\right].$$

(b) If $T\Vdash$"$\tau\in M$", then $\exists S\leqslant T\; F\subseteq[S]\;\exists G\in M$ countable and $S\Vdash$"$\tau\in G$".

Proof. Choose $n<\omega$ so that for every $x,y\in F$ $(x\neq y\Rightarrow x\restriction n\neq y\restriction n)$. For $x\in F$ let

$$R_x=\{t\in T\text{: }\exists m\geqslant n\; t=x\restriction m^\frown\langle 1-x(m)\rangle\}$$

and $R=\bigcup\{R_x\text{: } x\in F\}$. Choose $T'\leqslant T$ so that $R\subseteq T'$ and for all $s\in R$ $\exists m<N\; T'_s\Vdash$"Θ_m"(for (b): $\forall s\in R\,\exists x_s\in M\; T'_s\Vdash$"$\tau=x_s$" then let $S=T'$ and $G=\{x_s\text{: } s\in R\}$). Since $N<\omega$ $\forall x\in F\,\exists m_x<N\,\exists R'_x\subseteq R_x$ infinite so that for all $s\in R'_x$ $T'_s\Vdash$"Θ_{m_x}". Let $G=\{m_x\text{: } x\in F\}$ and $S=\bigcup\{T'_s: s\in\bigcup\{R'_x: x\in F\}\}$.

The stem of T is the unique $s\in T$ such that $T_s=T$ and $s^\frown\langle 0\rangle, s^\frown\langle 1\rangle\in T$. The nth level of T (Lev_n (T)) is defined by induction on $n<\omega$. Lev_0 $(T)=\{$stem of $T\}$.

$$\mathrm{Lev}_{n+1}(T)=\{\text{stem of } T_{s^\frown\langle i\rangle}\text{: } s\in\mathrm{Lev}_n(T) \text{ and } i=0,1\}.$$

For any $s\in T$ define x_s^T to be the lexicographical least element of $[T_s]$.

Definition. $T \leqslant^n S$ iff

(a) $T \leqslant S$ and $\mathrm{Lev}_n(T) = \mathrm{Lev}_n(S)$.

(b) $\forall t \in \mathrm{Lev}_n(S)\, x_t^S \in [T]$.

(c) $\forall t \in \mathrm{Lev}_n(S)$ if $x_t^S \in C_\alpha$ (α is necessarily unique if it exists, since the C_α are disjoint), then $\exists s \supseteq t\ s \in \mathrm{Lev}_{n+1}(T)$ such that $[T_s] \cap C_\alpha = \varnothing$.

Lemma 9. *If for each* $n<\omega \quad T^{n+1} \leqslant^n T^n$, *then* $\bigcap \{T_n \colon n<\omega\} = T \in \mathbb{P}$.

Proof. Since $\forall n \forall m [m \geqslant n \rightarrow \mathrm{Lev}_n(T^m) = \mathrm{Lev}_n(T)]$, T is perfect. Suppose for some $\alpha < \omega_1$ and $s \in T, [T_s] \subseteq C_\alpha$. Choose $n<\omega$ so that $s \subseteq t \in \mathrm{Lev}_n(T)$. By (b) $x_t^{T^n} \in [T]$, so $x_t^{T^n} \in C_\alpha$. But by (c) $\exists r \in \mathrm{Lev}_{n+1}(T^{n+1}) = \mathrm{Lev}_{n+1}(T)$ such that $[T_r^{n+1}] \cap C_\alpha = \varnothing$, contradiction.

Lemma 10. *Let* $T \in \mathbb{P}$ *and* $n<\omega$.

(a) *If* $T \Vdash$"$W_{i<N} \Theta_i$," *where* $N<\omega$, *then* $\exists S \leqslant^n T \exists G \subseteq N$ $\mathrm{card}\, G \leqslant 2^{n+1}$ *and* $S \Vdash$"$W_{i \in G} \Theta_i$".

(b) *If* $T \Vdash$"$\tau \subseteq M$ *is countable*", *then* $\exists S \leqslant^n T \exists G \in M$ *countable and* $S \Vdash$"$\tau \subseteq G$".

Proof. (a) Let $F = \{x_s^T \colon s \in \mathrm{Lev}_{n+1}(T)\}$. Applying Lemma 8(a) get $R \leqslant T$ with $F \subseteq [R], G \subseteq N, \mathrm{card}\, G \leqslant 2^{n+1}, R \Vdash$"$W_{i \in G} \Theta_i$". Since $F \subseteq [R]$ $\mathrm{Lev}_n(R) = \mathrm{Lev}_n(T)$. Let $D = \bigcup \{C_\alpha \colon F \cap C_\alpha \neq \varnothing\}$. Since this is a finite union D is nowhere dense in $[R]$. $\forall s \in \mathrm{Lev}_n(R)$ find $t_s \in R$ such that $t \supseteq s^\frown\langle 1 \rangle$ and $[t] \cap D = \varnothing$. Let $S = \bigcup \{R_{s^\frown\langle 0 \rangle}, R_{t_s} \colon s \in \mathrm{Lev}_n(R)\}$.

(b) Let $T_0 = T$. Using Lemma 8(b) and the argument above, build a sequence $T_{m+1} \leqslant^m T_m, G_m \in M$ countable for $m<\omega$ such that $T_m \Vdash$" the m^{th} element of τ is in G_m." Then by Lemma 9 $S = \bigcap_{m<\omega} T_m \in \mathbb{P}$ and $S \Vdash$"$\tau \subseteq \bigcup_{m<\omega} G_m$". If in addition $\forall i<n\ T_{i+1} \leqslant^n T_i$, then $S \leqslant^n T$.

Let $X = \{f \in \omega^\omega \colon \forall n\, f(n) < 2^{n^2}\}$. Suppose $T \Vdash$"$\tau \in X$", then using Lemma 10(a) build a sequence $T^{n+1} \leqslant^n T^n, T^0 = T, G^n \subseteq \omega$ with $\mathrm{card}\, G^n \leqslant 2^{n+1}$, and $T^{n+1} \Vdash$"$\tau(n) \in G_n$". Let $S = \bigcap_{n<\omega} T^n$, so $S \in \mathbb{P}$ by Lemma 9, and $S \Vdash$"$\forall n\, \tau(n) \in G_n$". But $C = \{f \in X \colon \forall n\ f(n) \in G_n\}$ is closed nowhere dense in X. Thus if G is $\mathbb{P}$-generic over M, then

$$M[G] \vDash \text{“}X = \cup \{C \colon C \text{ closed nowhere dense in } X \text{ and coded in } M\}\text{”}.$$

But X is homeomorphic to 2^ω, so

$$M[G] \Vdash \text{“}2^\omega = \cup \{C \colon C \text{ closed nowhere dense in } 2^\omega \text{ and coded in } M\}.\text{”}$$

We will do a Laver style iteration argument (Laver, 1976). Assume for each $\alpha < \omega_2$ we have a partial order $\mathbb{P}_\alpha$ and a term $\langle C_\beta^\alpha \colon \beta < \omega_1 \rangle$ so that $\Vdash_\alpha$"$\langle C_\beta^\alpha \colon \beta < \omega_1 \rangle$ are disjoint nowhere dense closed subsets of 2^ω". Then for each $\alpha \leqslant \omega_2$ [$p \in \mathbb{P}_\alpha$ iff $\forall \beta < \alpha\ p\restriction_\beta \Vdash$ "$p(\beta) \in \mathbb{P}(\langle C_\gamma^\beta\colon \gamma < \omega_1 \rangle)$" and for all but countably many γ (called the support of p) $p(\gamma)$ is a canonical term for $2^{<\omega}$. Lemma 5 thru Lemma 10 of LAVER (1976) are proved in this case mutatis mutandis. (Change Lemma 6(i) to read: If $k < \omega$ and $p \Vdash$ "$W_{j<k}\Theta_j$", then there is an $I \subseteq \{0, 1, \ldots, k-1\}$ with card $I \leqslant 2^{(n+1)i}$ and a p' such that $p' \leqslant_F^n p$ and $p' \Vdash$ "$W_{j \in I}\Theta_j$." Also $\leqslant$ is reversed in LAVER (1976).)

In particular for any G $\mathbb{P}_{\omega_2}$-generic over $M, M[G] \vDash$ "$\forall x \in \omega^\omega$ if $\forall n$ $x(n) < 2^{n^4}$, then $\exists g \in M \forall n$ card $g(n) \leqslant 2^{n^3}$ and $\forall n\ x(n) \in g(n)$". Hence as above $M[g] \vDash$ "2^ω is the ω_1 union of meager sets". Also there is a sequence $\langle W_\beta\colon \beta < \omega_2 \rangle$ in M such that for each β, W_β is dense in $\mathbb{P}_\beta$ and card$(W_\beta/\equiv) \leqslant \aleph_1$. So by a bookkeeping argument we can insure that $M[G] \vDash$ "For every sequence $C_\alpha\colon \alpha < \omega_1$ of closed disjoint nowhere dense subsets of $2^\omega, \exists \beta < \omega_2 \langle C_\alpha\colon \alpha < \omega_1 \rangle = \langle C_\alpha^\beta\colon \alpha < \omega_1 \rangle$."

Remark. It easily follows from arguments similar to those above that no real in $M[G]$ is random over M, so $M[G] \vDash$ "2^ω is the ω_1 union of measure zero sets" (see MILLER (1980)).

Tall remarks that Booth (1968, unpublished) proved that MA implies the closed unit interval is not the union of less than $|2^\omega|$ disjoint nonempty closed sets, and Weiss (1972, unpublished) rediscovered this and proved, for example, that MA implies no compact perfectly normal space is the union of κ many disjoint closed sets for any κ with $\omega < \kappa < |2^\omega|$.

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