Absoluteness of convexly orderable

A. Miller
Feb 2012

During a talk by Vincent Guingona [1], Uri Andrews raised the question of whether the property of being convexly orderable is set theoretically absolute. We show that it is absolute when the language is countable but it isn't for uncountable languages.

Definition 1 *An L-structure M is convexly orderable iff there is a linear order $\leq$ on M and a function $\phi \mapsto k_\phi$ from L-formulas to ω such that for any $\phi(x, \vec{y})$ every set of the form $\phi(M, \vec{a})$ for some $\vec{a} \in M$ can be written as the union of $\leq k_\phi$ convex sets.*

Theorem 2 *Suppose V is a transitive model of set theory, $M \in V$, and V models that M is a structure in a countable language L which is not convexly orderable. Then any transitive model of set theory $W \supseteq V$ also models that M is not convexly orderable.*

Proof
In V let ϕ_n for $n < \omega$ list all L-formulas and suppose that in W there is a function $f : \omega \to \omega$ and linear order on M such that each set of the form $\phi_n(M, \vec{a})$ is the union of $\leq f(n)$ convex sets. For each $N < \omega$ note that V models there is a linear order on M such that for every $n < N$ and each set of the form $\phi_n(M, \vec{a})$ is the union of $\leq f(n)$ convex sets. This follows easily from the compactness theorem of propositional logic using predicate symbols for the linear order and predicate symbols for convex sets required.

In V define a subtree $T \subseteq \omega^{<\omega}$ by $s \in T$ iff there exists a linear order on M such that for every $n < |s|$ each set of the form $\phi_n(M, \vec{a})$ is the union of $\leq s(n)$ convex sets.

Since $f \in [T]$ is branch thru T by absoluteness of well-foundedness T must have a branch g in V. Again by compactness g is a witness showing that V models that M is convexly orderable.
QED

Theorem 3 *There is a structure M in a language of size ω_1 which is not convexly orderable but in every extension of the universe V in which ω_1^V is countable M is convexly orderable.*

Lemma 4 *Let L be the language consisting of countably many unary predicate symbols. Every model in the language of L is convexly orderable.*

Proof
Let P_n for $n < \omega$ be the unary predicate symbols. For each $s \in 2^{<\omega}$ let

$$\rho_s(x) = (\wedge_{s(i)=0} P_i(x)) \wedge (\wedge_{s(i)=1} \neg P_i(x))$$

Using the order on M induced by lexicographical order on 2^ω shows that each $\rho_s(M)$ is convex. The definable subsets of M are boolean combinations of these sets and singletons.
QED

Lemma 5 *Suppose $A_n \subseteq X$ for $n < \omega$ is an independent family. Then there does not exist $k < \omega$ and a linear order on X such that every A_n is the union of $\leq k$ convex subsets.*

Proof
Suppose not. By a Lowenheim-Skolem argument we may assume that X is countable. Hence we may embed it in the unit interval $[0, 1]$. Taking the obvious convex closures we may assume that X is $[0, 1]$. Since changing the elements of an independent family mod finite does not effect independence we may assume each A_n is the union of $\leq k$ open intervals. Cutting down to an infinite subfamily we may assume that there are x_i^n, y_i^n for $i < k$ such that

1. $0 < x_0^n < y_0^n < x_1^n < y_1^n < \cdots < x_{k-1}^n < y_{k-1}^n < 1$
2. $A_n = \bigcup_{i<k}(x_i^n, y_i^n) = \bigcup_{i<k} I_n^i$

Using the compactness of the unit interval we pass to subsequence which such that each x_i^n converges to some x_i and is either strictly increasing, strictly decreasing, or constant. The same is true for the y_i^n. Note that

$$0 \leq x_0 \leq y_0 \leq x_1 \leq y_1 \leq \cdots \leq x_{k-1} \leq y_{k-1} \leq 1$$

and some or all may be equal. Define

$$L_n^i = (x_i^n, x_i^{n+1}] \text{ and } R_n^i = [y_i^{n+1}, y_i^n)$$

where we agree that $[a, b)$ and $(a, b]$ are empty when $b \le a$. Note that x_i is not in L_n^i (by monotonicity) and L_n^i converges to x_i. Similarly for R_n^i and y_i. If $I_n^i = (x_i^n, y_i^n)$ then

$$I_n^i \setminus I_{n+1}^i \subseteq L_n^i \cup R_n^i \text{ and } A_n \setminus A_{n+1} \subseteq \bigcup_{i<k} I_n^i \setminus I_{n+1}^i$$

For large enough n there will be an $\epsilon > 0$ such that $\bigcup_{i<k} L_n^i \cup R_n^i$ is disjoint from $\bigcup_{i<k} B_\epsilon(x_i) \cup B_\epsilon(y_i)$. But then we can choose $m >> n$ such that

$$\bigcup_{i<k} L_m^i \cup R_m^i \subseteq \bigcup_{i<k} B_\epsilon(x_i) \cup B_\epsilon(y_i)$$

and so $(A_n \setminus A_{n+1})$ and $(A_m \setminus A_{m+1})$ are disjoint, contradicting independence.
QED

To prove Theorem 3 let M be the following model in the language consisting of ω_1 unary predicate symbols. $|M| = 2^{\omega_1}$ and for each $\alpha < \omega_1$ the unary predicate

$$A_\alpha = \{x \in 2^{<\omega_1} \ : \ x(\alpha) = 1\}.$$

M is not convexly orderable since if it were there would be a $k < \omega$ and an infinite set of α such that every A_α is the union of $\le k$ convex subsets. Contradicting Lemma 5. On the other hand if ω_1^V is countable, then by Lemma 4 the structure M is convexly orderable.
QED

References

[1] Vincent Guingona, Logic Colloquium UW Madison, Feb 2012.