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WHEN THE CONTINUUM HAS COFINALITY ω_1

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In this paper we consider models of set theory in which the continuum has cofinality ω_1. We show that it is consistent with $\neg$CH that for any complete boolean algebra B of cardinality less than or equal to c (continuum) there exists an ω_1-generated ideal J in $P(\omega)$ (power set of ω) such that B is isomorphic to $P(\omega) \bmod J$. We also show that the existence of generalized Luzin sets for every ω_1-saturated ideal in the Borel sets does not imply Martin's axiom.

Introduction. In §1 we prove our main result that it is consistent with $\neg$CH that every complete boolean algebra of cardinality $\leq c$ is isomorphic to $P(\omega) \bmod J$ for some J ω_1-generated. We think of this as generalizing Kunen's theorem that it is consistent with $\neg$CH that there is an ω_1 generated nonprincipal ultrafilter on ω.

For I an ideal in the Borel subsets of the reals we say that a set of reals X is a κ-I-Luzin set iff X has cardinality κ and for every A in I, $A \cap X$ has cardinality less than κ. If c is regular, then it follows easily from Martin-Solovay **[9]** that MA is equivalent to the statement "for every ω_1-saturated σ-ideal I in the Borels there is a c-I-Luzin set". In §2 we show that the regularity of c is necessary. This answers a question of Fremlin **[5]**.

We also show that it is consistent with $\neg$CH that for every such I there exists an ω_1-I-Luzin set. These results can be thought of as a weak form of the following conjecture.

Conjecture. It is consistent with $\neg$CH that for every c.c.c partial order **P** of cardinality $\leq c$ there exist $\langle G_\alpha : \alpha < \omega_1 \rangle$ an ω_1-sequence of **P**-filters such that for every dense $D \subseteq$ **P** all but countably many G_α meet D.

Note that this is a trivial consequence of CH.

Next we give a result of Kunen that some restriction of the cardinality of **P** (e.g. $(2^{\omega_1})^+$) is necessary in our conjecture. We also show that for every c.c.c. **P** of cardinality $\leq \omega_2$ we can force (without adding reals) the existence of **P**-filters $\langle G_\alpha : \alpha < \omega_1 \rangle$ eventually meeting each dense subset of **P**.

1. ω_1-generated ideals in $P(\omega)$. Sikorski [**12**] showed that every complete boolean algebra of cardinality $\leq c$ is isomorphic to $P(\omega)/J$ for some ideal J. Kunen (see [**7**], p. 289) showed it is consistent with $\neg$CH that there exists a nonprincipal ω_1-generated ultrafilter U on ω, i.e. $P(\omega)$ mod the dual of U is the two element boolean algebra.

THEOREM 1. *It is consistent with* ZFC + $\neg$CH *that for every complete boolean algebra* **B** *of cardinality* $\leq c$ *there exists an* ω_1 *generated nonprincipal ideal I such that* **B** *is isomorphic to* $P(\omega)/I$.

Proof. We begin by describing the model which will be used here and in the next section. Let M_0 be a countable transitive model of ZFC + GCH. Using the usual finite support forcing do an ω_1 iteration where at step $\alpha < \omega_1$ obtain $M_{\alpha+1}$ a model of MA + $c = \aleph_{\alpha+2}$. For $\alpha < \omega_1$ a limit, M_α just models $c = \aleph_{\alpha+1}$ but not MA. Finally M_{ω_1} models that $c = \aleph_{\omega_1}$ and is an ω_1 limit of models of MA. This model (or one very similar to it) was used by Stepräns [**13**] and Bell and Kunen [**2**]. A similar ω_1-iteration (without increasing c) was done by van Douwen and Fleissner [**4**] and also Roitman [**10**].

We will need the following two lemmas of Sikorski:

LEMMA 1.1. (*Sikorski* [**12**] 33.1, p. 141). *Suppose* **B** *is a complete boolean algebra, and* C_0 *is a subalgebra of a boolean algebra* C (*neither of which need be complete*). *Then any homomorphism from* C_0 *into* **B** *can be extended to a homomorphism of* C *into* **B**.

LEMMA 1.2. (*Sikorski* [**12**] 12.2, p. 36). *Suppose* A_0 *generates a Boolean algebra* A *and* h: $A_0 \to B$ *is an arbitrary map into a boolean algebra* B. *Then* h *extends to a homomorphism from* A *into* B *iff for every sequence* a_1, $a_2, \ldots, a_n$ *from* A_0 *and sequence* $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ *of signs* $+, -$;

$$\varepsilon_1 a_1 \wedge \cdots \wedge \varepsilon_n a_n = 0 \Rightarrow \varepsilon_1 h(a_1) \wedge \varepsilon_2 h(a_2) \wedge \cdots \wedge \varepsilon_n h(a_n) = 0.$$

We will be using the proof of the following lemma so we include it here.

LEMMA 1.3. (*Sikorski*) *If* **B** *is any complete boolean algebra of cardinality* $\leq c$, *then there exists an ideal* J *in* $P(\omega)$ *such that* **B** *is isomorphic to* $P(\omega)/J$.

Proof. Let $\mathscr{F}$ be a family of c independent subsets of ω (see Kunen [7], p. 257). that is given any finite sequences $A_1, A_2, \ldots, A_n$ and $B_1, B_2, B_3, \ldots, B_n$ of distinct elements of $\mathscr{F}$

$$A_1 \cap A_2 \cap \cdots \cap A_n \cap (\omega \setminus B_1) \cap (\omega \setminus B_2) \cap \cdots \cap (\omega \setminus B_m)$$

is infinite. Let h be any map from $\mathscr{F}$ onto **B**. By Lemma 1.2 h extends to the subalgebra of $P(\omega)$ generated by $\mathscr{F}$ and by Lemma 1.1, h extends to $P(\omega)$. J is just the kernel of this extension. □

LEMMA 1.4. (*Martin-Solovay, Sikorski*) *Every* c.c.c. *complete boolean algebra of cardinality* $\leq c$ *is isomorphic to* F/I *where* F *is a* σ*-subfield of the Borel subsets of* 2^ω *and* I *is a* c.c.c. σ*-ideal.*

Proof. Theorem 2.3, page 155 of Martin Solovay **[9]** states that every c.c.c. complete boolean algebra of cardinality $\leq c$ is a complete subalgebra of a countably generated complete boolean algebra satisfying the c.c.c. According to Sikorski **[12]**, 31.6 page 136, every countably generated σ-boolean algebra is isomorphic to Borel$(2^\omega)/I$ for some σ-ideal I. So the given algebra **B** is isomorphic to a subalgebra of Borel$(2^\omega)/I$ for some I a c.c.c. σ-ideal. Now let F be a σ-subfield of Borel(2^ω) so that **B** is isomorphic to F/I. □

Note that in the model M_{ω_1}, $2^{\omega_1} > c$ and so any complete boolean algebra of cardinality $\leq c$ must have the c.c.c.. For $\alpha < \omega_1$ let $\Vdash_\alpha$ denote the forcing which has the ground model M_α and as the generic extension M_{ω_1}. Let **B** be any complete boolean algebra of cardinality $\leq c$ in M_{ω_1} and suppose

$$\Vdash_0 \text{“}\mathbf{B} \simeq F/I \text{ where } F \text{ is a } \sigma\text{-subfield of Borel}(2^\omega) \text{ and } I \text{ is a c.c.c. } \sigma\text{-ideal in Borel}(2^\omega).\text{”}$$

For $\alpha < \omega_1$ define (in M_α)

$$F_\alpha = \left\{ A \in \text{Borel}(2^\omega)^{M_\alpha} \colon \Vdash_\alpha \text{“}A \in F\text{”} \right\}$$

and

$$I_\alpha = \left\{ A \in \text{Borel}(2^\omega)^{M_\alpha} \colon \Vdash_\alpha \text{“}A \in I\text{”} \right\}.$$

LEMMA 1.5. (*Kunen*) (*In* M_α) F_α/I_α *is complete and* I_α *has the c.c.c.*

Proof. Clearly F_α is a σ-field and I_α is a σ-ideal so it suffices to show I_α has the c.c.c.. Suppose $\langle A_\beta: \beta < \omega_1\rangle \in M_\alpha$ and for all $\beta \neq \gamma$, $A_\beta \cap A_\gamma \in I_\alpha$. Then since

$$\Vdash_\alpha \text{“}I \text{ is c.c.c.”},$$

$I_\alpha \subseteq I$, and $\Vdash_\alpha$ is c.c.c. forcing, for all but countably many β

$$\Vdash_\alpha \text{“}A_\beta \in I\text{”}$$

and thus $A_\beta \in I_\alpha$. □

Working in M_{ω_1} we build a sequence of functions

$$h_\alpha: P(\omega) \cap M_\alpha \to F_\alpha$$

with $h_\alpha \in M_\alpha$ and such that for $\alpha < \beta$, h_β is an extension of h_α. They also have the following properties:

(i) the map $\hat{h}_\alpha: P(\omega) \cap M_\alpha \to F_\alpha/I_\alpha$ defined $\hat{h}_\alpha(A) = [h_\alpha(A)]_{I_\alpha}$ is a homomorphism;

(ii) for successor ordinals, $\alpha + 1$, $h_{\alpha+1}$ is onto $F_{\alpha+1}$; and

(iii) for successor ordinals, $\alpha + 1$, there exists X_α in the kernel of $\hat{h}_{\alpha+1}$ such that for all $A \in \text{kernel}(\hat{h}_\alpha)$ $A \subseteq^* X_\alpha$ (i.e. $A \setminus X_\alpha$ is finite).

Now suppose we already had the h_α and X_α as above and let us finish the proof of Theorem 1. Working in M_{ω_1} define

$$h: P(\omega) \to F$$

by $h = \bigcup_{\alpha<\omega_1} h_\alpha$ and let $\hat{h}: P(\omega) \to F/I$ be defined by $\hat{h}(A) = [h(A)]_I$. It follows from (i) that $\hat{h}$ is a homomorphism. Since $F = \bigcup_{\alpha<\omega_1} F_\alpha$ it follows from (ii) that $\hat{h}$ is onto. Since kernel $\hat{h} = \bigcup_{\alpha<\omega_1} \text{kernel}(h_\alpha)$ and by (iii) it is ω_1-generated. Hence F/I is isomorphic to $P(\omega)/J$ for J ω_1-generated. Now we indicate how to construct the h_α. Let h_0 by any map from $P(\omega) \cap M_0$ into F_0 such that $\hat{h}_0$ is a homomorphism from $P(\omega) \cap M_0$ into F_0/I_0 and the kernel of $\hat{h}_0$ contains the finite sets. E. g. let U be any nonprincipal ultrafilter in M_0 and define $h_0(A) = 2^\omega$ if $A \in U$ and $h_0(A) = \varnothing$ if $A \notin U$. Suppose $\alpha < \omega_1$ is a limit ordinal and $\langle h_\beta: \beta < \alpha\rangle \in M_\alpha$. Let $Q = \bigcup_{\beta<\alpha}(P(\omega) \cap M_\beta)$ and let h: $Q \to F_\alpha$ be defined by $h = \bigcup_{\beta<\alpha} h\beta$. Let $\hat{h}$: $Q \to F_\alpha/I_\alpha$ be defined by $\hat{h}(A) = [A]_{I_\alpha}$. Since $\bigcup_{\beta<\alpha} I_\beta \subseteq I_\alpha$, the I_β are increasing, and the $\hat{h}_\beta$ are homomorphism, $\hat{h}$ is also a homomorphism.

Working in M_α we see that by Lemma 1.5, F_α/I_α is complete, and by Lemma 1.1 there exists (in M_α) a homomorphism $\hat{h}_\alpha: P(\omega) \cap M_\alpha \to F_\alpha/I_\alpha$ which extends $\hat{h}$. Since the I_α equivalence classes are bigger than the I_β for $\beta < \alpha$ we can pick a representing function h_α: $P(\omega) \cap M_\alpha \to F_\alpha$ for $\hat{h}_\alpha$

which extends h. This does the limit case. Now we do the successor case. Suppose we have h_α: $P(\omega) \cap M_\alpha \to F_\alpha$ with $h_\alpha \in M_\alpha$. Note that $|P(\omega) \cap M_\alpha| = \aleph_{\alpha+1}$ and $M_{\alpha+1}$ is a model of MA and $c = \aleph_{\alpha+2}$. Let P be the kernel of h_α, i.e. $P = h_\alpha^{-1}(I_\alpha)$. Let $Q = (P(\omega) \cap M_\alpha) \setminus P$. Note that for any $B \in Q$ and finite $K \subseteq P$ we have that $B \cap (\cup K)$ is infinite. By Solovay's almost disjoint forcing (see Rudin [**11**]) in $M_{\alpha+1}$ there exists $Y \subseteq \omega$ such that for all $B \in Q$, $B \cap Y$ is infinite and for all $B \in P$, $B \cap Y$ is finite. Let $X_\alpha = \omega \setminus Y$. Note that for all A in the kernel of $\hat{h}_\alpha$, $A \subseteq^* X_\alpha$ and for all B not in the kernel $B \setminus X_\alpha$ is infinite. Define $\hat{h}$: $(P(\omega) \cap M_\alpha) \cup \{X_\alpha\} \to F_\alpha/I_\alpha$ by extending $\hat{h}_\alpha$ and letting $\hat{h}(X_\alpha) = 0$. Let us use Lemma 1.2 to check that $\hat{h}$ extends to a homomorphism from $\mathscr{B}$ = smallest boolean algebra generated by $(P(\omega) \cap M_\alpha) \cup \{X_\alpha\}$ into F_α/I_α. Since $P(\omega) \cap M_\alpha$ is a boolean algebra it is enough to check for each $A \in P(\omega) \cap M_\alpha$:

(i) if $A \cap X_\alpha = \varnothing$, then $\hat{h}_\alpha(A) \wedge 0 = 0$; and

(ii) if $A \cap (\omega \setminus X_\alpha) = \varnothing$, then $\hat{h}_\alpha(A) \wedge \mathbf{1} = 0$.

But (i) is trivial and (ii) follows from the choice of X_α. Next we use the method of independent sets to make $\hat{h}_{\alpha+1}$ onto $F_{\alpha+1}/I_{\alpha+1}$. Using MA, in $M_{\alpha+1}$ there exists a family $\mathscr{F}$ of cardinality $c = \aleph_{\alpha+2}$ of independent mod Q subsets of ω. That is, for any infinite $A \in Q$ and distinct $Y_1, Y_2, \ldots, Y_n, Z_1, Z_2, \ldots, Z_m$ from $\mathscr{F}$

$$A \cap Y_1 \cap Y_2 \cap \cdots \cap Y_n \cap (\omega \setminus Z_1) \cap (\omega \setminus Z_2) \cap \cdots \cap (\omega \setminus Z_m)$$

is infinite. Construct the family $\mathscr{F}$ by induction using the easy consequence of MA that for any family H of infinite subsets of ω with $|H| < c$ there exists $X \subseteq \omega$ such that for all $A \in H$, $A \cap X$ and $A \setminus X$ are both infinite. Choose h: $\mathscr{B} \to F_\alpha$ so that $\hat{h}(A) = [h(A)]_{I_\alpha}$ and h extends h_α. Let k: $\mathscr{B} \cup \mathscr{F} \to F_{\alpha+1}$ extend h and take $\mathscr{F}$ onto $F_{\alpha+1}$. by Lemmas 1.1 and 1.2 there exists $h_{\alpha+1}$: $P(\omega) \cap M_{\alpha+1} \to F_{\alpha+1}$ which extends k and $\hat{h}_{\alpha+1}$: $P(\omega) \cap M_{\alpha+1} \to F_{\alpha+1}/I_{\alpha+1}$ is a homomorphism. This concludes the construction of the h_α's and thus the proof of Theorem 1. □

One question we were unable to answer with this method is the following: Is it consistent with ZFC that there exists an ω_1-generated ideal J in $P(\omega)$ such that $P(\omega_1)$ is isomorphic to $P(\omega)/J$?

Finally, we remark that in M_{ω_1} the measure algebra (the Borel sets modulo the sets of Lebesgue measure zero) has density ω_1, i.e. there is a collection D of ω_1 sets of positive measure such that every set of positive measure contains one from D. This follows from the fact that under MA given any collection F of sets of positive measure such that $|F| < c$, there exists a countable collection $\{C_n: n < \omega\}$ of sets of positive measure such that for every $A \in F$ there exists $n < \omega$ such that $C_n \subseteq A$. Thus in the model M_{ω_1} there exists an atomless finitely additive measure μ on $P(\omega)$ and a family $F \subseteq P(\omega)$ of cardinality ω_1 such that for all $X \subseteq \omega$ and $\varepsilon > 0$ there exists $X_0, X_1 \in F$ such that $X_0 \subseteq X \subseteq X_1$ and

$$\mu(X_1 \setminus X_0) < \varepsilon.$$

2. Luzin sets. Recall that for an ideal I in the Borel sets and a cardinal κ, a set of reals X is called κ-I-Luzin iff X has cardinality κ and every set in I meets X in a set of cardinality strictly less than κ.

THEOREM 2.1. *In the model of* ZFC, M_{ω_1} *of* §1 (*in which the continuum, c, is $\aleph_{\omega_1}$*), *for any nontrivial* c.c.c. *σ-ideal I in the Borel sets there are both c-I-Luzin sets and ω_1-I-Luzin sets.*

Proof. Letting $B_\alpha = \text{Borel}^{M_\alpha}$ and as before $I_\alpha = \{A \in B_\alpha: \Vdash_\alpha \text{“}A \in I\text{”}\}$ we know by Lemma 1.5 that in M_α, B_α/I_α has the c.c.c.. Since each $M_{\alpha+1}$ is a model of MA, by Martin-Solovay [**9**] the union $\bigcup I_\alpha$ cannot cover the reals of $M_{\alpha+1}$ (since $I_\alpha \subseteq I_{\alpha+1}$ a c.c.c. ideal of $M_{\alpha+1}$). Consequently we can find X_α of cardinality $\aleph_{\alpha+2}$ in $M_{\alpha+1}$ with

$$X_\alpha \cap \bigcup I_\alpha = \varnothing.$$

Then $X = \bigcup_{\alpha < \omega_1} X_\alpha$ is c-I-Luzin in M_{ω_1}. □

For $\mathbf{P}$ a partial order define $\langle G_\alpha: \alpha < \omega_1\rangle$ a sequence of $\mathbf{P}$-filters to be an ω_1-generic sequence for $\mathbf{P}$ iff for all dense $D \subseteq \mathbf{P}$ all but countably many G_α meet D. This is motivated by van Douwen and Fleissner [**4**].

Question. In the model M_{ω_1} is it true that for all $\mathbf{P}$ c.c.c. of cardinality $\leq c$ there exists an ω_1-generic sequence for $\mathbf{P}$?

Note that by a result of Martin and Solovary [**9**] it is enough to prove the above for $\mathbf{P}$ of the form Borel$/I$ for I a c.c.c. σ-ideal. The difficulty with the above arguments is that filters on B_α/I_α may not lift to filters on B/I since elements of B_α/I_α may turn out to be in I. When we were first considering this question it was not clear to us that any restriction of the cardinality of $\mathbf{P}$ is necessary. The following theorem of Kunen shows that there is. FIN(κ) is the partial order of functions whose domain is a finite subset of κ and whose range is $\{0,1\}$.

THEOREM 2.2. (*Kunen*) *If* $\kappa \geq (2^{\omega_1})^+$ *there is no* ω_1*-generic sequence for* FIN(κ).

Proof. Suppose $\langle G_\alpha: \alpha < \omega_1 \rangle$ is given with each $G_\alpha: \kappa \to 2$. For each $\lambda < \kappa$ define $H_\lambda: \omega_1 \to 2$ by $H_\lambda(\alpha) = G_\alpha(\lambda)$. Since $\kappa \geq (2^{\omega_1})^+$ there exists an infinite $\Sigma \subseteq \kappa$ such that $H_\lambda = H_{\lambda'}$ for each $\lambda, \lambda' \in \Sigma$. Now choose $i \in \{0,1\}$ and an uncountable $\Gamma \subseteq \omega_1$ so that for each $\lambda \in \Sigma$, $H_\lambda \restriction \Gamma$ is constantly i. But this means that for each $\alpha \in \Gamma$, $G_\alpha \restriction \Sigma$ is constant. But then the G_α do not eventually meet the dense set $D = \{ p \in \mathrm{FIN}(\kappa): \exists\lambda \in \Sigma\, p(\lambda) \neq i\}$.

It is clear from the above proof that all we need is that

$$\binom{\kappa}{\omega_1} \to \binom{\omega}{\omega_1}_2^{1,1}$$

to see that there are no ω_1-generic sequences for FIN(κ). This partition relation is known to be consistent with GCH for $\kappa = \omega_2$ (see Laver [**8**]) assuming the consistency of a huge cardinal.

Theorem 2.2 can be strengthened to: if $\kappa > 2^{\omega_1}$ there is no family $\langle G_\alpha: \alpha < \omega_1 \rangle$ of filters in FIN(κ) such that every dense set meets some G_α. This was pointed out by D. H. Fremlin. To see how to prove this let

$$G = \bigcup_{\alpha < \omega_1} G_\alpha.$$

If for every $p \in \mathrm{FIN}(\kappa)$ there is some $q \leq p$ with $q \neq G$, then G misses some dense set. Consequently there is some condition in FIN(κ) such that G contains everything beneath it. But it is well known that the compact space 2^κ has density $> \omega_1$ (see Juhasz [**6**] 6.8, p.68).

Our next theorem goes in the other direction.

THEOREM 2.3. *Assume CH. Suppose that* **P** *is a* c.c.c. *partial order of cardinality* ω_2. *Then there exists* **Q** *an* ω_2 c.c. *order which is countably closed and has the same cardinality as* **P** *which adds an* ω_1*-generic sequence for* **P**.

Proof. We can assume without loss of generality that **P** is a boolean algebra. Elements of **Q** have the form

$$\langle \langle H_\alpha: \alpha < \beta \rangle, \mathscr{D} \rangle$$

where $\beta < \omega_1$, $\mathscr{D}$ is a countable family of dense subsets of **P**, and each H_α

is a countably generated **P** filter. The order on **Q** is defined as follows:

$$\langle\langle \hat{H}_\alpha\colon \alpha < \hat{\beta}\rangle, \hat{\mathscr{D}}\rangle \leq \langle\langle H_\alpha\colon \alpha < \beta\rangle, \mathscr{D}\rangle$$

iff $\hat{\beta} \geq \beta$, $\hat{\mathscr{D}} \supset \mathscr{D}$, $\hat{H}_\alpha \supset H_\alpha$ for all $\alpha < \beta$, and for all γ with $\beta \leq \gamma < \hat{\beta}$ and $D \in \mathscr{D}$, $D \cap \hat{H}_\gamma \neq \varnothing$.

It is easy to check that Q is countably closed and that Q adds an ω_1-generic sequence for **P**. Now let us see that it has the ω_2 chain condition. By an obvious argument it is enough to show that Q^* has ω_2-c.c. where

$$Q^* = \{\langle H_n\colon n < \omega\rangle\colon \text{each } H_n \text{ is a countably generated } \mathbf{P}\text{-filter}\}$$

ordered by

$$\langle \hat{H}_n\colon n < \omega\rangle \leq \langle H_n\colon n < \omega\rangle \quad \text{iff for all } n,\ \hat{H}_n \supset H_n.$$

Suppose $\{P_\alpha = (H_n^\alpha\colon n < \omega)\colon \alpha < \omega_2\}$ are pairwise incompatible. And let for each n and α

$$H_n^\alpha = \{q \in \mathbf{P}\colon \exists m < \omega\ q < p_m^{\alpha,n}\}$$

where $p_{m+1}^{\alpha,n} \leq p_m^{\alpha,n}$ is a descending sequence.

Thus for each $\alpha \neq \beta$ there exists n and m such that

$$p_m^{\alpha,n} \wedge p_m^{\beta,n} = 0.$$

by the Erdos-Rado Theorem $(\omega_2 \to (\omega_1)^2_\omega)$ (see Kunen [7], p. 290) there exists n_0 and m_0 and $X \in [\omega_2]^{\omega_1}$ such that for all $\alpha \neq \beta \in X$ $p_{m_0}^{\alpha,n_0} \wedge p_{m_0}^{\beta,n_0} = 0$. This contradicts the countable chain condition for **P**. □

This forcing can probably be iterated to take care of all c.c.c. **P** of cardinality ω_2. Unfortunately this would blow up 2^{ω_1} to ω_3. A better way would be to try to deduce the existence of ω_1-generic sequences from morasses with built-in $\diamondsuit$. The "black box" theorems of Velleman and Shelah-Stanley do not seem to apply because of the lack of homogeneity.

Question. In L does every c.c.c. partial order of cardinality ω_2 have an ω_1-generic sequence?

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