

THE NUMBER OF TRANSLATES OF A CLOSED NOWHERE DENSE SET REQUIRED TO COVER A POLISH GROUP

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ABSTRACT. For a Polish group \mathbb{G} let $\mathbf{cov}_{\mathbb{G}}$ be the minimal number of translates of a fixed closed nowhere dense subset of \mathbb{G} required to cover \mathbb{G} . For many locally compact \mathbb{G} this cardinal is known to be consistently larger than $\mathbf{cov}(\mathcal{M})$ which is the smallest cardinality of a covering of the real line by meagre sets. It is shown that for several non-locally compact groups $\mathbf{cov}_{\mathbb{G}} = \mathbf{cov}(\mathcal{M})$. For example the equality holds for the group of permutations of the integers, the additive group of a separable Banach space with an unconditional basis and the group of homeomorphisms of various compact spaces.

The notion of translation invariants corresponding to the usual invariants of the continuum has been considered by various researchers and an introductory survey can be found in §2.7 of the monograph [1] by Bartoszyński and Judah. The key definition for the purposes of this article is the cardinal they denoted by $\mathbf{cov}^*(\mathcal{M})$. It is the least cardinal of a set $X \subseteq \mathbb{R}$ such that there is some meagre set $M \subseteq \mathbb{R}$ such that $X + M = \mathbb{R}$. It is asserted that the value of $\mathbf{cov}^*(\mathcal{M})$ will be the same if the group $(\mathbb{R}, +)$ is replaced in this definition by the Cantor set with its natural Boolean operation or an infinite product of finite cyclic groups. The goal of this note is to initiate a study of translation invariants for arbitrary Polish groups by establishing that not all Polish groups yield the same invariants and posing various questions which arise from this observation. Since many of the interesting questions in this area concern non-locally compact groups the measure-theoretic version is not easily formulated and, therefore, only the topological version will be considered.

Throughout, the statement that \mathbb{G} is a group will mean that $\mathbb{G} = (G, \cdot, ^{-1})$ but for x and y in G the operation $x \cdot y$ will usually be abbreviated to xy . Similarly, if $A \subseteq G$ and $B \subseteq G$ then AB will denote the set $\{xy \mid x \in A \text{ and } y \in B\}$. If $A = \{a\}$ then $\{a\}B$ will be abbreviated to aB .

To begin, a generalization of $\mathbf{cov}^*(\mathcal{M})$ will be defined for arbitrary group actions.

Definition 1. Let \mathbb{G} be a group acting on a Polish space X with the action denoted by $\alpha : G \times X \rightarrow X$. Define \mathbf{cov}_{α} to be the least cardinal of a set $Q \subseteq G$ such that there is some closed nowhere dense set $C \subseteq X$ such that $\alpha(Q, C) = \{\alpha(h, c) \mid h \in Q \text{ and } c \in C\} = X$. Define \mathbf{cov}_{α}^* to be the least cardinal of a set $Q \subseteq G$ such that there is some meagre set $M \subseteq X$ such that $\alpha(Q, M) = X$. In the special case of a Polish group \mathbb{G} acting on itself by left translation these invariants will be denoted by $\mathbf{cov}_{\mathbb{G}}$ and $\mathbf{cov}_{\mathbb{G}}^*$. Note that the definitions of $\mathbf{cov}_{\mathbb{G}}$

1991 *Mathematics Subject Classification.* 03E17.

Key words and phrases. translations, nowhere dense sets, Polish groups, cardinal invariants.

Research for this paper was partially supported by NSERC of Canada.

and $\mathbf{cov}_{\mathbb{G}}^*$ do not change if left translation is replaced by right translation since $(QC)^{-1} = C^{-1}Q^{-1}$.

The first observation applies to arbitrary groups.

Proposition 2. *If \mathbb{G} is an arbitrary group, X is σ -compact, second countable and has no isolated points, $\alpha : G \times X \rightarrow X$ is a group action all of whose orbits are dense in X and $\alpha(g, \cdot)$ is continuous for each fixed $g \in G$, then $\mathbf{cov}_{\alpha} = \mathbf{cov}_{\alpha}^*$.*

Proof. It suffices to show that if M is meagre then there is a closed nowhere dense set C such that $\alpha(Y, C) \supseteq M$ for some countable set Y . Since X is σ -compact it may be assumed that $M = \bigcup_{n=0}^{\infty} D_n$ where each D_n is compact and nowhere dense. Let $\{B_n\}_{n=0}^{\infty}$ be a base for X consisting of non-empty open sets. Select Y_k and U_k by induction on k such that

- Y_k is a finite subset of G
- $U_k \subseteq B_k$ is a non-empty open set
- if $j \leq k$ and $d \in D_j$ then there is some $y \in Y_j$ such that $\alpha(y, d) \notin \bigcup_{i=0}^k U_i$
- the closure of $\bigcup_{j=0}^k U_j$ is not equal to X

If this can be done then let C be the complement of $\bigcup_{i=0}^{\infty} U_i$ and take Y countable including $\bigcup_{n=0}^{\infty} Y_n$ and closed under inverses. Then given $x \in M$ there is some n such that $x \in D_n$ and hence there exists $y \in Y_n$ such that $\alpha(y, x) \in C$ and so

$$x = \alpha(y^{-1}, \alpha(y, x)) \in \alpha(\{y^{-1}\}, C) \subseteq \alpha(Y, C).$$

To carry out the induction suppose this has been done for $i < k$. Since α is a continuous group action, $\alpha(\{g\}, D_j)$ is closed nowhere dense for any $g \in G$ and so it is possible to choose a non-empty open $U_k \subseteq B_k$ disjoint from

$$\bigcup_{i < k} \bigcup_{y \in Y_i} \alpha(\{y\}, D_i).$$

Moreover, it is possible to choose U_k so that there is some non-empty open set W witnessing the last induction clause; in other words, $W \cap \bigcup_{i=0}^k U_i = \emptyset$. Since D_k is compact and $\alpha(\{d\}, G)$ is dense for each $d \in D_k$ there is a finite set $Y_k \subseteq G$ such that for each $d \in D_k$ there is some $y \in Y_k$ such that $\alpha(y, d) \in W$. \square

Proposition 3. $\mathbf{cov}_{\mathbb{G}} = \mathbf{cov}_{\mathbb{G}}^*$ for every non-discrete Polish \mathbb{G} .

Proof. If \mathbb{G} is compact, then apply Proposition 2. On the other hand, Birkhoff and Kakutani proved that every Polish group admits a left invariant¹ metric d inducing the same topology. While this metric d might not be complete, it is shown in Corollary 1.2.2 of [3] that the metric D defined by $D(x, y) = d(x, y) + d(x^{-1}, y^{-1})$ is a complete metric compatible with the topology. However D might not be left or right invariant. Nevertheless, since \mathbb{G} is not compact and D is complete it follows that D is not totally bounded; in other words, there is some $\epsilon > 0$ and a sequence $\{x_n\}_{n \in \omega}$ such that $D(x_n, x_m) > \epsilon$ for all pairs of distinct integers n and m . A routine application of Ramsey's Theorem yields an infinite subset $S \subseteq \omega$ such that either $d(x_n, x_m) > \epsilon/2$ for all pairs of distinct integers n and m in S or $d(x_n^{-1}, x_m^{-1}) > \epsilon/2$ for all pairs of distinct integers n and m in S . From the left invariance of the metric

¹In other words, $d(y, z) = d(xy, xz)$ for every $x, y, z \in G$. For a proof see [4] §7.1 page 115.

d it follows that letting U be the $\epsilon/3$ ball around the identity either $\{x_n U\}_{n \in \omega}$ are pairwise disjoint sets or $\{x_n^{-1} U\}_{n \in \omega}$ are pairwise disjoint sets. Without loss of generality assume the former alternative.

As in the argument for Proposition 2, it suffices to prove the following: For any family $\{C_n\}_{n \in \omega}$ of nowhere dense subsets of \mathbb{G} there exists a countable $Q \subseteq \mathbb{G}$ and a nowhere dense C such that $\bigcup_{n \in \omega} C_n \subseteq QC$. Choose $\{q_n\}_{n \in \omega} \subseteq \mathbb{G}$ so that $\bigcup_{n \in \omega} q_n U = \mathbb{G}$. This is possible since $\{xU \mid x \in G\}$ covers G and G is Lindelof. Let $k : \omega \times \omega \rightarrow \omega$ be a bijection and define $r_{n,m}$ so that $r_{n,m} q_n = x_{k(n,m)}$ and define

$$C = \bigcup_{m \in \omega} \bigcup_{n \in \omega} r_{n,m} (q_n U \cap C_m).$$

Note that $r_{n,m} (q_n U \cap C_m) \subseteq x_{k(n,m)} U$ and since $\{x_n U\}_{n \in \omega}$ are disjoint, C is nowhere dense. On the other hand if $Q = \{r_{n,m}^{-1}\}_{n,m \in \omega}$, then QC contains $q_n U \cap C_m$ for each n and m and hence $C_m \subseteq QC$. \square

The proof of the following fact can be found in [1] in Lemma 2.4.2.

Theorem 4.

$$\mathbf{cov}(\mathcal{M}) = \min \{ |\mathcal{F}| \mid \mathcal{F} \subseteq \omega^\omega \text{ and } \forall g \in \omega^\omega \exists f \in \mathcal{F} \forall n f(n) \neq g(n) \}.$$

Corollary 5. $\mathbf{cov}_{\mathbb{Z}^\omega} = \mathbf{cov}(\mathcal{M})$.

Proof. For any non-discrete Polish group \mathbb{G} the inequality $\mathbf{cov}_{\mathbb{G}} \geq \mathbf{cov}(\mathcal{M})$ holds. Let

$$C = \{f \in \mathbb{Z}^\omega \mid \forall n f(n) \neq 0\}.$$

Note that C is closed nowhere dense. Take $\mathcal{F} \subseteq \omega^\omega \subseteq \mathbb{Z}^\omega$ such that $|\mathcal{F}| = \mathbf{cov}(\mathcal{M})$ and such that

$$\forall g \in \mathbb{Z}^\omega \exists f \in \mathcal{F} \forall n f(n) \neq g(n).$$

But this means that $\mathcal{F} + C = \mathbb{Z}^\omega$. Hence $\mathbf{cov}_{\mathbb{Z}^\omega} \leq |\mathcal{F}| = \mathbf{cov}(\mathcal{M})$. \square

To generalize this to other groups the following lemma will play a key role. It is stated using standard notation about trees. If $T \subseteq {}^\omega \omega$ is a tree then

$$[T] = \{f \mid \forall n \in \omega f \upharpoonright n \in T\}$$

and T is infinite branching if for each $t \in T$ the set $\Sigma(t) = \{n \in \omega \mid t \frown (n) \in T\}$ is infinite.

Lemma 6. *If $T \subseteq {}^\omega \omega$ is an infinite branching tree then there is \mathcal{F} such that*

- $|\mathcal{F}| = \mathbf{cov}(\mathcal{M})$
- $\mathcal{F} \subseteq [T]$
- for each $g : \omega \rightarrow \omega$ there is $f \in \mathcal{F}$ such that $f(n) \neq g(n)$ for all integers n .

Proof. Using Theorem 4 let $\mathcal{F}^* \subseteq \omega^\omega$ be a family of cardinality $\mathbf{cov}(\mathcal{M})$ such that for every $g \in \omega^\omega$ there is $f \in \mathcal{F}^*$ such that $f(n) \neq g(n)$ for every integer n . By choosing an infinite branching subtree of T if necessary, it may be assumed that $\Sigma(t) \cap \Sigma(s) = \emptyset$ unless $s = t$. Let $\psi_t : \omega \rightarrow \Sigma(t)$ be a bijection for each $t \in T$. For any $f : \omega \rightarrow \omega$ define f_ψ by inductively setting

$$f_\psi(n) = \psi_{f \upharpoonright n}(f(n))$$

and note that $f_\psi \upharpoonright n \in T$ for all n . Then let $\mathcal{F} = \{f_\psi \mid f \in \mathcal{F}^*\}$.

Let $\Sigma = \cup_{s \in T} \Sigma(s)$. Given $g : \omega \rightarrow \omega$ and $k \in \omega$ if $g(k) \in \Sigma$ let $s_k \in T$ be the unique node in T with $g(k) \in \Sigma(s_k)$.

Now define $G : \omega \rightarrow \omega$ by

$$G(k) = \begin{cases} \psi_{s_k}^{-1}(g(k)) & \text{if } g(k) \in \Sigma \\ 0 & \text{otherwise} \end{cases}$$

and choose $f \in \mathcal{F}^*$ such that $f(n) \neq G(n)$ for all n . It suffices to show that $f_\psi(n) \neq g(n)$ for all n . Assume that $f_\psi(n) = g(n)$. Then necessarily $g(n) = f_\psi(n) \in \Sigma(f_\psi \upharpoonright n)$ and so $s_n = f_\psi \upharpoonright n$ and therefore

$$G(n) = \psi_{s_n}^{-1}(g(n)) = \psi_{f_\psi \upharpoonright n}^{-1}(f_\psi(n)) = f(n)$$

which is a contradiction. □

Theorem 7. *Let \mathbb{G} be a Polish group such that there are $\{B_k, A_k^j\}_{j,k \in \omega}$ such that:*

- (1) B_k and A_k^j are all subsets of G
- (2) there is an infinite branching tree $T \subseteq {}^\omega\omega$ such that $\bigcap_{k \in \omega} A_k^{b(k)} \neq \emptyset$ for $b \in [T]$
- (3) $\bigcup_{k \in \omega} B_k$ is dense open
- (4) $(A_k^i B_k) \cap (A_k^j B_k) = \emptyset$ unless $i = j$

then $\mathbf{cov}_{\mathbb{G}} = \mathbf{cov}(\mathcal{M})$.

Proof. Using Lemma 6 let \mathcal{F} be a set of branches through T such that $|\mathcal{F}| = \mathbf{cov}(\mathcal{M})$ and for each $g : \omega \rightarrow \omega$ there is $f \in \mathcal{F}$ such that $f(n) \neq g(n)$ for all integers n . For $b \in [T]$ let $b^* \in \bigcap_k A_k^{b(k)}$ and let $X = \{b^* \mid b \in \mathcal{F}\}$. Let $C = G \setminus \bigcup_k B_k$. Then C is closed and nowhere dense and $|X| = \mathbf{cov}(\mathcal{M})$. Hence it suffices to show that $XC = G$.

To this end let $g \in G$ and define $\Gamma : \omega \rightarrow \omega$ such that $g \in A_n^{\Gamma(n)} B_n$ if there is any i with $g \in A_n^i B_n$. By Hypothesis 3 the choice of $\Gamma(n)$ is unique if it exists at all. Let $f \in \mathcal{F}$ be such that $f(n) \neq \Gamma(n)$ for all integers n . If $g \notin XC$ then $g \notin f^*C$ and so there is some integer m such that $g \in f^* B_m \subseteq A_m^{f(m)} B_m$. Hence $\Gamma(m) = f(m)$ which is impossible. □

For example, for the group \mathbb{Z}^ω one could take $A_k^j = \{\alpha \mid \alpha(k) = j\}$, $B_k = A_k^0$, and $T = {}^\omega\omega$ or, for the group \mathbb{R}^ω take $B_k = \{\alpha \in \mathbb{R}^\omega \mid |\alpha(k)| < 1/3\}$ and let A_k^j and T be the same.

Corollary 8. *If \mathbb{G} is the group of all permutations of the integers then $\mathbf{cov}_{\mathbb{G}} = \mathbf{cov}(\mathcal{M})$.*

Proof. Let $A_i^k = \{p \mid p(2i) = 2k\}$ and $B_k = A_k^k$. Letting T be the tree of all one-to-one sequences satisfies the hypotheses of Theorem 7. □

For any compact metric space X let $\mathbb{H}(X)$ be the group of autohomeomorphisms of X using composition as the group operation and the topology induced by the uniform metric.

Corollary 9. $\mathbf{cov}_{\mathbb{H}([0,1])} = \mathbf{cov}(\mathcal{M})$.

Proof. Construct a family of open intervals $\{I_s \mid s \in \mathbb{Z}^{<\omega}\}$ so that:

- $I_\emptyset = (0, 1)$
- $I_s = \bigcup_{n \in \mathbb{Z}} \overline{I_{s \smallfrown (n)}}$ and the right-hand endpoint of $I_{s \smallfrown (n)}$ is the left-hand endpoint of $I_{s \smallfrown (n+1)}$
- the length of I_s is less than $\frac{1}{|s|}$.

In other words, I_s is partitioned into contiguous subintervals in order type \mathbb{Z} and their endpoints. Define for $i, k \in \omega$ the open sets

$$U_k^i = \cup \{I_s \mid s \in \mathbb{Z}^{k+1} \text{ and } s(k) = i\}$$

Fix any $p_0 \in (0, 1)$. Define

$$B_k = \{h \in \mathbb{H} \mid h(p_0) \in U_k^0\}$$

and let $U = \bigcup_{k \in \omega} U_k^0$ and note that U is open dense and

$$\bigcup_{k \in \omega} B_k = \{h \in \mathbb{H} \mid h(p_0) \in U\}$$

is open dense in \mathbb{H} . For any $k, i \in \omega$ define

$$A_k^i = \{h \in \mathbb{H} \mid h(U_k^0) \subseteq U_k^i\}.$$

It is clear that $\{A_k^i \circ B_k\}_{i \in \omega}$ are pairwise disjoint because if $h \in A_k^i \circ B_k$ then $h(p_0) \in U_k^i$ and $\{U_k^i\}_{i \in \omega}$ are pairwise disjoint.

Now let $T = {}^\omega\omega$ and suppose $b \in [T]$. In order to find $h \in \bigcap_{k \in \omega} A_k^{b(k)}$ let a_s denote the left-hand endpoint of I_s and define $Q = \{a_s \mid s \in \mathbb{Z}^{<\omega}\}$. Note that Q is dense in $[0, 1]$ and its ordering is exactly the same as the lexicographical ordering $<_{lex}$ on $\mathbb{Z}^{<\omega}$: Define $s <_{lex} t$ if and only if $s \subset t$ or there exists i such that $s(i) < t(i)$ and $s \upharpoonright i = t \upharpoonright i$. Then $a_s < a_t$ if and only if $s <_{lex} t$. for all $s, t \in \mathbb{Z}^{<\omega}$.

For each $s \in \mathbb{Z}^n$ define $s + b = t \in \mathbb{Z}^n$ by pointwise addition and define $h(a_s) = a_{s+b}$. Clearly the mapping $s \mapsto s + b$ is a bijection preserving the lexicographical order and so $h : Q \rightarrow Q$ is an order preserving bijection. Thus it extends uniquely to an order preserving bijection h^* on $[0, 1]$. Note that $h(I_{s \smallfrown (0)}) \subseteq I_{t \smallfrown (b(k))}$ for any t, s and b where $t = s + b$ and $k = |s|$. So $h^*(U_k^0) \subseteq U_k^{b(k)}$ for each k . Thus $h^* \in \bigcap_{k \in \omega} A_k^{b(k)}$. \square

Corollary 10. *Suppose there exists $\pi : X \rightarrow [0, 1]$ which is*

- (1) *continuous, onto, and open*
- (2) $\forall h \in \mathbb{H}([0, 1]) \exists \hat{h} \in \mathbb{H}(X) \forall x \in X \pi(\hat{h}(x)) = h(\pi(x))$
- (3) *if $U \subseteq (0, 1)$ is open dense, then there exists $p_0 \in X$ such that*

$$\{h \in \mathbb{H}(X) \mid h(p_0) \in \pi^{-1}(U)\}$$

is open dense

then $\mathbf{cov}_{\mathbb{H}(X)} = \mathbf{cov}(\mathcal{M})$.

Proof. Using the same set $U = \bigcup_{k \in \omega} U_k^0 \subseteq (0, 1)$ as defined in the proof of Corollary 9 let $p_0 \in X$ be such that

$$\{h \in \mathbb{H}(X) \mid h(p_0) \in \pi^{-1}(U)\}$$

is open dense. Define

$$B_k = \{h \in \mathbb{H}(X) \mid \pi \circ h(p_0) \in U_k^0\}$$

and note that

$$\bigcup_{k \in \omega} B_k = \{h \in \mathbb{H}(X) \mid h(p_0) \in \pi^{-1}(U)\}$$

is open dense in $\mathbb{H}(X)$. For any integers k and i define

$$A_k^i(X) = \{h \in \mathbb{H}(X) \mid \pi \circ h(\pi^{-1}(U_k^0)) \subseteq U_k^i\}$$

reserving the notation A_k^i for the sets defined in the proof of Corollary 9. It is clear that $\{A_k^i(X) \circ B_k\}_{i \in \omega}$ are pairwise disjoint because if $h \in A_k^i(X) \circ B_k$ then $\pi \circ h(p_0) \in U_k^i$ and $\{U_k^i\}_{i \in \omega}$ are pairwise disjoint.

Now let $T = {}^\omega\omega$ and suppose $b \in [T]$. It has already been established in the proof of Corollary 9 that there is $h \in \bigcap_{k \in \omega} A_k^{b(k)}$. Then it is readily verified that $\hat{h} \in \bigcap_{k \in \omega} A_k^{b(k)}(X)$. \square

Corollary 11. *If X is either $[0, 1]^k$ for $k \in \omega$ or the infinite dimensional Hilbert cube $[0, 1]^\omega$ then $\mathbf{cov}_{\mathbb{H}(X)} = \mathbf{cov}(\mathcal{M})$.*

Proof. Use Corollary 10 applied to the mapping which projects a sequence to its first coordinate. Given $h \in \mathbb{H}([0, 1])$ take $\hat{h} \in \mathbb{H}([0, 1]^k)$ to be defined by

$$\hat{h}(x_0, x_1, x_2, \dots) = (h(x_0), x_1, x_2, \dots).$$

Let $p_0 = (1/2, 1/2, \dots, 1/2)$ and note that by a classical result of L.E.J. Brouwer (invariance of domain) $h(p_0)$ is in the interior of $[0, 1]^k$ and so its first coordinate is neither 0 or 1. It is easy to see that condition (3) of Corollary 10 holds. In the case of $[0, 1]^\omega$ it is possible that the first coordinate of $h(p_0)$ is 0 or 1 since the Hilbert cube is homogeneous. However, one of the standard proofs of homogeneity, see Lemma 6.1.4 page 252 in Van Mill [8], shows that given any $(u_n : n \in \omega) \in [0, 1]^\omega$, there exists a small $k \in \text{hom}([0, 1]^\omega)$ with $k(u_n : n \in \omega) = (v_n : n \in \omega)$ and $v_0 \in (0, 1)$. \square

Since the homeomorphism group of the Hilbert cube is universal — in the sense that any Polish group is homeomorphic to one of its closed subgroups — the following question is of central importance in this area.

Question 12. If \mathbb{G} is a non- σ -compact, closed subgroup of $\mathbb{H}([0, 1]^\omega)$ does the equality $\mathbf{cov}_{\mathbb{G}} = \mathbf{cov}(\mathcal{M})$ hold?

Corollary 13. *If S^n is the n dimensional Euclidean sphere then*

$$\mathbf{cov}_{\mathbb{H}(S^n)} = \mathbf{cov}(\mathcal{M}).$$

Similarly for any compact metric space X we have that

$$\mathbf{cov}_{\mathbb{H}(S^n \times X)} = \mathbf{cov}(\mathcal{M}).$$

Proof. Use Corollary 10 applied to the mapping which projects the sphere onto one of its diameters. \square

Corollary 14. *If \mathbb{G} is the additive group of a separable Banach space with an unconditional basis then $\mathbf{cov}_{\mathbb{G}} = \mathbf{cov}(\mathcal{M})$.*

Proof. Let $\{e_i\}_{i \in \omega}$ be an unconditional basis for the Banach space \mathbb{B} . Recall that this implies that for each $S \subseteq \omega$ the projection map

$$P_S \left(\sum_{n < \omega} \alpha_n e_n \right) = \sum_{n \in S} \alpha_n e_n$$

is a well-defined continuous linear operator.

Let $\{S_j\}_{j \in \omega}$ be pairwise disjoint infinite subsets of ω . Choose $\{\delta_i\}_{i \in \omega}$ positive reals such that $\sum_{i=0}^{\infty} \delta_i < \infty$. For each k the range of P_{S_k} is an infinite dimensional Banach space and hence the ball of diameter δ_k around the origin is not compact and therefore not totally bounded. In other words, it is possible to find some $\epsilon_k > 0$ and $\{u_k^i\}_{i \in \omega}$ contained in the range of P_{S_k} such that $\|u_k^i\| < \delta_k$ and $\|u_k^i - u_k^j\| > \epsilon_k$ for distinct i and j . Let

$$B_k = \{b \in \mathbb{B} \mid \|P_{S_k}(b)\| < \epsilon_k/2\} \text{ and } A_k^j = \{b \in \mathbb{B} \mid P_{S_k}(b) = u_k^j\}.$$

Finally, let $T = \bigcup_{n < \omega} \prod_{k < n} S_k$ and note that T is infinite branching.

To check that the other hypotheses of Theorem 7 are satisfied note that B_k is open because P_{S_k} is continuous. To see that $\bigcup_k B_k$ is dense note that the finite linear sums of the basis form a dense subset and if $b = \sum_{i=0}^N \alpha_i e_i$ and S_k is disjoint from $\{0, 1, \dots, N\}$ then $P_{S_k}(b) = \mathbf{0}$. To see that $\bigcap_k A_k^{b(k)}$ is non-empty for $b \in [T]$ note that $b^* = \sum_{k=0}^{\infty} u_k^{b(k)}$ is a convergent series by the choice of the δ_i . The sets $\{A_k^j + B_k\}_{j \in \omega}$ are pairwise disjoint by the choice of ϵ_k \square

Question 15. Is Corollary 14 true for Banach spaces without an unconditional basic sequence?

Definition 16. For $f : \mathbb{N} \rightarrow \mathbb{N}$ define $\mathbf{eq}(f)$ to be the least cardinal of a set $X \subseteq \prod_{i=0}^{\infty} f(i)$ such that for all $g \in \prod_{i=0}^{\infty} f(i)$ there is $x \in X$ such that $x(i) \neq g(i)$ for every $i \in \mathbb{N}$. Let \mathbf{eq} be the minimum of all $\mathbf{eq}(f)$ such that $\lim_{i \rightarrow \infty} f(i) = \infty$.

Proposition 17. (1) $\mathbf{cov}_{\mathbb{G}} = \mathbf{eq}$ when \mathbb{G} is either the product of finite cyclic groups or \mathbb{R} .

(2) Let $\mathbb{G} = \mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the circle group or any finite dimensional torus \mathbb{T}^n . Or let \mathbb{G} product of countably many nontrivial finite groups. Then $\mathbf{cov}_{\mathbb{G}} = \mathbf{eq}$.

Proof. Assertion (1) is proved in [1] and their argument generalizes to (2). \square

Lemma 18. Suppose \mathbb{G} and \mathbb{H} are topological groups and $h : \mathbb{G} \rightarrow \mathbb{H}$ is a homomorphism onto H .

- (a) If h is open and continuous, then $\mathbf{cov}_{\mathbb{H}} \geq \mathbf{cov}_{\mathbb{G}}$.
- (b) If h takes meagre sets to meagre sets, then $\mathbf{cov}_{\mathbb{H}}^* \leq \mathbf{cov}_{\mathbb{G}}^*$.

Proof. (a) Since h is open and continuous, the pre-image of a nowhere dense set is nowhere dense. Let $C \subseteq \mathbb{H}$ be nowhere dense and $Y \subseteq \mathbb{H}$ have the property that

$|Y| = \mathbf{cov}_{\mathbb{H}}$ and $YC = H$. Then $B = h^{-1}(C)$ is nowhere dense in \mathbb{G} and choose Y so that $h(Y) = X$ and $|X| = |Y|$. Since h is homomorphism it follows that $h(YB) = XC = H$. In order to see that $YB = G$ note that for any $g \in G$ there exists $y \in Y$ and $b \in B$ with $h(g) = h(yb)$. It follows that $h(y^{-1}g) = h(b) \in C$ so $y^{-1}g \in B$ and so $g = yb'$ for some $b' \in B$.

(b) Suppose h takes meagre sets to meagre sets and let $M \subseteq \mathbb{G}$ be meagre such that $\mathbf{cov}_{\mathbb{G}}^* = |X|$ for some X with $XM = G$. Then $h(X)h(M) = H$ and so $\mathbf{cov}_{\mathbb{H}}^* \leq |X| = \mathbf{cov}_{\mathbb{G}}^*$ □

So, for example, if \mathbb{G} and \mathbb{H} are Polish groups then $\mathbf{cov}_{\mathbb{H} \times \mathbb{G}} \leq \mathbf{cov}_{\mathbb{G}}$.

Proposition 19. *For any integer n the equality $\mathbf{cov}_{\mathbb{R}^n} = \mathfrak{eq}$ holds.*

Proof. The quotient mapping from \mathbb{R}^n to $\mathbb{R}^n/\mathbb{Z}^n$ is both open and it takes nowhere dense sets to meagre sets. It follows from Lemma 18 that $\mathbf{cov}_{\mathbb{R}^n} \leq \mathbf{cov}_{\mathbb{R}^n/\mathbb{Z}^n}$ and $\mathbf{cov}_{\mathbb{R}^n}^* \geq \mathbf{cov}_{\mathbb{R}^n/\mathbb{Z}^n}^*$.

Since $\mathbb{R}^n/\mathbb{Z}^n$ is isomorphic to \mathbb{T}^n it follows from Proposition 17 and Corollary 3 that $\mathbf{cov}_{\mathbb{R}^n} = \mathfrak{eq}$. □

There are many models of set theory where $\mathbf{cov}(\mathcal{M}) < \mathfrak{eq}$; examples can be found using modified Silver forcing as in §7 of [6].

Theorem 20. *It is relatively consistent with ZFC that*

- $2^{\aleph_0} = \aleph_2$,
- $\mathbf{cov}_{\mathbb{G}} = \aleph_2$ for every infinite compact group \mathbb{G}
- and $\mathbf{cov}(\mathcal{M}) = \aleph_1$.

In fact, the last equality will follow from the fact that $\mathfrak{d} = \aleph_1$ holds in the model constructed.

Proof. Fix \mathbb{G} an infinite compact group and $C \subseteq G$ closed nowhere dense. Let $h : 2^\omega \rightarrow G$ be a continuous onto map. Recall the notation preceding Lemma 6 concerning trees. If p is a tree and $s \in p$ define $p_s = \{t \in p \mid t \subseteq s \text{ or } s \subseteq t\}$.

Definition 21. Define the partially ordered set $\mathbb{P} = \mathbb{P}_{(h,C,\mathbb{G})}$ as follows: $p \in \mathbb{P}$ if and only if $p \subseteq {}^\omega 2$ is a perfect tree such that xC is relatively nowhere dense in $h([p_s])$ for every $x \in G$ and $s \in p$. This is equivalent to saying for every $x \in G$ and $s \in p$ there exists $t \in p$ with $s \subseteq t$ and $h([p_t]) \cap xC = \emptyset$. The ordering on \mathbb{P} is inclusion considering a subtree as a stronger condition.

First note that the forcing is not vacuous. For $s \in {}^\omega 2$ let $B_s = \{x \in 2^\omega \mid s \subseteq x\}$. If $p = \{s \in {}^\omega 2 \mid h(B_s) \text{ is non-meagre in } G\}$ then $p \in \mathbb{P}$. It will be shown that the forcing has the following three properties:

- (a) $\mathbb{P}_{(h,C,\mathbb{G})}$ is a proper forcing notion.
- (b) Forcing with $\mathbb{P}_{(h,C,\mathbb{G})}$ adds no \leq^* unbounded sequence in ω^ω .
- (c) In the generic extension there exists $z \in G$ such that $z \notin XC$ where X is the set of elements of G in the ground model.

Claim 1. *Given any $p \in \mathbb{P}$ and $n < \omega$ there exists N with $n < N < \omega$ such that for every $s \in 2^n \cap p$ and $x \in G$ there exists $t \in 2^N \cap p_s$ such that $h([p_t]) \cap xC = \emptyset$.*

Proof. To prove this claim, fix $s \in p$ and $x \in G$. Since xC is nowhere dense in $h[p_s]$ there exists $s_x \in p$ with $s \subseteq s_x$ and $h[p_{s_x}] \cap xC = \emptyset$. By the continuity of the group operation and the compactness of xC , there exists an open neighbourhood U_x of x with $h[p_{s_x}] \cap U_x C = \emptyset$. By compactness, finitely many of these U_x cover and so there is some N larger than the length of the finitely many corresponding s_x for each of the finitely many s in $2^n \cap p$. \square

Now suppose a sequence (p^n, k_n) has been constructed satisfying for every $n < \omega$:

- (1) $p^n \in \mathbb{P}$,
- (2) $k_n < k_{n+1} < \omega$,
- (3) $p^{n+1} \leq p^n$,
- (4) $2^{k_n} \cap p^{n+1} = 2^{k_n} \cap p^n$,
- (5) any $s \in 2^{k_n} \cap p^{n+1}$ has incomparable extensions in $2^{k_{n+1}} \cap p^{n+1}$, and
- (6) for each $s \in p^{n+1} \cap 2^{k_n}$ and $x \in G$ there exists $t \in p^{n+1} \cap 2^{k_{n+1}}$ with $s \subseteq t$ and $h([p_t^{n+1}]) \cap xC = \emptyset$.

Then $p = \bigcap_{n \in \omega} p^n$ is in \mathbb{P} . Except for the last condition 6 the fusion conditions described are identical to those used by Baumgartner and Laver in [2] for Sacks forcing. The last condition guarantees that for every $x \in G$ and $s \in p$ that xC is nowhere dense in $h([p_s])$.

In order to verify the properties (a), (b), and (c) note that this fusion property is close enough to a property-A forcing to see easily that \mathbb{P} is proper. Next it will be shown that forcing with \mathbb{P} is ω^ω -bounding. Suppose

$$p^0 \Vdash \text{“}\tau : \omega \rightarrow \text{Ordinals”}.$$

Construct a fusion sequence p^n, k_n as follows. At stage n given p^n, k_n find $p^{n+1} \leq p^n$ such that for every $s \in 2^{k_n} \cap p^n$ the condition p^{n+1} decides $\tau(n)$. Hence there exists a finite set F_n such that

$$p^{n+1} \Vdash \text{“}\tau(n) \in \check{F}_n \text{”}.$$

Now pick k_{n+1} large enough to satisfy Condition 5 and Condition 6. The fusion $p = \bigcap_{n \in \omega} p^n$ then satisfies:

$$p \Vdash \text{“}\forall n \tau(n) \in \check{F}_n \text{”}.$$

Lastly, if G is $\mathbb{P}_{(h,C,G)}$ -generic over a ground model V and $x \in 2^\omega$ is the real determined by G — in other words, $x \in [p]$ for all $p \in G$ — then $z = h(x)$ has the property that $z \notin XC$ where $X = V \cap G$. This follows from any easy density argument, since if $p \in \mathbb{P}$ and $s \in p$ then $p_s \in \mathbb{P}$.

To prove the Theorem note that the countable support iteration of proper forcings which are ω^ω -bounding is ω^ω -bounding (Shelah, see [1] 6.3.5). Hence a countable support ω_2 -iteration over a model of GCH will satisfy $\mathfrak{d} = \aleph_1$. By dovetailing it is easy to arrange that every infinite compact group and closed nowhere dense subset in the final model is forced with (or rather a code for such) at unboundedly many stages in the iteration. It follows by standard arguments that in the final model $\mathbf{cov}_{\mathbb{G}} = \aleph_2$ for every infinite compact group \mathbb{G} . \square

Question 22. Is it consistent to have a compact group \mathbb{G} such that $\mathbf{cov}_{\mathbb{G}} > \mathfrak{eq}$?

Question 23. Is it true that for any infinite compact group \mathbb{G} that $\mathbf{cov}_{\mathbb{G}} \geq \mathfrak{eq}$?

Question 24. Is it true that for every non-discrete Polish group \mathbb{G} that $\mathbf{cov}_{\mathbb{G}} = \mathfrak{eq}$ or $\mathbf{cov}_{\mathbb{G}} = \mathbf{cov}(\mathcal{M})$?

The question of general group actions has only been hinted at but there are many questions concerning these as well. For example, it is easy to see that if α_n is the natural action of the isometry group on \mathbb{R}^n and $n \leq m$ then $\mathbf{cov}_{\alpha_n} \leq \mathbf{cov}_{\alpha_m}$. However the following question is unanswered.

Question 25. Is it true that $\mathbf{cov}_{\alpha_n} = \mathbf{cov}_{\alpha_m}$ for all m and n ?

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Appendix

This is not intended for publication but only for the electronic version.

Details of the proof of Proposition 17 (2).

Proposition 26. *If \mathbb{G}_n is a nontrivial finite group for each $n \in \mathbb{N}$ and $\mathbb{G} = \prod_{n=0}^{\infty} \mathbb{G}_n$ with the product topology then $\mathbf{cov}_{\mathbb{G}} = \mathbf{eq}$.*

Proof. To begin, note that if $f \leq g$ then $\mathbf{eq}(f) \geq \mathbf{eq}(g)$. Choose $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $\mathbf{eq}(f) = \mathbf{eq}$. Now let $\{k_n\}_{n=0}^{\infty}$ be an increasing sequence of integers such that $g(n) = \prod_{i=k_n}^{k_{n+1}-1} |G_i| \geq f(n)$ for each n . Clearly $\mathbf{eq}(g) = \mathbf{eq}$ as well. Now let $C = \{x \in G \mid (\forall n \in \mathbb{N})(\exists i \in [k_n, k_{n+1})) x(i) \neq e_i\}$ and note that C is closed nowhere dense. Using the definition of $\mathbf{eq}(g)$ it is possible to find $X \subseteq G$ such that for all $y \in G$ there is $x \in X$ such that $y \upharpoonright [k_n, k_{n+1}) \neq x \upharpoonright [k_n, k_{n+1})$ for all n . Therefore $\prod_{n=0}^{\infty} (x(n)y(n)) \in C$; in other words $XC = G$ and so $\mathbf{cov}_{\mathbb{G}} \leq \mathbf{eq}$.

On the other hand, suppose that $C \subseteq G$ is closed nowhere dense and there is $W \subseteq G$ such that $|W| < \mathbf{eq}$ and $WC = G$. Then it is possible to find disjoint intervals of integers $\{I_n\}_{n=0}^{\infty}$ such that, letting $\pi_n : G \rightarrow \prod_{i \in I_n} G_i$ be the projection map, for all n there is $x_n \in \prod_{i \in I_n} G_i$ such that $\pi_n^{-1}\{x_n\} \cap C = \emptyset$. Let

$$C^* = \left\{ y \in \prod_{n=0}^{\infty} \prod_{i \in I_n} G_i \mid (\forall n) y(n) \neq x_n \right\}$$

and let $g(n) = \prod_{i \in I_n} |G_i|$. Let $\{g_m\}_{m=1}^{g(n)}$ enumerate $\prod_{i \in I_n} G_i$. For $w \in W$ let $f_w : \mathbb{N} \rightarrow \mathbb{N}$ be defined by $f_w(n) = j$ if and only if $x_n \pi_n(w) = g_j$.

Since $|W| < \mathbf{eq}(g)$ it is possible to find $h : \mathbb{N} \rightarrow \mathbb{N}$ such that for all $w \in W$ there is some $n \in \mathbb{N}$ such that $f_w(n) = h(n)$. Let $z = \prod_{n=0}^{\infty} g_{h(n)}$. It suffices to note that $z\pi_n(w)^{-1} \notin C^*$ for all $w \in W$ because it then follows that $\pi^{-1}\{z\}\{w^{-1}\} \cap C = \emptyset$ for all $w \in W$ and so $WC \neq G$. To see that $z\pi_n(w)^{-1} \notin C^*$ let n be such that $h(n) = f_w(n)$. Then $x_n \pi_n(w) = g_{h(n)} = z \upharpoonright I_n$. Hence $z\pi_n(w)^{-1} = x_n$ and so $z\pi_n(w)^{-1} \notin C^*$. \square

Proposition 27. *For any integer n the equality $\mathbf{cov}_{\mathbb{T}^n} = \mathbf{eq}$ holds.*

Proof. While there is a continuous and open mapping from $2^{\mathbb{N}}$ onto the circle \mathbb{T} , this mapping is not a homomorphism. Nevertheless, it is sufficiently close to a homomorphism to be able to do the proof.

This follows the pattern of Proposition 26 taking care of carry digits. To be more precise, given $\psi : \mathbb{N} \rightarrow \mathbb{N}$ such that $\mathbf{eq}(\psi) = \mathbf{eq}$ let $\{k_n\}_{n=0}^{\infty}$ be an increasing sequence of integers such that $\prod_{i=k_n}^{k_{n+1}-1} |G_i| \geq 2\psi(n)$ for each n . Consider \mathbb{T} to be \mathbb{R}/\mathbb{Z} so that \mathbb{T} is identified with $[0, 1]$. For $t \in [0, 1]$ let $x_t : \mathbb{N} \rightarrow 2$ be chosen so that $\sum_{i=0}^{\infty} x_t(i)/2^i = t$ and there are infinitely many i such that $x_t(i) = 0$. Now let

$$C = \{t \in [0, 1] \mid (\forall n \in \mathbb{N})(\exists i \in [k_n, k_{n+1})) x_t(i) \neq 0\}$$

and note that C is closed nowhere dense. Now, for any two functions $f : [k_n, k_{n+1} - 1) \rightarrow 2$ and $g : [k_n, k_{n+1} - 1) \rightarrow 2$ define $f \equiv_n g$ if and only if there is some j such that

- $f(i) \neq g(i)$ for $i \geq j$
- $f(i) = g(i)$ for $i < j$
- $f \upharpoonright [j+1, k_{n+1})$ is constant (and hence so is $g \upharpoonright [j+1, k_{n+1})$).

It is immediate that \equiv_n is an equivalence relation whose equivalence classes are all pairs. Let S_n be the set of equivalence classes of \equiv_n . Hence $\phi(n) = |S_n| \geq \psi(n)$.

Using the definition of $\mathbf{eq}(\phi)$ it is possible to find $Y \subseteq \prod_{n=0}^{\infty} S_n$ such that for all $t \in [0, 1]$ there is $y \in Y$ such that $x_t \upharpoonright [k_n, k_{n+1}) \not\equiv_n y(n)$ for all n . For each $y \in Y$ let y^* be a choice function which selects an element of $y(n)$ for each n and let $\bar{y} = \sum_{n=0}^{\infty} \sum_{i=k_n}^{k_{n+1}-1} y^*(i)/2^i$. Observe that $x_t \upharpoonright [k_n, k_{n+1}) \not\equiv_n x_{\bar{y}} \upharpoonright [k_n, k_{n+1})$ and hence, regardless of whether digits are carried from the right, $x_{t-\bar{y}} \upharpoonright [k_n, k_{n+1})$ is not identically 0. In other words, $t \in \bar{y} + C$.

On the other hand, suppose that $C \subseteq [0, 1]$ is closed nowhere dense and there is $W \subseteq [0, 1]$ such that $|W| < \mathbf{eq}$ and $W + C = [0, 1]$. Then it is possible to find disjoint intervals of integers $\{I_n\}_{n=0}^{\infty}$ such that, letting \equiv_n be the equivalence relation defined on functions from I_n to 2 and S_n be as before, for all n there is $[s_n]_{\equiv_n} \in S_n$ such that

$$\{t \in [0, 1] \mid x_t \upharpoonright I_n \equiv_n s_n\} \cap C = \emptyset.$$

Let

$$C^* = \{t \in [0, 1] \mid (\forall n) x_t \upharpoonright I_n \not\equiv_n s_n\}$$

and let $g(n) = 2^{|I_n|}$. Let $\{g_m\}_{m=1}^{g(n)}$ enumerate 2^{I_n} . For $w \in W$ let $f_w : \mathbb{N} \rightarrow \mathbb{N}$ be defined by $f_w(n) = j$ if and only if $g_j - s_n = x_w \upharpoonright I_n$.

Since $|W| < \mathbf{eq}(g)$ it is possible to find $h : \mathbb{N} \rightarrow \mathbb{N}$ such that for all $w \in W$ there is some $n \in \mathbb{N}$ such that $f_w(n) = h(n)$. Now argue as in Proposition 26. \square

The following is another example of a corollary to 10.

Corollary 28. *If X is an orientable 2-manifold then $\mathbf{cov}_{\mathbb{H}(X)} = \mathbf{cov}(\mathcal{M})$.*

Proof. An orientable 2-manifold is homeomorphic to the surface of the unit cube with holes drilled along axes perpendicular to the xy -plane. Use Corollary 10 applied to the mapping which projects the surface to the z -axis. \square

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