A hierarchy of clopen graphs on the Baire space

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We say that $E \subseteq X \times X$ is a *clopen graph* on X iff E is symmetric and irreflexive and clopen relative to $X^2 \setminus \Delta$ where $\Delta = \{(x,x) : x \in X\}$ is the diagonal. Equivalently $E \subseteq [X]^2$ and for all $x \neq y \in X$ there are open neighborhoods $x \in U$ and $y \in V$ such that either $U \times V \subseteq E$ or $U \times V \subseteq X^2 \setminus E$.

For clopen graphs E_1, E_2 on spaces X_1, X_2, we say that E_1 *continuously reduces* to E_2 iff there is a continuous map $f : X_1 \to X_2$ such that for every $x, y \in X_1$

$$(x, y) \in E_1 \text{ iff } (f(x), f(y)) \in E_2.$$

Note that f need not be one-to-one but there should be no edges in the preimage of a point. If f is a homeomorphism to its image, then we say that E_1 *continuously embeds* into E_2.

Theorem 1 *There does not exist countably many clopen graphs on the Baire space, ω^ω, such that every clopen graph on ω^ω can be continuously reduced to one of them. However, there are ω_1 clopen graphs on ω^ω such that every clopen graph on ω^ω continuously embeds into one of them.*

Since one can take a countable clopen separated union of countably many clopen graphs on ω^ω, having countably many is the same as having one universal graph.

Definition 2 *For $R \subseteq \omega^\omega \times \omega^\omega$, C and D clopen subsets of ω^ω, and α an ordinal define*

1. $\text{rank}_R(C \times D) = 0$ *iff* $C \times D \subseteq R$ *or* $R \cap (C \times D) = \emptyset$
2. $\text{rank}_R(C \times D) \leq \alpha$ *iff there are partitions of C and D into clopen sets: $C = \sqcup_{i<\omega} C_i$ and $D = \sqcup_{j<\omega} D_j$ such that* $\text{rank}_R(C_i \times D_j) < \alpha$ *for all i, j in ω.*

We use $\sqcup$ to mean disjoint union.

Since we allow C_i's and D_j's to be empty, it is clear that:

Proposition 3 *If* $\mathrm{rank}_R(C \times D) \leq \alpha$ *and* $C' \subseteq C$ *and* $D' \subseteq D$*, then* $\mathrm{rank}_R(C' \times D') \leq \alpha$*.*

More generally:

Proposition 4 *Suppose* $f : C \sqcup D \to C' \sqcup D'$ *is a continuous reduction of* $R \subseteq C \times D$ *to* $R' \subseteq C' \times D'$ *and* $f^{-1}(C') = C$ *and* $f^{-1}(D') = D$*. Then*

$$\mathrm{rank}_R(C \times D) \leq \mathrm{rank}_{R'}(C' \times D').$$

Proof
Since f is continuous, clopen partitions $C' = \sqcup C'_i$ and $D' = \sqcup D'_j$ induce clopen partitions $C = \sqcup_i f^{-1}(C'_i)$ and $D = \sqcup_j f^{-1}(D'_j)$.
QED

Definition 5 *For* E *a clopen graph on* ω^ω *define*

$$\mathrm{rank}(E) = \sup\{\mathrm{rank}_E(C \times D) \; : \; C \text{ and } D \text{ are disjoint clopen sets}\}.$$

Lemma 6 *If* E *is a clopen graph on* ω^ω*, then* $\mathrm{rank}(E) < \omega_1$*.*

Proof
Given incomparable $s_0, t_0 \in \omega^{<\omega}$ with the same length look at the tree T:

$(s, t) \in T$ iff

1. $s_0 \subseteq s$, $t_0 \subseteq t$, $|s| = |t|$, and
2. both $([s] \times [t]) \cap E$ and $([s] \times [t]) \setminus E$ are nonempty.

Let

$$T^* = \{(s_0, t_0)\} \cup \{(s\hat{}\langle i\rangle, t\hat{}\langle j\rangle) \; : \; (s,t) \in T \text{ and } i, j \in \omega\}.$$

Since $E \cap ([s_0] \times [t_0])$ is clopen, T^* is well-founded and $T^* \setminus T$ is the set of the terminal nodes of T^*. Let r be the standard rank function on T^*, i.e.,

- $r(s,t) = 0$ iff $(s,t) \in T^* \setminus T$
- $r(s,t) = \sup\{r(s\hat{}\langle i\rangle, t\hat{}\langle j\rangle) + 1 \; : \; i, j \in \omega\}$ if $(s,t) \in T$.

Note that $\mathrm{rank}_E([s] \times [t]) \leq r(s,t)$ for $(s,t) \in T^*$. Take any countable ordinal α such that for every $s \in \omega^{<\omega}$ and distinct $i,j \in \omega$ we have that $\mathrm{rank}_E([s\hat{\ }\langle i\rangle] \times [s\hat{\ }\langle j\rangle]) \leq \alpha$.

For any $s,t \in \omega^{<\omega}$ which are incomparable, let n be the least such that $s \upharpoonright n \neq t \upharpoonright n$. Then $[s] \subseteq [s \upharpoonright n]$ and $[t] \subseteq [t \upharpoonright n]$ and so by Proposition 3, $\mathrm{rank}_E([s] \times [t]) \leq \alpha$. But any nonempty open set U can be written as pairwise disjoint basic clopen sets, i.e., $U = \sqcup_{i<\omega}[s_i]$ where $s_i \in \omega^{<\omega}$. (To see this just take for any $x \in U$ the least n with $[x \upharpoonright n] \subseteq U$.) Hence for any disjoint clopen sets C, D we have that $\mathrm{rank}_E(C \times D) \leq \alpha + 1$. And so $\mathrm{rank}(E) \leq \alpha + 1$.
QED

Lemma 7 *For any $\alpha < \omega_1$ there exists a clopen graph E_α on ω^ω such that if E is any clopen graph on the Baire space such that E_α is continuously reducible to E, then* $\mathrm{rank}(E) \geq \alpha$.

Proof
Let $Q = \{in, out\}$ and α any countable limit ordinal. Put $\Gamma_\alpha = \omega \times (Q \cup \alpha)$ with the discrete topology and define a clopen relation $R_\alpha \subseteq \omega^\omega \times \Gamma_\alpha^\omega$ as follows. Given $x \in \omega^\omega$ and $y \in \Gamma_\alpha^\omega$ construct sequences $m_i, n_i \in \omega$ and $\alpha_i \in \alpha \cup Q$ as follows.

- $x(0) = m_0$ and $y(m_0) = (n_0, \alpha_0)$
- $x(n_{i-1}) = m_i$ and $y(m_i) = (n_i, \alpha_i)$ for $i \geq 1$.

To determine whether or not $(x,y) \in R_\alpha$ look at the first i such that either $\alpha_i \in Q$ or ($i > 0$ and $\alpha_i \in \alpha$ but **not** $\alpha_i < \alpha_{i-1}$). Note that such an i must always occur since otherwise we would get an infinite descending sequence of ordinals. Let i_0 be the first such i and put $(x,y) \in R_\alpha$ iff $\alpha_{i_0} = in$.

Note that R_α is clopen since given any $x, y \in \omega^\omega$ we can choose N sufficiently large so that every pair in $[x \upharpoonright N] \times [y \upharpoonright N]$ will terminate the same way (x,y) did.

Claim 7.1. Suppose $s \in \omega^{<\omega}$ and $t \in \Gamma_\alpha^{<\omega}$ have the property that we can define the sequences m_i and (n_i, α_i) for $i < N$ using the same prescription as above:

1. $s(0) = m_0$ and $t(m_0) = (n_0, \alpha_0)$,

2. $s(n_{i-1}) = m_i$ and $t(m_i) = (n_i, \alpha_i)$ for $1 \le i < N$,

3. $\alpha_0 > \alpha_1 > \cdots > \alpha_{N-1}$ are all ordinals, and

4. $s(n_{N-1}) = m_N$,

5. however $m_N \ge |t|$ so we have not yet determined α_N.

Then $\mathrm{rank}_{R_\alpha}([s] \times [t]) \ge \alpha_{N-1}$.

Proof
Suppose that α_{N-1} is the least ordinal for which this could be false (for any s, t, N) and let $\mathrm{rank}_{R_\alpha}([s] \times [t]) = \beta < \alpha_{N-1}$. It is easy to check that if $\alpha_{N-1} > 0$ then β cannot be zero since we may find extensions t_{in}, t_{out} of t with $t_{in}(m_N) = (\cdot, in)$ and $t_{out}(m_N) = (\cdot, out)$.

Let $[s] = \sqcup_i C_i$ and $[t] = \sqcup_j D_j$ be clopen partitions with $\mathrm{rank}_{R_\alpha}(C_i \times D_j) < \beta$ for all i, j. Extend $s \subseteq s'$ so that $[s'] \subseteq C_i$ for some i. Extend t to t' so that $t'(m_N) = (|s'|, \beta) =^{def} (n_N, \alpha_N)$ and $[t'] \subseteq D_j$ for some j. Finally extend s' by putting $s'' = s'\hat{\ }\langle |t'| \rangle$ so $s''(\alpha_N) = m_N = |t'|$. Now we are in the same situation as before except $\alpha_N = \beta$ is now defined. But

$$\mathrm{rank}_{R_\alpha}([s''] \times [t']) \le \mathrm{rank}_{R_\alpha}(C_i \times D_j) < \beta = \alpha_N.$$

This violates the minimality of α_{N-1} and so proves the Claim.
QED

Now for any limit ordinal α and $\beta < \alpha$ let $|s| = 2$ and $|t| = 1$ be defined by $s(0) = 0, s(1) = 1$ and $t(0) = (1, \beta)$. By the claim $\mathrm{rank}_{R_\alpha}([s] \times [t]) \ge \beta$ and since these exist for every $\beta < \alpha$, it follows that $\mathrm{rank}(R_\alpha) \ge \alpha$.

Now we adjust R_α to make its domain and range disjoint. Identify Γ_α with ω and define $S_\alpha = \{(0\hat{\ }\langle x \rangle, 1\hat{\ }\langle y \rangle) \;:\; (x, y) \in R_\alpha\}$. Then $S_\alpha \subseteq C \times D$ where $C = [\langle 0 \rangle]$ and $D = [\langle 1 \rangle]$ are disjoint clopen sets. Clearly $\mathrm{rank}(S_\alpha) \ge \alpha$ as it is a copy of R_α. Let C_1 and D_1 be nonempty clopen sets such that $C \sqcup C_1 \sqcup D \sqcup D_1 = \omega^\omega$. Let

$$P_\alpha = S_\alpha \cup (C_1 \times D) \cup (C \times D_1)$$

Since $P_\alpha \cap (C \times D) = S_\alpha$ we know $\mathrm{rank}(P_\alpha) \ge \alpha$. If we let $A = C \cup C_1$ and $B = D \cup D_1$ then A and B are complementary clopen sets with $P_\alpha \subseteq A \times B$ and for every $x \in A$ there is a $y \in B$ with $(x, y) \in P_\alpha$ and for every $y \in B$ there is an $x \in A$ with $(x, y) \in P_\alpha$. (This property that everything is connected to something else might have already been true of R_α but if not, in this step we have added it.)

Finally we define the clopen graph E_α. We put

$$(x,y) \in E_\alpha \text{ iff } (x,y) \in P_\alpha \text{ or } (y,x) \in P_\alpha.$$

Then E_α is a true clopen graph, i.e., E_α is a clopen relation in $(\omega^\omega)^2$ which is symmetric and irreflexive. Also $\omega^\omega = A \sqcup B$ where every element ω^ω is connected to something else, but neither A nor B contain two elements which are connected.

Claim 7.2. Suppose α a countable limit ordinal and there is a continuous reduction of E_α to a clopen graph E. Then $\text{rank}(E) \geq \alpha$.
Proof
Let $f : \omega^\omega \to \omega^\omega$ be a continuous reduction and suppose for contraction that $\text{rank}(E) = \beta < \alpha$. Then in particular for every $(x,y) \in A \times B$

$$(x,y) \in P_\alpha \text{ iff } (f(x), f(y)) \in E.$$

Let $A' = f(A)$ and $B' = f(B)$ We show that not only are these sets disjoint but they have a stronger separation property.

For every $z \in A' \cup B'$ there exists some n such that $f^{-1}([z \restriction n]) \subseteq A$ or $f^{-1}([z \restriction n]) \subseteq B$. To see why let $z = f(x)$ for some $x \in A$. By our construction of P_α there is a $y \in B$ with $(x,y) \in P_\alpha$. By the reduction $(f(x), f(y)) \in E$ and since E is clopen $[f(x) \restriction n] \times [f(y) \restriction n] \subseteq E$ for some n. So if $f(u) \in [f(x) \restriction n]$ then $(f(u), f(y)) \in E$ and so $(u,y) \in P_\alpha$. But this implies $u \in A$ since $y \in B$.

Now define $\Sigma \subseteq \omega^{<\omega}$ by

$$\Sigma = \{s \in \omega^{<\omega} \ : \ f^{-1}[s] \subseteq A \text{ or } f^{-1}[s] \subseteq B\}$$

and let

$$\Sigma_0 = \{s \in \Sigma \ : \ \forall t \in \Sigma \ \ t \subseteq s \to t = s\}.$$

Note that the elements of Σ_0 are pairwise incomparable and that

$$A = \sqcup_{s \in \Sigma_0} \{f^{-1}[s] \ : \ f^{-1}[s] \subseteq A\}$$

and

$$B = \sqcup_{t \in \Sigma_0} \{f^{-1}[t] \ : \ f^{-1}[t] \subseteq B\}$$

are clopen partitions of A and B. Since $\text{rank}(E) \leq \beta$ for any distinct $s, t \in \Sigma_0$ we have that $\text{rank}_E([s] \times [t]) \leq \beta$. By Proposition 4 we get that $\text{rank}(P_\alpha) \leq \beta + 1 < \alpha$, which is a contradiction. This proves Lemma 7.

QED

Lemma 8 *There exists U_α for $\alpha < \omega_1$ clopen graphs on ω^ω such that for every clopen graph E on ω^ω there exists $\alpha < \omega_1$ such that E continuously embeds into U_α.*

Proof

For any $s \in \omega^{<\omega}$ except the trivial sequence $\langle\rangle$ let s^* be the parent of s, i.e., the unique $s^* \subseteq s$ and $|s^*| = |s| - 1$.

Let α be a countable ordinal, $Q = \{in, out\}$ (or more generally any countable set). A pair (T, l) is an *α-tree* iff T is a subtree of $\omega^{<\omega}$ and $l : D \to \alpha \cup Q$ where $D = \{\{s,t\} \in [T]^2 \ : \ |s| = |t| \text{ and } s \neq t\}$ and l satisfies:

if $(s,t) \in D$ and $s^* \neq t^*$ then

1. if $l(s^*, t^*) \in \alpha$ then $l(s,t) < l(s^*, t^*)$ or $l(s,t) \in Q$
2. if $l(s^*, t^*) \in Q$ then $l(s,t) = l(s^*, t^*)$.

Note that l is only defined on pairs with $s \neq t$ of the same length. Also if $s^* = t^*$, then $l(s,t)$ can be anything in $\alpha \cup Q$. A compact way of stating the above two conditions would be by taking the binary relation $\lhd$ on $\alpha \cup Q$ defined by $x \lhd y$ iff

1. $x, y \in \alpha$ and $x < y$,
2. $x \in Q$ and $y \in \alpha$, or
3. $x, y \in Q$ and $x = y$.

Then our condition on l is equivalent to:

if $(s,t) \in D$ and $s^* \neq t^*$ then $l(s,t) \lhd l(s^*, t^*)$.

Given any clopen graph E we describe the canonical α-tree $(\omega^{<\omega}, l)$ associated with it. For any distinct s, t of the same length if $[s] \times [t] \subseteq E$, then put $l(s,t) = in$, if $([s] \times [t]) \cap E = \emptyset$, then put $l(s,t) = out$.

Let $P = \{(s,t) \ : \ l(s,t) \in Q\}$ and note that P is closed downward. For any s and distinct $i, j \in \omega$ the tree

$$T_{s,i,j} = \{(t_1, t_2) : s\hat{\ }\langle i\rangle \subseteq t_1,\ s\hat{\ }\langle j\rangle \subseteq t_2,\ |t_1| = |t_2|,\ \text{and } (t_1, t_2) \notin P\}$$

is a well-founded tree because E is clopen. Let $l \upharpoonright T_{s,i,j}$ be its rank function. Picking α large enough makes $(\omega^{<\omega}, l)$ an α-tree.

Next we construct a universal α-tree $(\omega^{<\omega}, L)$. It will be very strongly universal in the following sense: Suppose that (T, l) is any α-tree. Then there will exists $\sigma : \omega^{<\omega} \to \omega^{<\omega}$ which is tree embedding, i.e,

1. σ is one-to-one and level preserving, i.e., $|\sigma(s)| = |s|$
2. σ preserves the tree ordering, i.e., $s \subseteq t$ implies $\sigma(s) \subseteq \sigma(t)$
3. σ preserves the labeling on edges, i.e., $l(s,t) = L(\sigma(s), \sigma(t))$ for any distinct s, t of the same length.

It easy to see that σ induces a continuous embedding $f : [T] \to \omega^\omega$ by $f(x) = \bigcup_{n<\omega} \sigma(x \upharpoonright n)$ which reduces the graph associated to l to the one associated with L.

We construct L to have the following property:

> For any $n < \omega$, $p \in \omega^n$, finite $F \subseteq \omega^{n+1}$, and $f : F \to \alpha \cup Q$ consistent with L, there will be infinitely many $t \in \omega^{n+1}$ with $t^* = p$ such that $L(t, s) = f(s)$ for all $s \in F$. By f consistent with L we mean: for all $s \in F$ if $s^* \neq p$, then $f(s) \lhd L(p, s^*)$.

First let us check that it is possible to construct L with this property. Let (p_n, F_n, f_n) list with infinitely many repetitions all triples (p, F, f) with $p \in \omega^{<\omega}$, $F \subseteq T \cap \omega^{k+1}$ where $|p| = k$, and $f : F \to \alpha \cup Q$ arbitrary. Construct (T_n, L_n) an α-tree with T_n finite, $T_n \subseteq T_{n+1}$ and $L_n \subseteq L_{n+1}$ and if $p_n \in T_n$, $F_n \subseteq T_n$, and f_n consistent with L_n, then there exists $t \in T_{n+1} \setminus T_n$ with $p_n = t^*$ such that $L_{n+1}(s,t) = f_n(s)$ for all $s \in F_n$. This can be done as follows: choose any $t \notin T_n$ with $t^* = p_n$. For $s \in F_n$ define $L_{n+1}(s,t) = f_n(s)$. For all other $s \in T_n$ with $|s| = |t|$ and $s^* \neq t^*$ put $L_{n+1}(s,t) = q$ for any $q \in Q$ with $q \lhd L_n(s^*, t^*)$.

Second let us check that this property is all that is needed for universality. Write any α-tree as an increasing union of finite subtrees T_n gotten by adding one new child to some node from T_n, i.e., $T_{n+1} = T_n \cup \{r_n\}$ where $r_n^* \in T_n$ but $r_n \notin T_n$. The map σ is constructed by extending $\sigma \upharpoonright T_n$ to T_{n+1} by defining σ at r_n. Without all the subscripts one step looks like this:

Suppose (T, l) is a finite α-tree and $r \in T$ has no child and let $T_0 = T \setminus \{r\}$. Suppose that $\sigma : T_0 \to \omega^{<\omega}$ is a tree embedding of $(T_0, l \upharpoonright [T_0]^2)$ into $(\omega^{<\omega}, L)$.

Suppose $|r| = n$, $F = \sigma(T_0) \cap \omega^n$, $p = \sigma(r^*)$, and $f : F \to \alpha \cup Q$ is defined by $f(\sigma(s)) = l(r,s)$. By our property there are infinitely many t such that we can extend σ to T by defining $\sigma(r) = t$. This proves the Lemma.[1]
QED

Theorem 1 follows immediately from the three Lemmas.

Remarks

Theorem 1 settles a question of Stefan Geschke [1]. It was motivated by his result that the smallest cardinality of a family of clopen graphs on the Cantor space, 2^ω, such that every such graph can be continuously embedded into some member of the family is exactly $\mathfrak{d}$, the dominating number. Geschke also showed that there is a clopen graph on ω^ω universal for all clopen graphs on 2^ω.

The family of U_α in Lemma 8 are also universal for all clopen graphs on closed subsets of ω^ω and hence for all clopen graphs on zero dimensional Polish spaces.

Recall that a clopen graph E on X is *true clopen* iff $E \subseteq X^2$ is symmetric irreflexive and clopen in X^2 - not just clopen in $X^2 \setminus \Delta$. The proof of Lemma 7 shows that in fact there is no clopen graph on ω^ω which is universal for all true clopen graphs on ω^ω. Note that if E_1 is continuously reducible to E_2 and E_2 is true clopen, then E_1 is true clopen. Also if E is true clopen, then there exists a clopen partition $\omega^\omega = \sqcup_{i<\omega} C_i$ such that $C_i^2 \cap E = \emptyset$ for each $i < \omega$. Using this we can vary the proof of Lemma 8 to produce true clopen U'_α for $\alpha < \omega_1$ such that every true clopen graph continuously embeds into one of them. Construct a α-universal tree L' similar to L but satisfying: if s, t are distinct, $|s| = |t| = n > 1$, and $s(0) = t(0)$, then $L'(s,t) = out$. Hence we are thinking of replacing C_i with $[\langle i \rangle]$.

In the case of unary predicates continuous reducibility is called Wadge reducibility, i.e., for $A, B \subseteq \omega^\omega$ define $A \leq_W B$ iff there exists a continuous $f : \omega^\omega \to \omega^\omega$ such that $x \in A$ iff $f(x) \in B$. For a generalization of Wadge reducibility to Borel labellings in a better-quasi-order see van Engelen, Miller, and Steel [3]. Louveau and Saint-Raymond [2] contains some results about the quasi-order of Borel linear orders under embeddability. Even for finite

[1]This type of argument is familiar to model theorists who would refer to it as joint embedding, amalgamation, and universal Fraisse structure. Set theorists would say its like Cantor's proof that every countable linear order embeds into the rationals.

graphs the n-cycles are pairwise incomparable under graph embedding, so we don't get a well-quasi-order. However there are weaker notions of reducibility under which finite graphs are well-quasi-ordered, see Robertson and Dale [4]. Perhaps there is a natural notion of reducibility for clopen graphs that gives a well-quasi-ordering.

References

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