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A CHARACTERIZATION OF THE LEAST CARDINAL FOR WHICH THE BAIRE CATEGORY THEOREM FAILS

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ABSTRACT. Let κ be the least cardinal such that the real line can be covered by κ many nowhere dense sets. We show that κ can be characterized as the least cardinal such that "infinitely equal" reals fail to exist for families of cardinality κ.

Let Baire (κ) stand for:

> "The real line is not the union of κ many nowhere dense sets (a set is nowhere dense iff its closure has no interior)".

The property was extensively studied in Miller (**1981**) and Miller (**1982**). It is easily seen (see Kuratowski (**1966**)) that: *not* Baire (κ) iff some separable, completely metrizable space is the union of κ many nowhere dense sets iff every separable, completely metrizable space without isolated points is the union of κ many nowhere dense sets.

For example, we may replace the real line by Cantor space (2^ω), or Baire space (ω^ω). Recall that ω is the first infinite ordinal and is equal to its set of predecessors (i.e. the nonnegative integers), $2 = \{0, 1\}$, X^Y is the set of functions mapping Y into X, 2 and ω have the discrete topology, and 2^ω and ω^ω have the product topology.

Let Uniformity (κ) stand for the proposition

> "Every subset of the real line of cardinality less than or equal to κ is meager (a set is meager iff it is the union of countably many nowhere dense sets)".

Let us recall some standard terminology: $|X|$ is the cardinality of X, (for any cardinal κ) $[X]^\kappa = \{Y \mid Y \subseteq X, |Y| = \kappa\}$, $[X]^{\leqslant\kappa} = \{Y \mid Y \subseteq X, |Y| \leqslant \kappa\}$, $\forall^\infty n$ abbreviates "for all but finitely many n", and $\exists^\infty n$ abbreviates "there are infinitely many n".

Consider the following properties:

> Different (κ) iff $\forall A \in [\omega^\omega]^{\leqslant\kappa}\ \exists X \in [\omega]^\omega\ \exists f \in \omega^\omega\ \forall g \in A$ $\forall^\infty n \in X\, f(n) \neq g(n)$;
>
> Equal (κ) iff $\forall A \in [\omega^\omega]^{\leqslant\kappa}\ \forall B \in [[\omega]^\omega]^{\leqslant\kappa}\ \exists f \in \omega^\omega\ \forall g \in A$ $\forall X \in B\ \exists^\infty n \in X\, (f(n) = g(n))$.

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In Miller (**1981**) it was shown that Uniformity (κ) iff Different (κ). A less satisfactory property was found equivalent to Baire (κ). The purpose of this note is to prove

THEOREM. *Baire* (κ) *iff Equal* (κ).

To see that Baire (κ) implies Equal (κ), note that for any $g \in \omega^\omega$ and $X \in [\omega]^\omega$, $\{f \in \omega^\omega \mid \forall n \in X\, f(n) \neq g(n)\}$ is closed nowhere dense in ω^ω.

Now let us prove that Equal (κ) implies Baire (κ). Let Independent (κ) stand for:

$$\text{“}\forall B \in \left[[\omega]^\omega\right]^{\leqslant \kappa} \exists Z \in [\omega]^\omega \, \forall X \in B \mid X \cap Z \mid = \mid X - Z \mid = \omega\text{”}.$$

LEMMA 1. *Equal* (κ) *implies Independent* (κ).

PROOF. Let $B = \{X_\alpha \mid \alpha < \kappa\}$ and choose $X^i_\alpha \in [X_\alpha]^\omega$ for $i = 0, 1$ so that $X^0_\alpha \cap X^1_\alpha = \varnothing$. Choose $g^i_\alpha \in \omega^\omega$ so that $g^i_\alpha(X^i_\alpha) = \{i\}$. By Equal ($\kappa$) let $f \in \omega^\omega$ be such that for all α and i, $\exists^\infty n \in X^i_\alpha\, f(n) = g^i_\alpha(n) = i$. Then $Z = f^{-1}\{0\}$ does the job. □

DEFINITION. $Z \in [\omega]^\omega$ is l-uncrowded iff $\forall n, m \in Z n \neq m \rightarrow \mid n - m \mid \geqslant l$.

DEFINITION. l-Uncrowded (κ) iff $\forall B \in [[\omega]^\omega]^{\leqslant \kappa}$ $\exists Z$ l-uncrowded $\forall X \in B \mid X \cap Z \mid = \omega$.

LEMMA 2. *The following are equivalent.*

(A) *Independent* (κ),

(B) 2-*Uncrowded* (κ),

(C) *for all* $l < \omega$ l-*Uncrowded* (κ).

PROOF. Let us first prove that (A) implies (B). I claim there exists $T \in [\omega]^\omega$ such that for every $X \in B$, $\mid X - T \mid = \omega$. To see this, note that Independent (κ) implies $\kappa < c = \mid \omega^\omega \mid$. A well-known theorem of Sierpinski (**1928**) says that there exists an almost disjoint family of cardinality c, i.e. there exists $M_\alpha \in [\omega]^\omega$ for $\alpha < c$ such that for all $\alpha \neq \beta$, $\mid M_\alpha \cap M_\beta \mid < \omega$. Since the M_α are almost disjoint and $\kappa < c$, for some $\alpha < c$ for all $X \in B \mid X - M_\alpha \mid = \omega$. Let T be any such M_α.

Let E be the even integers and O the odd integers. Without loss of generality we may assume that for all $X \in B$, $X \subseteq E - T$ or $X \subseteq O - T$. Let $T = \{a_n : n < \omega\}$ be an enumeration in increasing order and for any a and b let $(a, b) = \{n \in \omega \mid a < n < b\}$. For each $X \in B$ let $X^* = \{n \mid (a_n, a_{n+1}) \cap X \neq \varnothing\}$. Let W be independent with respect to $\{X^* \mid X \in B\}$ (i.e. for all X^*, $\mid X^* - W \mid = \mid X^* \cap W \mid = \omega$). Let

$$Z = \bigcup \{(a_n, a_{n+1}) \cap E \mid n \in W\} \cup \{(a_n, a_{n+1}) \cap O \mid n \notin W\}.$$

It is easily checked that Z is 2-uncrowded and for all $X \in B$, $\mid Z \cap X \mid = \omega$.

(B) implies (C) is proved by induction on l. Suppose Z is l-crowded and for all $X \in B$, $\mid X \cap Z \mid = \omega$. Let $Z = \{a_n : n < \omega\}$ (increasing order) and for each $X \in B$, $X^* = \{n \mid a_n \in X\}$. Let Q be a 2-uncrowded set such that for all $X \in B$, $\mid X^* \cap Q \mid = \omega$. Then $\{a_n \mid n \in Q\}$ is a $2l$-uncrowded set meeting each element of B in an infinite set.

Now we prove (B) implies (A). Let $\{W_\alpha \mid \alpha < \kappa\} \subseteq [\omega]^\omega$. For each $\alpha < \kappa$, let $W^e_\alpha = \{2n \mid n \in W_\alpha\}$ and $W^0_\alpha = \{2n + 1 \mid n \in W_\alpha\}$. Let Z be a 2-uncrowded set such that for each $\alpha < \kappa$, $\mid Z \cap W^e_\alpha \mid = \mid Z \cap W^0_\alpha \mid = \omega$. Let $Q = \{n \mid 2n \in Z\}$. Then

$|Z \cap W_\alpha^e| = \omega$ implies $|Q \cap W_\alpha| = \omega$, and $|Z \cap W_\alpha^0| = \omega$ implies $|W_\alpha - Q| = \omega$ since $2n + 1 \in Z$ implies $2n \notin Z$. □

LEMMA 3. (*Equal* (κ)) $\forall F \in [\omega^\omega]^{\leq\kappa}$ *there exists a sequence* $n_k < \omega$ *for* $k < \omega$ *such that* $n_{k+1} > \Sigma_{i=0}^k n_i$ *and for every* $f \in F$, $\exists^\infty k\ f(n_k) < n_{k+2}$.

PROOF. We may assume without loss of generality that each $f \in F$ is strictly increasing. Choose $g \in \omega^\omega$ such that for every $f \in F$, $\exists^\infty n\ g(n) = f(n)$. Construct a sequence n_k for $k < \omega$ so that $n_{k+1} > \Sigma_{i=0}^k n_i$ and for every $i \leq n_k$, $g(i) < n_{k+1}$. Then for every $f \in F\ \exists^\infty k\ f(n_k) < n_{k+2}$. □

Now we finish proving the theorem. Let us review some standard terminology. Let $2^{<\omega} = \bigcup_{n<\omega} 2^n$ and for $s \in 2^{<\omega}$ let $|s|$ be the length of s (i.e. that n such that $s \in 2^n$). For s and t elements of $2^{<\omega}$ let $s^\frown t$ be their concatenation. A basic clopen subset of 2^ω is of the form $[s] = \{x \in 2^\omega \mid s \subseteq x\}$ for some $s \in 2^{<\omega}$.

Suppose $D_\alpha \subseteq Z^\omega$ for $\alpha < \kappa$ are dense open sets. We must show that $\bigcap_{\alpha<\kappa} D_\alpha \neq \varnothing$. Construct f_α: $\omega \to 2^{<\omega}$ such that for every $s \in 2^{\leq n}$, $[s^\frown f_\alpha(n)] \subseteq D_\alpha$. This is done by successively extending $|2^{\leq n}|$ times. By Lemma 3 there exists a sequence n_k for $k < \omega$ with $n_{k+1} > \Sigma_{i=0}^k n_i$ and for each α, $\exists^\infty k\ |f_\alpha(n_k)| < n_{k+2}$. By Equal ($\kappa$) there exists g: $\omega \to 2^{<\omega}$ such that for each $\alpha < \kappa$,

$$X_\alpha = \{k < \omega \mid f_\alpha(n_k) = g(k) \text{ and } |f_\alpha(n_k)| < n_{k+2}\}$$

is infinite. We may assume that for all k, $|g(k)| < n_{k+2}$. Now let Z be a 3-uncrowded set such that for all $\alpha < \kappa$, $|X_\alpha \cap Z| = \omega$ and let $Z = \{k_n \mid n < \omega\}$. Define $h \in 2^\omega$ to be the infinite concatenation

$$g(k_0)^\frown g(k_1)^\frown g(k_2)^\frown \ldots.$$

Then $h \in D_\alpha$ for each α, because if $k_n \in Z \cap X_\alpha$, then

$$|g(k_0)^\frown g(k_1)^\frown \cdots{}^\frown g(k_{n-1})| \leq \sum_{i=0}^{n-1} n_{k_i+2} \leq n_{k_{n-1}+3} \leq n_{k_n}$$

and $g(k_n) = f_\alpha(n_{k_n})$. □

The notion of independent family is due to Fichtenholz and Kantorovitch (**1934**). The property Independent (κ) is due to R. Price (**1979**). The notion of uncrowded set is new here, therefore let us scrutinize some variations of it.

First, we may weaken this notion by saying that $Z \subseteq \omega$ is loosely packed iff there exists $N < \omega$ such that for all i, $(i, i + N) - Z \neq \varnothing$ (i.e. Z does not contain a block of $N - 1$ consecutive integers). Call a set $Z \in [\omega]^\omega$ ∞-uncrowded iff $Z = \{a_n : n < \omega\}$ and $\lim_{n\to\infty}(a_{n+1} - a_n) = \infty$. Define the two properties Loosely packed (κ) and ∞-Uncrowded (κ) by requiring that for every $B \in [[\omega]^\omega]^{\leq\kappa}$ there exists Z loosely packed (∞-uncrowded) such that for all $X \in B$, $|Z \cap X| = \omega$.

THEOREM. (A) *Independent* (κ) *iff Loosely packed* (κ).

(B) *Independent* (ω_1) $\nRightarrow$ ∞-*Uncrowded* (ω_1) $\nRightarrow$ *Baire* (ω_1).

To prove part (A) left to right, just note that a 2-uncrowded set is loosely packed. Now suppose that Independent (κ) fails. Then there exists $B \in [[\omega]^\omega]^{\leq\kappa}$ such that for every finite partition of ω, $\{X_0, X_1, X_2, \ldots, X_{n-1}\}$, there exists $i < n$ and $X \in B$

such that $X \subseteq X_1$. The easiest way to obtain such a B is by a Lowenheim-Skolem argument. (Those readers unfamiliar with the logic involved are invited to find their own proof.) Let $B_0 \in [[\omega]^\omega]^{\leqslant \kappa}$ witness that Independent (κ) fails, i.e. $\forall Z \in [\omega]^\omega$ $\exists X \in B_0\ X \cap Z$ is finite or $X - Z$ is finite. Let M be an elementary substructure of $(H(c^+), \varepsilon)$ such that $\{B_0\} \cup B_0 \subseteq M$ and $|M| \leqslant k$. Let $B = M \cap [\omega]^\omega$. For any $Z \in M \cap [\omega]^\omega$ and $Y \subseteq Z$ there exist $X \in M \cap [\omega]^\omega$ such that $X \subseteq Z$ or $X \cap Y = \varnothing$. This is because there is, in M, a bijection between Z and ω. Now by an easy induction on $n < \omega$, for every partition $X_0, X_1, X_2, \ldots, X_{n-1}$ of ω suppose Z is loosely packed and N is such that for every i, $(i, i + N) - Z \neq \varnothing$. Define X_l for $l < N$ by $k \in X_l$ iff $(k \cdot N + l) \notin Z$. Clearly $\bigcup_{l<N} X_l = \omega$. So $\exists Q \in B, \exists l < N\, Q \subseteq X_l$. But then $\{k \cdot N + l \mid k \in Q\} \cap Z = \varnothing$, so not Loosely packed (κ).

Next we show that Independent (ω_1) does not imply ∞-Uncrowded (ω_1). This fact is demonstrated by the random real model of Solovay. This model is obtained by forcing with the measure algebra B of 2^κ for some $\kappa \geqslant \omega_2$ over a model of CH. One easily shows that if R is a random subset of ω then for every $X \in [\omega]^\omega$ in the ground model, $|X - R| = |X \cap R| = \omega$. Now suppose

$$\mu([\![\tau \subseteq \omega \wedge \tau \text{ } \infty\text{-uncrowded}]\!]) = 1.$$

By standard arguments obtain an increasing sequence n_k in the ground model such that $\mu([\![\forall k\ (\tau - n_k)$ is k-uncrowded$]\!] = c) \geqslant \frac{1}{2}$. Letting $b_i = [\![i \in \tau]\!]$ we see that for any $i < j < k$, $\mu(b_{n_k+i} \cdot b_{n_k+j} \cdot c) = 0$. Thus we can find an infinite $W \subseteq \omega$ such that $\mu(\Sigma_{i \in W} b_i) \leqslant \frac{1}{4}$. It follows that $\mu([\![\tau \cap W = \varnothing]\!]) \geqslant \frac{1}{4}$.

Now let us see that ∞-Uncrowded (ω_1) does not imply Baire (ω_1).

DEFINITION. Bounded (κ) iff $\forall F \in [\omega^\omega]^{\leqslant \kappa}\ \exists g \in \omega^\omega\ \forall f \in F\ \forall^\infty n\ f(n) < g(n)$.

DEFINITION. Weak Bounded (κ) iff $\forall F \in [\omega]^{\leqslant \kappa}$, $\exists g \in \omega^\omega\ \forall f \in F\ \exists^\infty n\ f(n) < g(n)$.

LEMMA 1. (1)[2] *Bounded* (κ) $\Rightarrow$ *Independent* (κ).

(2) *Independent* (κ) + *Weak Bounded* (κ) $\Rightarrow$ ∞-*Uncrowded* (κ).

PROOF. (1) This is a generalization of Theorem 2 of Solomon (1977). Given $B \in [[\omega]^\omega]^{\leqslant \kappa}$ define for each $X \in B$, $g_X \in \omega^\omega$ by letting $g_X(n)$ be least element of B greater than n. By Bounded (κ) find $f \in \omega^\omega$ which eventually dominates each g_X for $X \in B$. Let

$$Z = \bigcup_{n<\omega} [f(2n), f(2n + 1)).$$

Then for all $X \in B$,

$$|X \cap Z| = |X - Z| = \omega.$$

(2) Given $B \in [[\omega]^\omega]^{\leqslant \kappa}$ find using Independent (κ) a sequence $Z_{n+1} \subseteq Z_n$ such that Z_n is n-uncrowded and for all $X \in B$, $|Z_n \cap X| = \omega$. For each $X \in B$ define $f_x \in \omega^\omega$ by requiring that for each $n < \omega$, $[n, f_x(n)) \cap X \cap Z_n \neq \varnothing$. By Weak

[2] This was discovered also by P. Nyikos, F. Galvin, and G. Gruenhage.

Bounded (κ) let $g \in \omega^\omega$ infinitely often dominate each f_x and put

$$Z = \bigcup \{[n, g(n)) \cap Z_n \mid n < \omega\}. \quad \square$$

In Laver's model (Laver (**1976**)), it is well known that Bounded (ω_1) holds but Baire (ω_1) fails. $\square$

Some remarks and questions. Can one drop the set X from Different (κ) or the family B from Equal (κ)? If one changes the definition of Weak Bounded (κ) to

$$\forall F \in [\omega^\omega]^{\leqslant\kappa}\, \forall B \in \big[[\omega]^\omega\big]^{\leqslant\kappa}\, \exists g \in \omega^\omega\, \forall X \in B\, \forall f \in F\, (\exists^\infty n \in X\, f(n) < g(n)),$$

one gets an equivalent property (Roitman (**1979**)).

The property Bounded (κ) was defined and studied by Rothberger (**1939**, **1941**, **1952**). See also R. Solomon (**1977**). The property Weak Bounded (κ) has been studied in connection with p-points, see Ketonen (**1976**) and in connection with box products, see Williams (**1976**), Roitman (**1979**), and van Douwen (**1981**).

For some models in which Independent (ω_1) fails, see Baumgartner and Laver (**1979**) and Kunen (**1980**) (in particular Exercises A10, A12, and A13 on p. 289).

Can the least κ for which Independent (κ) fails have cofinality ω?

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